Phase synchronization effects in a lattice of nonidentical Rössler oscillators

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We study phase synchronization in a chain of weakly coupled chaotic oscillators. In the synchronous state, the phases of oscillators are locked, while the amplitudes remain chaotic. We demonstrate that the coexistence of several clusters of mutually synchronized elements and global synchronization of all oscillators is possible. Two mechanisms of the transition to global synchronization are shown. The dynamics of spatiotemporal defects is discussed for the cases of phase-coherent and funnel Rössler oscillators. [S1063-651X(97)01803-7]

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I. INTRODUCTION

Collective dynamics in large ensembles of periodic self-excited oscillators has been intensively studied in the past two decades [1]. Such models have been used for the description of Josephson-junction arrays [2], multimode lasers and laser arrays [3,4], relativistic magnetrons [5], and other contexts [6–8]. Arrays of oscillators also appear in studies of biological rhythms of the heart [9], nervous system [9,10], intestines [11], pancreas [12], and other biological problems [13–16]. Coupled oscillator models are also widely used as general models in studies of complex dynamics in a nonequilibrium medium [17–19].

One of the most interesting and practically important phenomena in large ensembles is mutual synchronization of oscillators [20–24]. With the spread of studies of chaos, the notion of mutual synchronization has been extended to the case of chaotic oscillations. In this context, there are different phenomena that are usually referred to as ''synchronization.'' First, due to the interaction of at least two identical chaotic systems their states can coincide, while the dynamics in time remains chaotic [25,26]. In this case one can speak about ''full synchronization'' of chaotic oscillations. This effect can be easily generalized to the case of slightly nonidentical systems [26,27] or interacting subsystems [28]. In another approach synchronization in chaotic systems has been defined as the overlap of power spectra of respective signals [29,30], thus drawing the analogy to the coincidence of frequencies of synchronized periodic systems. It has been shown that due to an interaction, the widths of the peaks of the power spectra become practically equal and the peaks become closer in frequency. The appearance of synchronization was quantified by means of cross-correlation functions [29] or cross spectra [31]. Another approach is based on the calculation of the attractor dimension of the whole system and its comparison with the partial dimensions calculated in the phase subspaces formed by the coordinates of each interacting oscillator [32,33].

Recently, the phenomenon of *phase synchronization* of chaotic systems has been found [34–37]. This effect is a direct extension of the classical definition of synchronization of periodic oscillators, where only the phase locking is important, while no restriction on the amplitudes is imposed. It has been shown that, at least for some paradigmatic models, the notion of phase can be introduced for chaotic self-sustained oscillators as well. Hence a phase synchronization of a chaotic system can be defined as the appearance of entrainment between the phases of interacting systems, while the amplitudes remain chaotic and, in general, noncorrelated. This effect has been demonstrated in [34] for two coupled nonidentical systems, and Rössler oscillators in particular, for an ensemble of globally coupled chaotic oscillators [35], and for the case of external synchronization by periodic or noisy forcing [36]. Phase synchronization was also observed in a physical experiment with the electronic model of two coupled Rössler oscillators [37]. Similar to the synchronization of periodic oscillators, phase synchronization of chaotic systems appears for very weak or even vanishing coupling if the detuning between interacting oscillators is small.

In this paper we study cooperative behavior in a chain of diffusively coupled nonidentical Rössler oscillators. We are interested in whether the phenomena usually encountered in the networks of periodic oscillators can be observed for chaotic systems as well. The main effect is the existence of a regime of global synchronization, i.e., all elements of the chain are synchronized, or the existence of several clusters of synchronized oscillators [38,39]. We also investigate the properties of the collective behavior inherent in chaotic networks.

A work very similar to our study was done in [19,40], where one- and two-dimensional lattices of *identical* Rössler oscillators have been considered. Observed in [19], the effect of the appearance of a macroscopic mean field for very small couplings can be interpreted in our terms as the appearance of a phase-synchronous state. In the case of *nonidentical* oscillators the transition is, however, nontrivial, as we will show below.

The paper is organized as follows. In Sec. II we decsribe the model under study, briefly introduce the notion of a phase for the chaotic Rössler oscillator, and discuss criteria of synchronization in an oscillator network. In Sec. III we discuss mutual synchronization of two Rössler oscillators.

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'In Secs. IV and V we present the results of the numerical study of synchronization in the one-dimensional lattices (chains) of coupled Rössler oscillators with linear and random distribution of natural frequencies. Section VI is devoted to synchronization of identical Rössler oscillators with the so-called funnel attractor. The results are summarized in Sec. VII.

II. BASIC MODEL

A. Lattice of Rössler oscillators

Our basic model is the chain of coupled nonidentical Rössler oscillators [41] with a nearest-neighbor diffusive coupling. It can be written as a set of ordinary differential equations

$$\dot{x}_j = -\omega_j y_j - z_j,$$
$$\dot{y}_j = \omega_j x_j + a y_j + \varepsilon(y_{j+1} - 2y_j + y_{j-1}), \tag{1}$$
$$\dot{z}_j = 0.4 + (x_j - 8.5) z_j .$$

Here the index $j=1,\ldots,N$ is the position of an oscillator in the lattice and ε is the coupling coefficient. The parameter a determines the topology of the attractor; its significance is discussed below. The parameter ω_j corresponds to the natural frequency of the individual oscillator. This can be easily seen if the Rössler equations

$$\dot{x} = -\omega y - z,$$
$$\dot{y} = \omega x + a y, \tag{2}$$
$$\dot{z} = 0.4 + z(x - 8.5)$$

are rewritten in the form

$$\ddot{y} - a\dot{y} + \omega^2 y = -\omega z,$$
$$\dot{z} + 8.5 z = 0.4 + z(\dot{y} - a y)/\omega. \tag{3}$$

To study synchronization in the lattice of nonidentical oscillators, we introduce a linear distribution of natural frequencies ω_j,

$$\omega_j = \omega_1 + \delta(j-1), \tag{4}$$

where δ is the frequency mismatch between neighboring systems. Another variant considered below is a random distribution of natural frequencies in the range $[\omega_1, \omega_1 + \delta(N-1)]$. We also assume free boundary conditions

$$y_0(t) = y_1(t), \quad y_{N+1}(t) = y_N(t). \tag{5}$$

Because the Rössler system typically shows windows of periodic behavior as the parameter ω is changed, we choose ω_1 and δ in such a way that large periodic windows are avoided.

