Self-modulation of a strong electromagnetic wave in a positron-electron plasma induced by relativistic temperatures and phonon damping

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The modulational instability of a linearly polarized, strong, electromagnetic wave in a (unmagnetized) positron-electron plasma is analyzed using relativistic two-fluid hydrodynamics to properly account for physical regimes of very high temperatures. The effect of phonon damping is also included in the treatment. The theory can be reduced to a pair of extended Zakharov equations. The envelope modulation is then studied by deriving the corresponding nonlinear Schrödinger (NLS) equation, using multiscale perturbation analysis. According to the intensity of the damping three different types of NLS are obtained. The main results are (a) that relativistic temperatures modify the stability result found in the literature for low temperature, zero damping, e_+-e_- plasmas, and (b) that phonon damping also produces substantial changes in the NLS, which then predict unstable envelopes. This work extends previous analyses, showing that if the phonon damping is $O(\epsilon^0)$ or $O(\epsilon^1)$ (ϵ is the perturbation parameter), a modulational instability appears in the electron-positron case in all ranges of temperature and wave frequencies. Thus presence of some amount of sound absorption helps to produce an envelope decay. When the phonon damping is very small $[O(\epsilon^2)]$ the self-modulational instability occurs in a finite band near the reduced plasma frequency, for ultrarelativistic temperatures. [S1063-651X(97)14302-1]

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I. INTRODUCTION

The literature on waves and nonlinear processes in positron-electron plasmas has grown rapidly in recent times in view of possible applications in the following fields. Relativistic positron-electron plasmas are encountered in pulsar magnetospheres and in active galactic nuclei. Surveys of these fields can be found, for instance, in Refs. [1,2] (pulsars) and [3,4] active galactic nuclei (AGN). Interesting scenarios for e_+-e_- plasmas are conjectured in the physics of the early time Universe, i.e., $10^{-4}<t<1$ s after the big bang [5–7]. In the laboratory, nonrelativistic electron and positron trapping in magnetic mirror experiments are presently actively pursued [8,9].

Linear and nonlinear waves in e_+-e_- plasmas have many properties different from electron-ion plasmas [10], due to the absence of high and low frequency scales associated with the electron-ion mass difference. A survey of linear waves in e_+-e_- plasmas can be found in [11], while [12] reviews nonlinear relativistic effects in plasmas, including the e_+-e_- case. Other surveys on relativistic nonlinear effects in waves for ordinary and e_+-e_- plasmas, related to the physical mechanisms discussed in this paper, can be found in Refs. [13] and [14].

Propagation and nonlinear processes associated with a strong, linearly polarized, electromagnetic wave in an unmagnetized electron-positron plasma have been studied by several authors (see, e.g., [14–20]) since pioneering work by Chian and Kennel [15] proposed a mechanism to explain very short intensity variations (micropulses) of pulsar radio emission. They suggested that a self-modulational instability of the electromagnetic wave may be a natural process for amplitude modulation. Gangadhara *et al.* [18] examined, instead, a parametric instability of a weakly relativistic, electromagnetic wave in the e_+-e_- plasma, to explain also the short-time variability of the radio sources. Kates and Kaup [17] in a careful analysis of the modulational instability, based on multiple time-space scale perturbation theory of the nonlinear electromagnetic wave in an electron-ion plasma, concluded that in the zero temperature limit the special case of a e_+-e_- plasma is stable. When a finite (classical) temperature is considered, only a vanishingly small frequency interval (at $\omega\approx\omega_p$) appears, where the instability is possible in a e_+-e_- plasma. The divergence with the results of [15] is explained by the absence of the ponderomotive force and harmonic generation effects in that reference. In an electron-ion plasma, however, the modulational envelope instability of the electromagnetic wave is again possible. The nonlinear Schrödinger equation (NLS) derived in [16], which was supportive of the instability found in [15] and which included longitudinal density variations, was found to be incorrect in [17].

We have reexamined the problem of the self-modulational instability of a linearly polarized, large-amplitude, electromagnetic wave in a (unmagnetized) positron-electron plasma, using a two-fluid model and taking into account the three nonlinear effects considered in [17], i.e., (i) relativistic correction due to mass variation, (ii) the ponderomotive force that produces density changes, and (iii) harmonic generation. Indeed, the three mentioned effects have the same order of magnitude in a perturbation treatment and they must all be included at the same level in the theory.

We first obtain a set of extended Zakharov-type equations [see Eqs. (31) and (32)] for the vector potential of the wave A coupled with phonons, i.e., with the sum of the perturbed electron and positron density $\mathcal{N}$ [here $A=eA_x/mc^2$, $\mathcal{N}=(n_p+n_e-2n_0)/n_0$, n_0 is the unperturbed density, c the speed of light, $e>0$, m the positron charge and mass, Gaussian units are used throughout]. These equations are correct up to order A^3 in the wave amplitude. The derivation is carried through using a fully relativistic hydrodynamic two-fluid model so that both ultrarelativistic and classical random thermal energies can be considered. In addition, we introduce a phenomenological damping in the phonon equation.

Early Universe plasma and active galactic nuclei plasma (which may or may not be magnetized) have very large temperatures. The plasma of the early Universe has ultrarelativistic temperatures, i.e., the electron and positron energy much larger than the rest mass energy, so that electrons and positrons behave dynamically as photons in the time interval $10^{-4}<t<1$ sec from the big bang [7]. The damping of phonons and longitudinal waves in a relativistic e^+-e^- plasma is treated in [7] and [21].

We carry through a multiscale perturbation analysis [parameter $\epsilon\sim O(A)$], and obtain three different types of nonlinear Schrödinger equations according to the case of finite (order one), $O(\epsilon^0)$, weak, $O(\epsilon^1)$, or ultraweak damping, $O(\epsilon^2)$. In the first two cases we prove that absorption in the longitudinal oscillations induces a modulational instability of the electromagnetic wave in a e^+-e^- plasma. In the third case, where the NLS coincides for classical temperatures with the one obtained in [17] with zero damping, the envelope instability appears again in a finite band near the relativistically reduced plasma frequency when the temperature is ultrarelativistic.

Our study, thus, complements and extends the analysis of [17] (which is limited to classical temperatures and does not include damping) showing that at relativistic temperatures, or when the damping of the acoustic waves becomes relevant, a modulational instability appears also in the e_+-e_- case. In spite of the incomplete derivation, the process conjectured in [15] may occur after all. However, as noted by most of the cited authors, for pulsar applications more work is needed to take into account the presence of a very strong magnetic field. Our paper is focused on the study of general theoretical consequences of relativistic temperatures and phonon damping on the modulational instability, and we do not intend to analyze here the implications for a particular physical scenario. However, we hope the results may be useful in several applications.

The work is presented in the following way. Section II describes the relativistic hydrodynamic fluid model. Section III presents the basic equations for a linearly polarized, finite amplitude plane wave in a e^+-e^- plasma. In Sec. IV we derive the basic equation set (31) and (32) and show its validity up to third order in A. In Sec. V we report the multiple scale perturbation analysis and the derivation of three types of NLS, according to the damping intensity. For order one damping we find a NLS with a complex cubic coefficient. For weak damping the NLS becomes an integrodifferential equation, with the integral in the cubic (real) term. For ultraweak damping we get a NLS with a real cubic coefficient (similar to that of Ref. [17]). In Sec. VI the stability of constant envelope solutions of the three types of NLS is analyzed. The basic solutions are perturbed with a spatial modulation, and the dispersion relation is examined. Sections VI A, VI B, and VI C deal with order one, weak, and ultraweak damping, respectively. In all three cases a modulational instability is found. Discussions and conclusions, with a table that summarizes the NLS derived in the paper, are given in Sec. VII.

II. RELATIVISTIC POSITRON-ELECTRON TWO-FLUID MODEL

We examine here the equations for the relativistic motion of an e^+-e^- plasma. We are interested in the behavior of the wave in systems with very high plasma temperatures, where the internal energy associated with random motion contributes substantially to the inertia of the fluid. We extend, therefore, the analysis of the references quoted in the Introduction, to a relativistic hydrodynamic treatment of the two-fluid plasma model. Thus, we work with equations valid both for classical temperatures, $p\ll nmc^2$ (p,n the pressure and number of particles density, respectively) as well as for ultrarelativistic energies, $p\gg nmc^2$.

We use the following signature for the metric tensor $ds^2=g_{\mu\nu}dx^\mu dx^\nu$, $g_{\mu\nu}=\mathrm{diag}(1,-1,-1,-1)$, $\mu,\nu=0,1,2,3$, where $dx^\mu=(cdt,dx^l)$ ($l=1,2,3$, roman indices for ordinary vectors) and denote with $u^\mu=\gamma(c,v^l)$ the average tetravelocity of a fluid element, $\gamma=[1-(\mathbf{v}/c)^2]^{-1/2}$. The energy-momentum tensor for an ideal fluid is

$$T^{\mu\nu}=\frac{h}{c^2}\,u^\mu u^\nu-pg^{\mu\nu}, \tag{1}$$

where h,p are the enthalpy and pressure fields, measured in the rest frame of each element of the fluid [$\overline{u^\mu}=(c,0)$] (for this and other basic equations see, e.g., [23,24]). It is also convenient to introduce n, the number particle density and the total energy density $\mathcal{E}$, both as quantities in the proper frame. Then, $h=\mathcal{E}+p$, and $\mathcal{E}=nmc^2+\overline{\epsilon}$, where $\overline{\epsilon}$ is the internal energy of the fluid (m the proper mass of the particle). In general, equations of state link $\overline{\epsilon}$ and p with density n and temperature T (T the temperature in the rest frame, in energy units)

$$\overline{\epsilon}=\overline{\epsilon}(n,T), \quad p=p(n,T). \tag{2}$$

In a low energy plasma (classical limit) we have $\mathcal{E}\gg p$, $\overline{\epsilon}=(3/2)nT$, and $p=(2/3)\overline{\epsilon}$, while in the ultrarelativistic limit $p\gg nmc^2$ and $p=(1/3)\mathcal{E}$.

The basic equation of a charged fluid in the presence of electromagnetic fields is

$$\partial_\nu T^{\mu\nu}=\frac{1}{c}\,j_\nu F^{\nu\mu}, \tag{3}$$

where $j^\nu=qnu^\nu$ (q the electric charge of the particles) and $F^{\nu\mu}=\partial^\nu A^\mu-\partial^\mu A^\nu$, indicating the tetrapotential with $A^\mu=(\phi,A^l)$ (ϕ the scalar potential, A^l the vector potential). The electric and magnetic fields are given as usual by $\mathbf{E}=-(1/c)\partial\mathbf{A}/\partial t-\mathrm{grad}\phi$, $\mathbf{B}=\mathrm{rot}\mathbf{A}$. The conservation of the number of particles (and, therefore, also of charge) is written as

$$\partial_\nu(nu^\nu)=0. \tag{4}$$

From $u_\mu\partial_\nu T^{\mu\nu}=0$ one obtains the equation of adiabatic motion

$$\frac{d\mathcal{E}}{dt}=\frac{h}{n}\frac{dn}{dt}. \tag{5}$$

In the classic limit, $h=nmc^2+(5/3)\overline{\epsilon}$, and we readily obtain $\overline{\epsilon}/\overline{\epsilon_0}=p/p_0=(n/n_0)^{5/3}$ (denoting with a subscript zero any reference state). In the ultrarelativistic case, $h=(4/3)\mathcal{E}$, and we have instead, $\overline{\epsilon}/\overline{\epsilon_0}=p/p_0=(n/n_0)^{4/3}$. Positron and electrons behave dynamically as photons at ultrarelativistic temperatures. For very high temperatures, $T\gg mc^2$, positrons and electrons coexist with a high frequency $\hbar\omega\sim T$ photon gas. This can be modeled with a radiation pressure $p_r=\text{const}\times n^{4/3}$ to be added to the gas pressure. However, positrons and electrons follow the same adiabatic law, therefore this addition affects only the constant of the 4/3 adiabatic law. In the following we assume that this change has been taken into account. The low frequency ($\hbar\omega\ll T$) electromagnetic wave, instead, is treated via Maxwell equations and plasma collective effects.

The effective collision frequency in the e^+-e^- plasma, which includes recombinations and photon annihilations, is assumed to be much smaller than the plasma frequency. The validity conditions of wave equations and collective plasma processes in a similar physical scenario are discussed in [7].

Therefore, we may write $p/p_0=(n/n_0)^\Gamma$, with a polytropic index $4/3\leq\Gamma\leq 5/3$, and using Eq. (5) we obtain $h=\Gamma p/(\Gamma-1)+nmc^2$, $\overline{\epsilon}=p/(\Gamma-1)$ as interpolation formulas that include both limits, with the understanding that as Γ approaches the value 5/3 we must set $p\ll nmc^2$.

The temporal component of Eq. (3) gives an equation for the time change of γ,

$$h\frac{d\gamma}{dt}=\frac{1}{\gamma}\frac{\partial p}{\partial t}-\gamma\frac{dp}{dt}+qn\mathbf{v}\cdot\mathbf{E}. \tag{6}$$

The spatial components of Eq. (3) [using Eqs. (5) and (6)] lead to the momentum equation of the relativistic fluid:

$$\frac{h}{c^2}\frac{d}{dt}(\gamma\mathbf{v})=-\frac{1}{\gamma}\,\text{grad}p-\frac{\mathbf{v}}{c^2}\gamma\frac{dp}{dt}+nq\left(\mathbf{E}+\frac{1}{c}\mathbf{v}\times\mathbf{B}\right). \tag{7}$$

We may note the enhancement of inertia (h/c^2 instead of nm) due to internal energy and pressure, and the presence of the extra force term originated from dp/dt in the right-hand side (RHS). There will be two equations like Eq. (7) for positrons and electrons, and we shall write $q=\sigma e$ ($e>0$, positron charge) where $\sigma=1$ for positrons and $\sigma=-1$ for electrons. All rest frame thermodynamic quantities in the absence of waves ($\mathbf{E}=\mathbf{B}=\mathbf{0}$) $n_0,p_0,h_0,\mathcal{E}_0$ are assumed to be constant and equal for electrons and positrons (neutral equilibrium state with equal random energy in both species).

III. PLANE ELECTROMAGNETIC WAVE OF FINITE AMPLITUDE

We consider now a disturbance of a uniform equilibrium, in which all the physical quantities depend on one spatial coordinate z, the direction of propagation of a linearly polarized, finite amplitude wave, with $E_x=E_x(z,t)$ and $B_y=B_y(z,t)$. Therefore we have $\mathbf{A}=(A_x(z,t),0,0)$ and $\mathbf{v}=(v_x(z,t),0,v_z(z,t))$, where $v_z(z,t)$ is generated by the v_xB_y term of the Lorentz force. The longitudinal motion $v_z(z,t)$ coupled with the wave is accompanied by a density variation $n(z,t)$, and a longitudinal electric field $E_z(z,t)=-\partial\phi(z,t)/\partial z$. In the following we use a nondimensional form for the potentials $A\equiv eA_x/mc^2$, $\varphi\equiv e\phi/mc^2$, and for the sake of simplicity in notation we use the same symbol for positron and electron quantities like $\mathbf{v}$ or n, although it is clear that they may be different during the evolution of the disturbance.

The assumed symmetry leads to an important invariant of the motion by considering the x component of Eq. (7):

$$\frac{h}{c}\frac{d}{dt}(\gamma v_x)=-\frac{\gamma v_x}{c}\frac{dp}{dt}-\sigma nmc^2\frac{dA}{dt}. \tag{8}$$

Noting that Eq. (5) is equivalent to

$$\frac{1}{n}\frac{dp}{dt}=\frac{d}{dt}\left(\frac{h}{n}\right) \tag{9}$$

we write Eq. (8) in the form

$$\frac{d}{dt}\left[\frac{h}{nmc^2}\left(\gamma\frac{v_x}{c}\right)+\sigma A\right]=0. \tag{10}$$

Considering that the fluid velocity v_x must be zero when $A=0$, we can write

$$\frac{h}{nmc^2}\gamma\frac{v_x}{c}=-\sigma A. \tag{11}$$

Finally from Eq. (11) it follows that

$$\gamma^2=\frac{1+(nmc^2/h)^2A^2}{1-(v_z/c)^2}, \tag{12}$$

The z component of Eq. (7) can also be elaborated in a convenient form,

$$\frac{h}{c^2}\frac{d}{dt}(\gamma v_z)=-\frac{1}{\gamma}\frac{\partial p}{\partial z}-\frac{\gamma v_z}{c^2}\frac{dp}{dt}+nmc^2\sigma\left(-\frac{\partial\varphi}{\partial z}+\frac{v_x}{c}\frac{\partial A}{\partial z}\right). \tag{13}$$

Using Eqs. (9) and (11) we obtain

$$\frac{1}{c}\frac{d}{dt}\left(\frac{h}{nmc^2}\gamma\frac{v_z}{c}\right)=-\frac{1}{\gamma nmc^2}\frac{\partial p}{\partial z}-\sigma\frac{\partial\varphi}{\partial z}-\frac{nmc^2}{\gamma h}\frac{\partial}{\partial z}\left(\frac{A^2}{2}\right). \tag{14}$$

The pressure gradient term in Eq. (14) contains the number density as measured in the laboratory $n_L=\gamma n$ that appears also in the continuity equation

$$\frac{\partial}{\partial t}(\gamma n)+\frac{\partial}{\partial z}(\gamma n v_z)=0, \tag{15}$$

which follows from Eq. (4).

Finally the potentials of the wave satisfy the equations

$$\left(\frac{\partial^2}{\partial t^2}-c^2\frac{\partial^2}{\partial z^2}\right)A=\frac{4\pi e^2}{m}\sum\sigma n_L\frac{v_x}{c}, \tag{16}$$

$$c\frac{\partial^2\varphi}{\partial z\partial t}=\frac{4\pi e^2}{m}\sum\sigma n_L\frac{v_z}{c}, \tag{17}$$

where the sum symbol is over positron and electron quantities. The set (9), (11), (14)–(17) and the thermodynamic properties (Sec. II) complete the set of equations for a plane, linearly polarized, finite amplitude, electromagnetic wave in a relativistic positron-electron plasma.

IV. WEAKLY NONLINEAR WAVES

Here we derive a reduced set of equations for the slow motion approximation of relativistic dynamics, i.e., when γ can be approximated by $\gamma\cong 1+(1/2)(\mathbf{v}/c)^2$. The basic idea is that when $\varepsilon\sim O(Anmc^2/h)$ is assumed to be small then according to Eq. (11), $v_x/c\sim O(\varepsilon)$. We can define an effective particle mass, m^*, enhanced by the relativistic temperature effect as $m^*c^2=h/n$, and note that the expansion parameter ε is a measure of the "quiver velocity" (peak oscillation velocity of e_+-e_- in the driving field) in units of the speed of light, $\varepsilon=|eE_{0x}|/(m^*\omega_0 c)\ll 1$, ω_0 being the frequency and E_{0x} the electric field amplitude. For low temperatures, $p\ll nmc^2$, the effective mass coincides with the rest mass, so that ε is the ratio of the classical "quiver velocity" to the speed of light. Thus, $\varepsilon\sim O(A)\ll 1$ is also a measure of the amplitude of the wave. However, for ultrarelativistic temperatures, since $h\gg nmc^2$, A is not necessarily small and can take values of order one.

Hence, we consider now weakly nonlinear waves, keeping systematically all significant terms up to order ε^3, and neglecting $O(\varepsilon^4)$ contributions. To this order of approximation the equations of Sec. III reduce to a pair of nonlinear coupled equations for A and $\mathcal{N}\equiv\Sigma(n_L-n_0)/n_0$.

If $v_x/c\sim O(\varepsilon)$, we can verify from Eq. (14) that $v_z/c\sim O(\varepsilon^2)$. In fact, assuming that $(v_z/c)^2\sim O(\varepsilon^4)$, Eq. (12) shows that

$$\gamma^2=1+\left(\frac{nmc^2}{h}A\right)^2+O(\varepsilon^4) \tag{18}$$

and, therefore, from Eq. (11)

$$\frac{v_x}{c}=-\sigma\frac{nmc^2}{h}A\left[1-\frac{1}{2}\left(\frac{nmc^2}{h}A\right)^2\right], \tag{19}$$

neglecting terms in ε^4 only. Several terms drop in Eq. (14), given the assumption that v_z/c is already of order ε^2, i.e., γ can be taken equal to 1 and dv_z/dt equal to $\partial v_z/\partial t$, with errors of order ε^4, so that

$$\frac{1}{c}\frac{\partial}{\partial t}\left(\frac{h}{nmc^2}\frac{v_z}{c}\right)=-\frac{1}{nmc^2}\frac{\partial p}{\partial z}-\sigma\frac{\partial\varphi}{\partial z}-\frac{nmc^2}{h}\frac{\partial}{\partial z}\left(\frac{A^2}{2}\right). \tag{20}$$

From Eq. (20) we can see *a posteriori* that indeed $v_z/c\sim O(\varepsilon^2)$ as assumed, since the first and second terms on the RHS must be of the same order as the third one. These longitudinal quantities originate from the wave through the Lorentz force, here represented by the last term in the RHS. In fact, Eqs. (15) and (17) confirm these estimates; i.e., density perturbations and electric potential are of the same order as v_z.

Finally, we perform an expansion of the thermodynamic quantities about the (constant) equilibrium state as $n=n_0+n'$, $h=h_0+h'$, where n',h' are $O(\varepsilon^2)$. We obtain, using Eq. (9)

$$\frac{v_x}{c}=-\sigma\frac{mc^2n_0}{h_0}A\left[1-\frac{1}{2}\left(\frac{mc^2n_0}{h_0}\right)^2A^2+\frac{n'}{n_0}-\frac{h'}{h_0}\right]+O(\varepsilon^4), \tag{21}$$

$$\frac{1}{c}\frac{\partial}{\partial t}\frac{v_z}{c}=-\frac{1}{h_0}\left(\frac{dp}{dn}\right)_0\frac{\partial n'}{\partial z}-\sigma\left(\frac{n_0mc^2}{h_0}\right)\frac{\partial\varphi}{\partial z}-\frac{1}{2}\frac{\partial}{\partial z}\left[\left(\frac{n_0mc^2}{h_0}\right)^2A^2\right]+O(\varepsilon^4), \tag{22}$$

where $(dp/dn)_0$ is the adiabatic pressure derivative evaluated in the equilibrium state. We also need $n_L=\gamma n$ in Eqs. (15)–(17), and setting $n_L=n_0+n_L'$, we have

$$\frac{n_L'}{n_0}=\frac{n'}{n_0}+\frac{1}{2}\left(\frac{n_0mc^2}{h_0}\right)^2A^2+O(\varepsilon^4), \tag{23}$$

so that Eq. (15) reduces to

$$\frac{\partial}{\partial t}\frac{n_L'}{n_0}=-\frac{\partial}{\partial z}v_z. \tag{24}$$

It is convenient to define two constants δ and η as

$$\delta\equiv\frac{\Gamma p_0}{h_0},\qquad \eta\equiv\frac{n_0mc^2}{h_0}.$$

Taking into account that $p=\mathcal{D}_\Gamma n^\Gamma$ (with $\mathcal{D}_\Gamma\equiv p_0/n_0^\Gamma$) we find that

$$\frac{n'}{n_0}-\frac{h'}{h_0}=-\delta\frac{n'}{n_0}=-\delta\left(\frac{n_L'}{n_0}-\frac{1}{2}\eta^2A^2\right). \tag{25}$$

Eliminating v_z from Eqs. (22) and (24), we obtain

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2}-\delta\frac{\partial^2}{\partial z^2}\right)\frac{n_L'}{n_0}=\eta\sigma\frac{\partial^2\varphi}{\partial z^2}+\frac{1}{2}(1-\delta)\eta^2\frac{\partial^2}{\partial z^2}A^2. \tag{26}$$

Summing Eq. (26) for electrons and positrons, the electric potential is eliminated, and defining the normalized sum of the perturbed densities as $\mathcal{N}\equiv(n_{Lp}'+n_{Le}')/n_0$, we get

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2}-\delta\frac{\partial^2}{\partial z^2}\right)\mathcal{N}=(1-\delta)\eta^2\frac{\partial^2}{\partial z^2}A^2. \tag{27}$$

The difference of Eq. (26) for electrons and positrons eliminates A^2, and setting $\mathcal{M}\equiv(n_{Lp}'-n_{Le}')/n_0$, we have

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2}-\delta\frac{\partial^2}{\partial z^2}\right)\mathcal{M}=2\eta\frac{\partial^2\varphi}{\partial z^2}. \tag{28}$$

Since this equation is uncoupled from the electromagnetic wave, we can choose the solution $\mathcal{M}=\varphi=0$ if we want to study only the longitudinal disturbances generated by the wave. Then, $n'_{Lp}=n'_{Le}$ and $v_{zp}=v_{ze}$. Thus, Eq. (27) is the equation for the phonons driven by the ponderomotive force of the electromagnetic wave. There is no separation of charges and no electrostatic field in the positron-electron plasma, to this order of approximation. Plasmons are given by Eq. (28), but for our purposes we assume that they are not present in the system.

We now introduce the expressions (21), (23), and (24) in (16), and neglecting $O(\varepsilon^4)$ terms we find

$$\left(\frac{\partial^2}{\partial t^2}-c^2\frac{\partial^2}{\partial z^2}\right)A+2\omega_p^2\eta A=\omega_p^2(1-\delta)\eta A(\eta^2A^2-\mathcal{N}), \tag{29}$$

where $\omega_p^2\equiv 4\pi e^2 n_0/m$ is the square of the electron (positron) plasma frequency. It is therefore convenient to use new variables,

$$\hat{z}\equiv\sqrt{\eta}z,\quad \hat{t}\equiv\sqrt{\eta}t,\quad \hat{A}\equiv\eta A, \tag{30}$$

so that we finally write the coupled equations for the weakly nonlinear wave (correct to order ε^3) as

$$\left(\frac{\partial^2}{\partial\hat{t}^2}-c^2\frac{\partial^2}{\partial\hat{z}^2}\right)\hat{A}+2\omega_p^2\hat{A}=\omega_p^2(1-\delta)\hat{A}(\hat{A}^2-\mathcal{N}), \tag{31}$$

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial\hat{t}^2}-\delta\frac{\partial^2}{\partial\hat{z}^2}\right)\mathcal{N}=(1-\delta)\frac{\partial^2}{\partial\hat{z}^2}\hat{A}^2, \tag{32}$$

where $\hat{A}$ (and $\mathcal{N}$) must be small with respect to unity.

For low thermal energies, case (a),

$$\eta=1-\frac{3}{2}\delta,\quad \delta=\frac{v_s^2}{c^2+(3/2)v_s^2}\ll 1,$$

where $v_s^2=(5/3)p_0/(mn_0)$ is the classic speed of sound.

For ultrarelativistic (random) energies, case (b),

$$\delta=\frac{1}{3},\quad \eta\ll 1.$$

The speed of sound is given now by $v_s^2=\delta c^2=(1/3)c^2$.

V. DERIVATION OF THE NONLINEAR SCHRÖDINGER EQUATIONS

We introduce now an important new element in the model: a phenomenological damping term for the phonons. References [7,21] discuss the nature of the noncollisional absorption of sound and longitudinal waves in a relativistic positron-electron plasma. Here we focus our attention on the effect of a phonon damping rate ν on the self-modulation of the transverse wave.

The main result of the present section is that relativistic temperatures and phonon damping produce substantial changes in the NLS. The consequences for the stability of the envelope are then examined in Sec. VI.

To simplify the notation we write the equations of Sec. IV, adding now the damping as

$$(\partial_t^2-c^2\partial_z^2+2\omega_p^2)A=(1-\delta)\omega_p^2A(A^2-\mathcal{N}), \tag{33}$$

$$(\partial_t^2-v_s^2\partial_z^2+\nu\partial_t)\mathcal{N}=(1-\delta)c^2\partial_z^2A^2. \tag{34}$$

Here, in case (a) (nonrelativistic thermal energy) $\delta=v_s^2/c^2$, $v_s^2=(5/3)p_0/(mn_0)$, t and z are the natural variables and $A,\mathcal{N}$ are defined as in Secs. III and IV. However, for case (b) (ultrarelativistic thermal energy) $\delta=1/3$ and $v_s^2=c^2/3$. Moreover, A stands here for $\hat{A}$, and t,z represent the reduced variables $\hat{t},\hat{z}$ of Sec. IV. Thus for applications to case (b), the formulas of this section must be first rescaled to ordinary time-space variables and to the normalized potential A.

To derive the nonlinear Schrödinger equation we apply the time-space multiscale perturbation technique to Eqs. (33) and (34) [22].

We formally write

$$A=\varepsilon A_1+\varepsilon^2A_2+\varepsilon^3A_3+O(\varepsilon^4),$$
$$\mathcal{N}=\varepsilon\mathcal{N}_1+\varepsilon^2\mathcal{N}_2+\varepsilon^3\mathcal{N}_3+O(\varepsilon^4), \tag{35}$$
$$\nu=\nu_0+\varepsilon\nu_1+\varepsilon^2\nu_2+\varepsilon^3\nu_3+O(\varepsilon^4),$$

with $\varepsilon\ll 1$. The expansion of ν in powers of ε allows treatment of weaker dampings by taking $\nu_0=0$, $\nu_0=\nu_1=0$, etc. To the lowest significant order

$$A_1=ae^{i\theta}+a^*e^{-i\theta},\quad \mathcal{N}_1=0, \tag{36}$$

where $\theta=kz-\omega t$ is the fast variable and k,ω satisfy the dispersion relation

$$D(\omega,k)=-\omega^2+k^2c^2+2\omega_p^2=0, \tag{37}$$

for the transverse electromagnetic wave. For applications to case (b) (as commented above) we must note that true values for k,ω are $k/\sqrt{\eta}$, $\omega/\sqrt{\eta}$, since here z,t represent the reduced variables (30). The amplitudes a,a^*, in Eq. (36) are taken as slow variables, and the perturbative expansion assumes that $A_{2,3}=A_{2,3}(a,a^*,\theta)$, $\mathcal{N}_{2,3}=\mathcal{N}_{2,3}(a,a^*,\theta)$. The space-time dependence of the higher order corrections is through a, a^*, and θ only. The multiscale technique is introduced by the following expansions

$$\frac{\partial a}{\partial t}=\varepsilon T_1(a,a^*)+\varepsilon^2T_2(a,a^*)+\varepsilon^3T_3(a,a^*)+O(\varepsilon^4), \tag{38}$$

$$\frac{\partial a}{\partial z}=\varepsilon Z_1(a,a^*)+\varepsilon^2Z_2(a,a^*)+\varepsilon^3Z_3(a,a^*)+O(\varepsilon^4), \tag{39}$$

and the corresponding complex conjugates. We have

$$\partial_z=k\partial_\theta+\varepsilon(Z_1\partial_a+Z_1^*\partial_{a*})+O(\varepsilon^2), \tag{40}$$

$$\partial_t=-\omega\partial_\theta+\varepsilon(T_1\partial_a+T_1^*\partial_{a*})+O(\varepsilon^2). \tag{41}$$

Care must be taken to remove secularities at each perturbative order. Applying the perturbative procedure and separating powers of ε we obtain from Eq. (33) the equation for A_2 (for order ε^2),

$$2\omega_p^2\left(\frac{\partial^2 A_2}{\partial\theta^2}+A_2\right)-2i\omega(T_1+v_g Z_1)e^{i\theta}+\text{c.c.}=0, \quad (42)$$

where $v_g=c^2k/\omega$ is the group velocity derived from Eq. (37). To eliminate the secular solution of A_2 in Eq. (42) we require

$$T_1+v_g Z_1=0. \quad (43)$$

Consequently, we can write

$$\frac{\partial a}{\partial t_1}+v_g\frac{\partial a}{\partial z_1}=0, \quad (44)$$

neglecting $O(\varepsilon)$ terms. Therefore, up to order ε^2, the amplitude is constant in time if we move with the wave packet speed. The amplitude modulation is revealed at the next order of the perturbation theory. The solution of Eq. (42) is

$$A_2=Be^{i\theta}+B^*e^{-i\theta}, \quad (45)$$

where $B=B(a,a^*)$ is an arbitrary function of the slow variables.

Introducing the operator

$$\mathcal{P}=Z_1\partial_a+Z_1^*\partial_{a*}=-\frac{1}{v_g}(T_1\partial_a+T_1^*\partial_{a*}),$$

Eq. (34) is expressed as

$$\{(\omega\partial_\theta+\varepsilon v_g\mathcal{P})^2-v_s^2(k\partial_\theta+\varepsilon\mathcal{P})^2-\nu_0(\omega\partial_\theta+\varepsilon v_g\mathcal{P})\}\mathcal{N}$$
$$=(1-\delta)c^2(k\partial_\theta+\varepsilon\mathcal{P})^2A^2. \quad (46)$$

To order ε^2 we find that

$$(\omega^2-v_s^2k^2)\frac{\partial^2\mathcal{N}_2}{\partial\theta^2}-\omega\nu_0\frac{\partial\mathcal{N}_2}{\partial\theta}=-4(1-\delta)c^2k^2a^2e^{2i\theta}+\text{c.c.} \quad (47)$$

To avoid a secular behavior of $\mathcal{N}_2$ we must exclude the solution $\exp(\alpha_1\theta)$ with $\alpha_1=\omega\nu_0/(\omega^2-v_s^2k^2)$. Therefore, we get

$$\mathcal{N}_2=\alpha a^2e^{2i\theta}+\text{c.c.}+C(a,a^*), \quad (48)$$

where α is a complex parameter defined by

$$\alpha=\frac{(1-\delta)c^2k^2}{\omega(\omega+i\nu_0/2)-v_s^2k^2} \quad (49)$$

and $C(a,a^*)$ is an arbitrary real function of the slow variables.

From the order ε^3 in Eq. (34) we obtain

$$(\omega^2-v_s^2k^2)\frac{\partial^2\mathcal{N}_3}{\partial\theta^2}+\nu_0\omega\frac{\partial\mathcal{N}_3}{\partial\theta}$$
$$=2a[\alpha\nu_0v_gZ_1+i\alpha\nu_1\omega a-4i\alpha Z_1(v_g\omega-v_s^2k)$$
$$+4(1-\delta)c^2k(iZ_1-kB)]e^{2i\theta}+\text{c.c.}+\nu_0v_g\mathcal{P}C. \quad (50)$$

Examining Eq. (50) we find that to avoid secularities in N_3 it is necessary that

$$\nu_0\mathcal{P}C=0. \quad (51)$$

No other information is needed from Eq. (50).

Finally the equation for A to third order, taking into account the results already obtained, gives

$$\left[\frac{\partial^2 a}{\partial t_1^2}-c^2\frac{\partial^2 a}{\partial z_1^2}-2i\omega(T_2+v_gZ_2)\right]e^{i\theta}+\text{c.c.}+2\omega_p^2\left(\frac{\partial^2A_3}{\partial\theta^2}+A_3\right)=(1-\delta)\omega_p^2[(1-\alpha)a^3e^{3i\theta}+[(3-\alpha)|a|^2-C]ae^{i\theta}]+\text{c.c.} \quad (52)$$

From Eq. (52) it follows that the equation that eliminates the secularity of A_3 is

$$-2i\omega(T_2+v_gZ_2)+\frac{\partial^2 a}{\partial t_1^2}-c^2\frac{\partial^2 a}{\partial z_1^2}$$
$$=(1-\delta)\omega_p^2a[(3-\alpha)|a|^2-C]. \quad (53)$$

Using previous results and $\partial v_g/\partial k=(1/\omega)(c^2-v_g^2)$ we can write Eq. (53) in the form of a NLS equation

$$i\frac{\partial a}{\partial\tau}+\frac{1}{2}\frac{\partial v_g}{\partial k}\frac{\partial^2 a}{\partial\xi^2}+\frac{1}{2}(1-\delta)\frac{\omega_p^2}{\omega}a[(3-\alpha)|a|^2-C]=0, \quad (54)$$

where we have introduced the space-time variables

$$\tau=t_2=\varepsilon t_1=\varepsilon^2t,$$

$$\xi=\frac{1}{\varepsilon}(z_2-v_gt_2)=z_1-v_gt_1=\varepsilon(z-v_gt). \quad (55)$$

We shall show now that the value of C depends on the intensity of the damping. Basically, there are three different cases: (i) $\nu_0>0$ (order-one damping), (ii) $\nu_0=0$ and $\nu_1>0$ (weak damping), and (iii) $\nu_0=\nu_1=0$ and $\nu_2>0$ (ultraweak damping).

In case (i), $\nu_0>0$, to comply with Eq. (51) C cannot depend on a or a^*: it must be an absolute constant. It follows that $C=0$, since for $a=0$ the density disturbance must be absent, so that $\mathcal{N}_2=0$. Thus we obtain a NLS equation with complex coefficients, since α is complex [Eq. (54) with $C=0$].

In case (ii), when $\nu_0=0$, the elimination of secularities follows a different route; Eq. (50) no longer provides information about $C(a,a^*)$. The value of C can be determined from Eq. (34), noting that it is exact to order ε^3 for the θ-dependent terms, but that it is valid to order ε^4 for the slowly varying terms (those that depend on a and a^*, but not on θ). Thus, we find that

$$\left(\mathcal{P}-\frac{\nu_1 v_g}{v_g^2-v_s^2}\right)\mathcal{P}C(a,a^*)=\frac{2(1-\delta)c^2}{v_g^2-v_s^2}\,\mathcal{P}^2(aa^*). \quad (56)$$

Integration of this equation gives

$$C=C_0|a|^2-C_0C_1e^{C_1\xi}\int_\xi^\infty e^{-C_1\zeta}|a|^2d\zeta, \quad (57)$$

with $C_0=2(1-\delta)c^2/(v_g^2-v_s^2)$, and $C_1=\nu_1 v_g/(v_g^2-v_s^2)$.

Therefore the NLS equation takes the form

$$i\frac{\partial a}{\partial\tau}+\frac{1}{2}\frac{\partial v_g}{\partial k}\frac{\partial^2 a}{\partial\xi^2}+\frac{(1-\delta)\omega_p^2 v_g}{2kc^2}\left[\left(3-\frac{(1-\delta)c^2k^2}{\omega^2-v_s^2k^2}-C_0\right)\right.$$
$$\left.\times|a|^2+C_0C_1e^{C_1\xi}\int_\xi^\infty e^{-C_1\zeta}|a|^2d\zeta\right]a=0. \quad (58)$$

In case (ii), $C_1\neq 0$, the NLS becomes an integrodifferential equation.

Finally, for an ultraweak damping, case (iii), Eq. (57) holds with $C_1=0$, so that

$$C=\frac{2(1-\delta)c^2|a|^2}{v_g^2-v_s^2} \quad (59)$$

and α is real. This ultraweak damping result reproduces, for classical temperatures, the NLS equation obtained in [17], which was derived for a positron-electron plasma with nonrelativistic temperatures ($v_s^2\ll c^2$) without damping effects.

VI. EFFECT OF PHONON DAMPING AND RELATIVISTIC TEMPERATURES: MODULATIONAL INSTABILITY

A. Order $O(1)$ damping

We report now on the envelope instability for damping effects of order $O(1)$. For consistency we assume $\varepsilon\ll\nu_0/\omega\ll 1$. We have a nonlinear Schrödinger equation with complex coefficients, which we will write as

$$i\frac{\partial a}{\partial\tau}+p\frac{\partial^2 a}{\partial\xi^2}+qa|a|^2=0, \quad (60)$$

with

$$p=\frac{1}{2}\frac{\partial v_g}{\partial k}=\frac{\omega_p^2c^2}{\omega^3}, \quad (61)$$

$$q_r=\mathrm{Re}(q)=\frac{(1-\delta)\omega_p^2}{2\omega}\left(3-\frac{(1-\delta)c^2k^2(\omega^2-v_s^2k^2)}{(\omega^2-v_s^2k^2)^2+(\nu_0\omega/2)^2}\right), \quad (62)$$

and

$$q_i=\mathrm{Im}(q)=\frac{\nu_0}{4}\frac{(1-\delta)^2\omega_p^2c^2k^2}{(\omega^2-v_s^2k^2)^2+(\nu_0\omega/2)^2}, \quad (63)$$

where we assume that $|q_i|\ll|q_r|$.

Let us consider a solution of Eq. (60) of the form

$$\overline{a(\xi,\tau)}=\frac{a_0}{\sqrt{1+2q_ia_0^2\tau}}\exp\left\{i\left(k_0\xi-pk_0^2\tau+\int\dot{\phi}d\tau\right)\right\}, \quad (64)$$

where $\dot{\phi}(\tau)=q_ra_0^2/(1+2q_ia_0^2\tau)$, and a_0 is a real constant. This solution decays slowly (i.e., not exponentially) in time.

We now introduce small perturbations $\delta a_{1,2}$ to the solution

$$a(\xi,\tau)=\overline{a(\xi,\tau)}+\delta a_1\exp\left\{i\left[(K+k_0)\xi-pk_0^2\tau\right.\right.$$
$$\left.\left.-\int(\Omega-\dot{\phi})d\tau\right]\right\}+\delta a_2\exp\left\{-i\left[(K-k_0)\xi\right.\right.$$
$$\left.\left.+pk_0^2\tau-\int(\Omega^*+\dot{\phi})d\tau\right]\right\} \quad (65)$$

and linearize. The following equations hold:

$$(\Omega-2pk_0K+2iq_i|\overline{a}|^2-pK^2+q_r|\overline{a}|^2)\delta a_1+q|\overline{a}|^2\delta a_2^*=0, \quad (66)$$

$$q|\overline{a}|^2\delta a_1^*+(-\Omega^*+2pk_0K+2iq_i|\overline{a}|^2-pK^2+q_r|\overline{a}|^2)\delta a_2$$
$$=0. \quad (67)$$

Thus, $\Omega(K)$ is given by

$$\Omega^2-4(pk_0K-iq_i|\overline{a}|^2)\Omega+4(pk_0K-iq_i|\overline{a}|^2)^2$$
$$+(|q|^2+q_i^2)|\overline{a}|^4-|pK^2-q|\overline{a}|^2|^2=0, \quad (68)$$

or, alternatively,

$$\Omega=2(pk_0K-iq_i|\overline{a}|^2)\pm\sqrt{p^2K^4-2pq_r|\overline{a}|^2K^2-q_i^2|\overline{a}|^4}, \quad (69)$$

where $|\overline{a}|^2=a_0^2/(1+2q_ia_0^2\tau)$.

Conditions for instability arise for values of K where the expression $S(K)=p^2K^4-2pq_r|\overline{a}|^2K^2+3q_i^2|\overline{a}|^4$ becomes negative. Thus we find an instability band in the interval (K_1,K_2) with

$$K_{1,2}=\left[|\overline{a}|^2\frac{q_r}{p}\pm\left(|\overline{a}|^4\frac{q_r^2}{p^2}-3|\overline{a}|^4\frac{q_i^2}{p^2}\right)^{1/2}\right]^{1/2}. \quad (70)$$

Note that the growth rate is maximum when $S(K)$ is minimum, that is, for $K_M=(|\overline{a}|^2q_r/p)^{1/2}$, and its value is $\mathcal{I}(\Omega)=|\overline{a}|^2(|q|-2q_i)$. A stability diagram is shown in Fig. 1, where in a semilogarithmic graph the ordinates Y_1,Y_2 represent K_1,K_2, measured in units of $\varepsilon|\overline{a}|/d$, versus the ab-

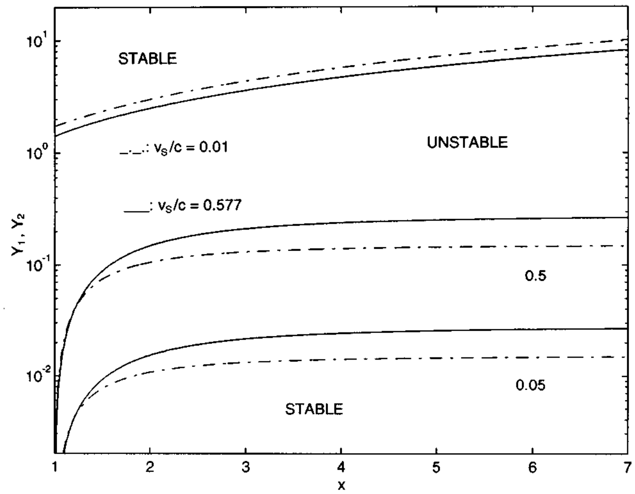

FIG. 1. Stability diagram for finite phonon damping, $\nu_0 \neq 0$. The semilogarithmic plot shows $(Y_1, Y_2) = (K_1, K_2) d/\epsilon |\overline{a}|$ ($d = c/\sqrt{2\eta}\omega_p$), as functions of $x = \omega/\sqrt{2\eta}\omega_p$. The instability occurs for Y, x values comprised between the upper line Y_2 and the lower line Y_1. The boundaries Y_1, Y_2 are shown for two values of $\nu/\sqrt{2\eta}\omega_p = 0.05$ and 0.5, to illustrate the effect of changes in the value of damping. The figure indicates also variations when temperature changes from classical to ultrarelativistic values. The lines Y_1, Y_2 are dash-dotted for $v_s/c = 0.01$, and full for $1/\sqrt{3}$.

scissa $x = \omega/\sqrt{2\eta}\omega_p$ [here $d = c/(\sqrt{2\eta}\omega_p)$]. The instability occurs for $Y_1 < Y < Y_2$. In the figure the lines are represented for two damping values, $\nu/\sqrt{2\eta}\omega_p = 0.05$ and 0.5. In addition the figure shows lines for two temperature values. The lines are dash-dotted for $v_s/c = 0.01$ and full for $1/\sqrt{3}$.

We can see in Fig. 1 that the temperature range, classic or ultrarelativistic, has some influence on the boundaries of the unstable region. The range $Y_1 - Y_2$, at a given frequency, is narrower at high energies. The lower boundary is also sensitive to changes of the parameter, $\nu/\sqrt{2\eta}\omega_p$, while the effect on the upper boundary is negligible. The boundary Y_1 increases by an order of magnitude when the damping is increased ten times, and so the unstable K interval decreases. The region of instability exists for all frequencies of the wave, but the growth rate tends to zero as $\omega \to \infty$. Note that, while the characteristics of the instability region depend on the presence of finite damping, the direct effect of ν_0 on the growth rate (in view of the assumption $|q_i| \ll |q_r|$) is not crucial. The instability persists even in the limit $\nu_0/\omega \to 0$.

We give now quick estimates of order of magnitude for the growth rate and characteristic wavelength of the modulational instability. Restoring true time and space variables, the dispersion relation of the electromagnetic wave is given by $\omega^2 = 2\omega_p^2 \eta + k^2 c^2$. At ultrarelativistic temperatures where $\eta \ll 1$, the plasma frequency cutoff decreases substantially. Waves can propagate at lower frequencies than in classical plasmas at the same density. This effect is due to the increased inertia, stored in very large thermal energies. In expressions (62) and (63) for q_r, q_i we must similarly replace ω_p^2 by $\omega_p^2 \eta$ when we return to ω, k values associated to the original t, z variables. We also have $p = \omega_p^2 \eta c^2/\omega^3$ from Eq. (61).

It is easy to see that over the whole k range, $q_r \sim O(\omega_p^2 \eta/\omega)$ (since $\nu/\omega \ll 1$ and v_s^2 can be at most $c^2/3$). Thus, passing from K_M to the true wave number, we find that $k_M \sim \varepsilon |\overline{a_0}| \omega/c$. We have introduced a collisionless skin depth $d = c/\sqrt{2\eta}\omega_p$ with the effective plasma frequency. Then, $\lambda_M/2\pi \sim d/\varepsilon|\overline{a_0}|$ when $k^2 d^2 \ll 1$, while $\lambda_M \sim \lambda/\varepsilon|\overline{a_0}|$ when $k^2 d^2 \gg 1$. The ultrarelativistic d is much larger than the classical value (at the same density). Thus, the length of the packet modulation increases with increasing temperatures in the relativistic domain.

Since $|q_i/q_r| \ll 1$ over the whole ω range, we estimate $|q| - 2q_i \sim q_r$. This gives us an order of magnitude for $g \equiv \mathcal{I}(\Omega)$ in true space-time scales, $g \sim \varepsilon^2 |\overline{a_0}|^2 q_r \sim \varepsilon^2 |\overline{a_0}|^2 \eta \omega_p^2/\omega$. Hence, we conclude that

$$g \sim \varepsilon^2 |\overline{a_0}|^2 \omega_p \sqrt{\eta} \qquad \text{for } k^2 d^2 \ll 1, \tag{71}$$

and

$$g \sim \varepsilon^2 |\overline{a_0}|^2 \omega_p \sqrt{\eta}/kd \qquad \text{for } k^2 d^2 \gg 1. \tag{72}$$

The modulational instability is faster for large wavelengths of the electromagnetic wave, and the growth rate is reduced by ultrarelativistic temperatures.

Finally, we check the consistency of the predicted instability with the decay of the amplitude $\overline{a(\tau)}$ given by Eq. (64). The decay rate μ of the chosen solution due to the imaginary part of q is given by $\mu \sim \varepsilon^2 |\overline{a}|^2 q_i$. Therefore, $g/\mu \sim q_r/q_i \sim \omega/\nu \gg 1$. The instability will develop much faster than the decay time of the amplitude. Thus, we confirm that the wave becomes modulationally unstable before decay by absorption

B. Weak damping

In the weak damping case ($\nu_0 = 0$, $\nu_1 \neq 0$) the corresponding NLS equation for the amplitude is given by Eq. (58). The analysis of this case shows greater complexity. The integro-differential NLS has a solution of constant envelope of the form $\overline{a(\xi,\tau)} = a_0 \exp\{i(k_0\xi - \omega_0\tau)\}$, with a_0 a real constant. This solution has the peculiarity that C given by Eq. (57) is zero. Thus the nonlinear wave with constant envelope is described by the same NLS of Sec. V, case (i), but with real α, since here $\nu_0 = 0$.

The small perturbations of this solution are a different matter, since the constant C given by Eq. (57) is not zero in this case. Perturbing the solution as

$$a(\xi,\tau) = \overline{a(\xi,\tau)} + \delta a_1 \exp\{i[(K + k_0)\xi - (\Omega + \omega_0)\tau]\} \tag{73}$$

$$+ \delta a_2 \exp\{-i[(K - k_0)\xi - (\Omega^* - \omega_0)\tau]\} \tag{74}$$

we find that in order to satisfy Eq. (58), Ω and K must be related by

$$\left(\Omega - 2pk_0K - (pK^2 - qa_0^2) + irC_0 a_0^2 \frac{K(C_1 + iK)}{C_1^2 + K^2}\right)\delta a_2 + \left(q + irC_0 \frac{K(C_1 + iK)}{C_1^2 + K^2}\right) a_0^2 \delta a_2^* = 0, \tag{75}$$

$$\left(q+irC_0\,\frac{K(C_1+iK)}{C_1^2+K^2}\right)a_0^2\delta a_2-\left(\Omega-2pk_0K+(pK^2-qa_0^2)-irC_0a_0^2\,\frac{K(C_1+iK)}{C_1^2+K^2}\right)\delta a_2^*=0, \tag{76}$$

where p is the same as in Eq. (61), q is equal to q_r of Eq. (62) with $\nu_0=0$, and $r=(1-\delta)\omega_p^2 v_g/2kc^2$. Consequently, the following dispersion relation holds:

$$\Omega=2pk_0K\pm\sqrt{p^2K^4-2p\left[q-rC_0\,\frac{K^2}{C_1^2+K^2}\right]a_0^2K^2-2irpC_0a_0^2\,\frac{C_1K^3}{C_1^2+K^2}}. \tag{77}$$

From this equation we conclude that conditions for the modulational instability can be achieved at any wave frequency. The imaginary term in the square root is a source of instability for all values of ω and K. This term depends on the damping ν_1 and disappears when $\nu_1\to 0$. The destabilizing effect of the imaginary term, for fixed frequency, increases at large values of K, but becomes negligible when $v_g\to 0$ instead, i.e., for large wavelengths, when ω is close to $\sqrt{2}\omega_p$.

On the other hand an instability can also arise, independently from the damping ν_1, when the sum of the real terms in the square root is negative. This may happen (under restrictive conditions on ω and K) even if C_1 is negligible. We shall comment upon the conditions for this instability in Sec. VI C. Finally, let us note that putting formally $C_0=0$ as a control of the dispersion relation, we reobtain the properties of the dispersion relation of Sec. VI A in the limit $\nu_0\to 0$. In fact setting $C_0=0$, Eq. (58) coincides with Eq. (60) with $\nu_0=0$.

C. Ultraweak damping

The ultraweak damping is ruled by a NLS equation similar to that obtained in [17]. However, when temperatures are relativistic, in our extended equation it is easier to satisfy the condition $v_g^2=c^2(c^2k^2)/\omega^2<v_s^2$, since $v_s^2=c^2/3$, so that C becomes a real negative coefficient in Eq. (54). Furthermore $\delta=1/3$, and so there is a reduction of C with respect to the classical value. It follows from basic NLS theory that when the coefficient of the cubic term is positive (since $\partial v_g/\partial k>0$) a constant wave envelope is modulationally unstable.

We find, after rescaling variables, an unstable frequency interval, $\sqrt{2\eta}\omega_p\leqslant\omega<1.2247\sqrt{2\eta}\omega_p$. When the temperature is nonrelativistic, the unstable frequency interval is $\sqrt{2}\omega_p\leqslant\omega<\sqrt{2}\omega_p(1-v_s^2/2c^2)$ instead. Since $v_s^2/c^2\ll 1$, the unstable frequencies are restricted to a very small interval close to $\sqrt{2}\omega_p$, whose width tends to zero for a cold plasma. Conversely, as the temperature grows the unstable frequency interval increases, and becomes a finite frequency band at ultrarelativistic temperatures.

Following the procedures of Secs. VI A and VI B (or taking the limit $\nu_1\to 0$ in the dispersion relation of Sec. VI B) it is easy to find that perturbations of the form $\exp\{i(K\xi-\Omega\tau)\}$ are unstable when $0<K<|a_0|\sqrt{2q/p}=K_c$ for $q>0$, with a growth rate given by $\mathcal{I}(\Omega)=\sqrt{2pq|a_0|K^2-p^2K^4}$. Here, we use the notation of Eq. (60), and a_0 is the amplitude of a constant envelope solution. The maximum growth rate, $\mathcal{I}(\Omega)=|a_0|^2q$, occurs at $K=K_c/\sqrt{2}$. The values of p,q to be used here (without rescaling) are $p=\omega_p^2c^2/\omega^3$, and $q=[(1-\delta)\omega_p^2v_g/2kc^2][3-(1-\delta)c^2k^2/(\omega^2-v_s^2k^2)-2(1-\delta)c^2/(v_g^2-v_s^2)]$.

The modulational instability described here is characteristic of a physical regime with zero phonon damping, since ν_2 does not enter in the equations at this order of the perturbation theory. Comparing the results of this section with those of Sec. VI B, we conclude that a small amount of damping [i.e., $\nu\sim O(\varepsilon)$] destabilizes the wave in K-ω regions that are stable in an ideal dissipationless system. Further increase of damping [i.e., $\nu\sim O(1)$] as in Sec. VI A, restricts again the instability to a particular K-ω region, which is nevertheless wider than that of the ideal case considered in this section, as shown in Fig. 1.

VII. DISCUSSION AND CONCLUSIONS

We have studied the modulational instability of a linearly polarized, strong electromagnetic wave, in an unmagnetized positron-electron plasma, using relativistic two-fluid hydrodynamics to properly account for physical regimes of very high temperatures. The nonlinear wave is coupled with longitudinal oscillations via the Lorentz force. A relativistic correction for slow motion, as well as the effect of density variations on wave propagation, are taken into account. We have also included different degrees of phonon damping in the treatment. The model can be reduced to a pair of extended Zakharov equations, (33) and (34). We may recall that the well-known Zakharov equations for longitudinal Langmuir waves in the usual ion-electron plasma, are derived by time averaging over fast variables, so that a term like $\partial_{zz}|E|^2$ appears as driver of sound waves. However, in the case of electromagnetic waves in an e_+-e_- plasma, we have obtained Eqs. (33) and (34) without averaging procedures, and they are exact to third order in the expansion parameter. Thus, a term $\partial_{zz}A^2$ appears in Eq. (34), without absolute value or time average.

The envelope modulation is then studied deriving the corresponding NLS equation, using multiscale perturbation analysis. According to the intensity of the damping we obtain three different types of NLS. The coefficients of the NLS also change with classical or relativistic temperatures. Table I summarizes all cases treated for easier reference. In view of possible applications, equations in Table I are written with ordinary time-space, and frequency–wave number variables, instead of the special scalings used for mathematical convenience in the text. The table gives the cubic NLS term for finite, weak, and ultraweak damping. In each case the explicit form, for classic and ultrarelativistic temperatures, is presented. Short notes on stability and conditions on

TABLE I. A summary of NLS equations.

$i\partial_{\bar\tau} + \bar p \partial^2_{\bar\xi\bar\xi} a + \Delta(a) = 0$, $\bar\tau = \varepsilon^2 t$, $\bar\xi = \varepsilon(z - v_g t)$, $v_f = \omega/k$

$\omega^2 = 2\eta\omega_p^2 + c^2k^2$, $v_g = kc^2/\omega$, $\bar p = \eta\omega_p^2 c^2/\omega^3$; t,z,ω,k: ordinary variables

Damping	T	$\Delta(a)$ cubic term	Notes
ν_0	Classic	$\Delta = \left[\frac{3}{2}\frac{\omega_p^2}{\omega}\left(1 - \frac{c^2(v_f^2 - v_s^2)}{(v_f^2 - v_s^2)^2 + (\nu_0 v_f/2k)^2}\right) + i\,\frac{\nu_0}{4}\,\frac{\omega_p^2 c^2/k^2}{(v_f^2 - v_s^2)^2 + (\nu_0 v_f/2k)^2}\right] a\lvert a\rvert^2$	$(v_s/c)^2 \ll 1$, $\eta = 1$ unstable for all ω
Finite	Ultrarelativistic	$\Delta = \left[\frac{\omega_p^2\eta}{\omega}\left(1 - \frac{2}{9}\frac{c^2(v_f^2 - c^2/3)}{(v_f^2 - c^2/3)^2 + (\nu_0 v_f/2k)^2}\right) + i\,\frac{\nu_0}{9}\,\frac{\omega_p^2\eta c^2/k^2}{(v_f^2 - c^2/3)^2 + (\nu_0 v_f/2k)^2}\right] a\lvert a\rvert^2$	$\eta \ll 1$ unstable for all ω
$\varepsilon\nu_1$	Classic	$\Delta = \frac{3}{2}\frac{\omega_p^2}{\omega}\left[\left(1 - \frac{1}{3}\frac{c^2 - v_s^2}{v_f^2 - v_s^2} - \frac{2}{3}\frac{c^2 - v_s^2}{v_g^2 - v_s^2}\right)\lvert a\rvert^2 + \frac{2}{3}\frac{c^2 - v_s^2}{v_g^2 - v_s^2} C_1 e^{C_1\xi}\int_\xi^\infty e^{-C_1\zeta}\lvert a\rvert^2 d\zeta\right] a$ $C_1 = \frac{\nu_1 v_g}{v_g^2 - v_s^2}$	$(v_s/c)^2 \ll 1$, $\eta = 1$ unstable for all ω
Weak	Ultrarelativistic	$\Delta = \frac{\omega_p^2\eta}{\omega}\left[\left(1 - \frac{2}{9}\frac{c^2}{v_f^2 - c^2/3} - \frac{4}{9}\frac{c^2}{v_g^2 - c^2/3}\right)\lvert a\rvert^2 + \frac{4}{9}\frac{c^2}{v_g^2 - c^2/3} C_1 e^{C_1\sqrt{\eta}\xi}\int_{\sqrt{\eta}\xi}^\infty e^{-C_1\zeta}\lvert a\rvert^2 d\zeta\right] a$ $C_1 = \frac{\nu_1 v_g}{v_g^2 - c^2/3}$	$\eta \ll 1$ unstable for all ω
$\varepsilon^2\nu_2$	Classic	$\Delta = \frac{3}{2}\frac{\omega_p^2}{\omega}\left(1 - \frac{1}{3}\frac{c^2 - v_s^2}{v_f^2 - v_s^2} - \frac{2}{3}\frac{c^2 - v_s^2}{v_g^2 - v_s^2}\right) a\lvert a\rvert^2$	$(v_s/c)^2 \ll 1$, $\eta = 1$ unstable for $\sqrt{2}\omega_p \leqslant \omega < \sqrt{2}\omega_p(1 + \frac{1}{2}\, v_s^2/c^2)$
Ultraweak	Ultrarelativistic	$\Delta = \frac{\omega_p^2\eta}{\omega}\left(1 - \frac{2}{9}\frac{c^2}{v_f^2 - c^2/3} - \frac{4}{9}\frac{c^2}{v_g^2 - c^2/3}\right) a\lvert a\rvert^2$	$\eta \ll 1$ unstable for $\sqrt{2\eta}\,\omega_p \leqslant \omega < 1.22\sqrt{2\eta}\,\omega_p$

parameters are added. We see that the cubic term is local in finite and ultraweak damping cases, but it becomes spatially nonlocal for weak damping, the corresponding NLS being integrodifferential. Two results can be emphasized. (i) Relativistic temperatures do alter the stability result found in [17] for low temperature (zero damping, e_+-e_-) plasmas, by enlarging the range of unstable frequencies, which now takes place from 1 to about 1.22 times the relativistic plasma frequency. (ii) Phonon damping also produces substantial changes in the NLS, which then predicts unstable envelopes at all frequencies, except in the ultraweak case, i.e., when the damping rate is of second order in the expansion parameter.

It is also interesting to note that weak damping produces the largest destabilizing effect. With zero damping the modulational instability occurs in a restricted ω interval and $0 < K < K_c$. With finite damping it happens for all frequencies in a limited band $K_1 < K < K_2$. But for weak damping a modulational instability occurs without restrictions, for all K, ω values. Thus, an ideal dissipationless system is less prone to the instability than a system with a small amount of damping. Analogies can be found in other branches of plasma physics: a small amount of resistivity destabilizes ideal magnetohydrodynamic modes; a small amount of viscosity affects the stability of ideal flows.

The unstable wavelengths of the perturbed envelope and the growth rate of the instability have been computed for the three cases of finite, weak, and negligible damping. It is generally recognized that the growth rate of the modulational instability can give estimates of the time of formation of solitons. The model with relativistic temperatures may be used for discussions on early Universe plasmas, from the phase of neutrinos decoupling and disappearance of muons to the time of annihilation of positron and electrons. That is from about 10^{-4} s to 1 s, or better, in temperature epochs from approximately 100 to 1 MeV. Radiation pressure, then, acts to provide ''springiness'' to acoustic oscillations. However, specific analysis of that system (or other astrophysical scenarios like AGN, where high temperature positron-electron plasmas are expected) exceeds the scope of this paper. We limit ourselves to a discussion of conditions of applicability of the theory.

In primordial e_+-e_- plasmas, densities are considered to be in the 10^{33}–10^{28}-cm^{-3} range, so that the plasma frequency, assuming $\eta=0.01$, is very high, above 10^{18} rad/s. The time scale of the instability in this scenario, assuming finite phonon damping, can be estimated to be as small as 10^{-11} s, taking $\varepsilon=0.01$ (see Sec. VI A) and therefore during that time the expansion of the Universe is negligible. Thus, low frequency electromagnetic waves, $\hbar\omega\ll T$, could generate density inhomogeneities and leave imprints in the plasma, at earlier times than the recombination era. These structures, which perhaps may act as seeds of forthcoming gravitational developments, should run completely undetected in the present highly isotropic 3-K radiation background, which reflects dominant radiation at $\hbar\omega\simeq T$ of earlier epochs [7]. On this subject Ref. [25] contains interesting recent work for unmagnetized e_+-e_- plasmas. This reference studies the formation of solitons in early Universe scenarios, using positron-electron fluid dynamics with relativistic temperatures, in the framework of circularly polarized electromagnetic waves.

The choice of a proper phonon damping to be employed in the study of a particular physical system requires further analysis. For this purpose, a kinetic treatment of acoustic oscillations of e_+-e_-, driven by the ponderomotive effect of an electromagnetic pump wave, should be developed to obtain an estimate of the collisionless damping. This is not available yet, as far as we know. For classical temperatures, free (not driven) acoustic modes without electric field do not exist in a collisionless e_+-e_- plasma [10], although such acoustic waves do appear in a fluid-theoretical treatment. Conversely, at relativistic temperatures and in the presence of radiation pressure, acoustic waves can be sustained in a collisionless e_+-e_- plasma.

For early Universe e_+-e_- plasmas, Ref. [7] provides an approximate dispersion relation for phonons [Eq. (115) in that reference] derived from Vlasov equations that incorporate radiation pressure as external force. Unfortunately, the approximation given in [7] is not relativistic, so that the phonon phase velocity does not correspond to $c/\sqrt{3}$ [from Eq. (115) in [7], the root $\omega/kc=2.1730-i0.7331$, can be obtained]. A relativistic derivation of the dispersion relation of phonons driven by radiation pressure in a e_+-e_- plasma is still not available.

For applications of our theory we may conjecture, nevertheless, taking the estimate of [7] as a rough trend, that phonon absorption is small when the frequency of the electromagnetic wave approaches the plasma frequency cutoff, i.e., at very small wave numbers. It becomes, then, increasingly large as the frequency of the electromagnetic wave grows above the plasma frequency, that is, for large wave numbers. Thus, for e_+-e_- plasmas with radiation pressure, we expect that the phonon damping regime changes with the frequency of the transverse wave: from ultraweak, for ω very close to the plasma frequency, to weak or finite, when we consider waves with increasingly larger frequencies. The modulational instability, therefore, is due to relativistic temperature effects (ultraweak case) for frequencies near $\sqrt{2\eta}\omega_p$. At higher frequencies the ultraweak case predicts stability, but we expect that the phonon damping becomes stronger. Thus, the modulational instability should appear also at those higher frequencies, induced now by phonon damping effects (weak and finite cases).

Several references consider the effect of an ambient magnetic field on nonlinear electromagnetic waves processes in e_+-e_- plasmas (see literature quoted in the Introduction). However, the work up to the present has been mainly on circularly polarized waves. Circularly polarized waves permit some simplifications of the calculations. Reference [26] contains an important contribution to the modulational instability of electromagnetic waves in magnetized plasmas with classical temperatures, for propagation parallel to the magnetic field. The concern of [26] is primarily the ordinary ion-electron plasma, and the positron-electron case is treated as a special limit. Thus, this work also deals with circularly polarized waves. A magnetic field introduces new features and the modulational instability occurs over a broad band of low frequencies, below the cyclotron resonance.

However, in a positron-electron plasma with a magnetic field, the natural electromagnetic modes for parallel propagation are linearly polarized waves. In Cartesian coordinates the dielectric tensor is diagonal, since off-diagonal terms compensate exactly. Circularly polarized solutions are a special linear superposition of natural modes, with a rather particular phase relationship. The situation is the opposite to that of more common ion-electron plasmas, where the circularly polarized representation diagonalizes the dielectric tensor. In fact, observations of pulsars, radio sources expected to have e_+-e_- magnetospheres with very large magnetic fields, often indicate dominance of linearly polarized waves [1]. In addition, it is not possible, of course, to use linear combination of solutions when dealing with nonlinear waves, so that results on the modulational instability of circularly polarized waves do not apply to linearly polarized waves, and vice versa. In this sense, the theory of self-modulational instability of electromagnetic waves in a magnetized e_+-e_- plasma is still not complete.

In physical regimes where the plasma frequency is much larger than the cyclotron frequency, and for high-frequency electromagnetic waves propagating parallel to the magnetic field above the plasma frequency, the dispersive properties approach those of an unmagnetized plasma. In the primordial plasma these conditions would apply since the magnetic field, if any, is supposed to have been very weak, while the plasma density was huge. Moreover, at relativistic temperatures the ratio of cyclotron to plasma frequencies is further reduced by the effect of inertia enhancement. Thus, our theory can be applied to early Universe scenarios, and to cases of unmagnetized plasmas in AGN. For configurations with important magnetic fields, such as e_+-e_- in pulsars, or confined in laboratory experiments, it may be relevant only under restricted conditions, i.e., in the high frequency limit.

At lower frequencies, linearly polarized Alfvén solitons in cold e_+-e_- plasmas have been recently derived [27].

Summarizing, our work extends previous analyses of linearly polarized electromagnetic waves in nonmagnetized plasmas, showing that if the phonon damping is $O(\varepsilon^0)$ or $O(\varepsilon^1)$, a modulational instability appears in the electron-positron case in all ranges of temperature and wave frequencies. Thus the presence of some amount of sound absorption helps to produce an envelope decay. When the phonon damping is very small $[O(\varepsilon^2)]$ the result of [17] is recovered, but if the temperature is ultrarelativistic the self-modulational instability is present again in a finite frequency range.

Finally, the set of equations (31) and (32) is more basic than the NLS for the nonlinear analysis of the electromagnetic wave properties and propagation in electron-positron plasmas. Thus, (31) and (32) provide a framework, exact to $O(A^3)$, for further analytical or numerical studies of the problem.

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