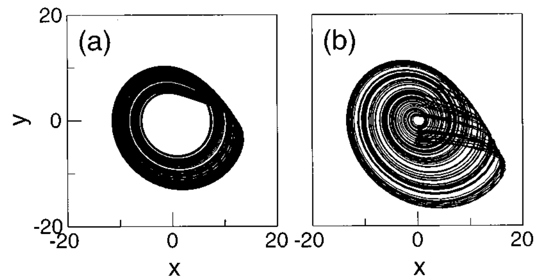

FIG. 1. (a) Phase coherent and (b) funnel Rössler attractors with parameters $a=0.15$ and $a=0.25$.

B. Phase of the Rössler oscillator

There is no unambiguous and general definition of a phase for chaotic systems. Roughly speaking, we want to define a phase as a variable that corresponds to the zero Lyapunov exponent of the continuous-time chaotic dynamical system, while other variables can be referred to as amplitudes. This can be achieved via partial Poincaré maps or via an analytic signal approach based on the Hilbert transform [42,43]. A detailed discussion and comparison of these approaches to the phase definition are presented elsewhere [36].

For the Rössler system (2) possible definitions of the phase depend on the topology of the attractor, which is determined by the value of the parameter a. For $a=0.15$ the phase coherent attractor is observed with rather simple topological properties [44,45]. If we project the attractor on the plane (x,y), this projection resembles the smeared limit cycle where the phase point always rotates around the origin [Fig. 1(a)]. Hence, the phase here can be introduced in a simple way (see also [35,40]), namely, as

$$\phi = \arctan(y/x). \tag{6}$$

The amplitude correspondingly can be defined as

$$A = \sqrt{x^2 + y^2}. \tag{7}$$

As the phase of a chaotic system is well defined, one can straightforwardly calculate the phase difference between neighboring oscillators $\phi_j - \phi_{j+1}$. If the phase difference does not grow with time but remains bounded, we have a 1:1 phase locking. (Generally, an n to m locking $[|n\phi_j(t) - m\phi_{j+1}(t)| < \text{const}]$ can be observed, but this case is not considered in this study.) A weaker condition of synchronization is the coincidence of the averaged partial frequencies defined as

$$\Omega_j = \langle \dot{\phi}_j \rangle = \lim_{T\to\infty} \frac{\phi_j(T) - \phi_j(0)}{T}. \tag{8}$$

For practical (experimental) applications it is important that the mean frequency of chaotic oscillations (observed frequency) Ω_j can be calculated as

$$\Omega_j = \lim_{T\to\infty} 2\pi \frac{M_T^j}{T}, \tag{9}$$

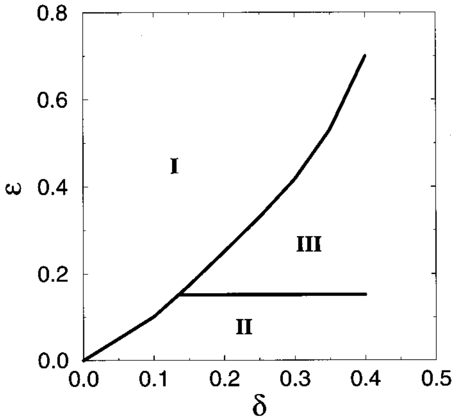

FIG. 2. Regions of nonsynchronous (II) and synchronous (I) motion and of oscillations quenching (III). The diagram is approximate; the windows of periodic behavior in regions I and II are not shown.

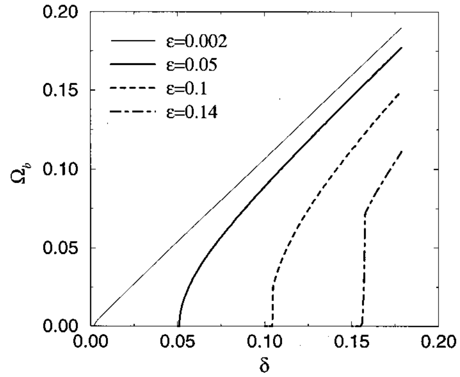

FIG. 3. Dependence of the beat frequency Ω_b on the frequency mismatch δ for different values of coupling ε. In full analogy to the classical case of two coupled periodic oscillators, the transition to (from) synchronization occurs smoothly or practically by a jump for weak and strong coupling, respectively.

where M_T^j is the number of rotations of the phase point around the origin during time T. This method can be directly applied to observed time series, when one, e.g., takes for M_T^j the number of maxima of $x_i(t)$. For the Rössler attractor, the estimates (8) and (9) practically coincide [36].

The topology of the Rössler attractor changes if the parameter a exceeds the value 0.21. The phase in this case (which is called the funnel attractor) is not well defined: there are large and small loops on the (x,y) plane [see Fig. 1(b)], and it is not evident which phase gain should be attributed to these loops. Thus we cannot calculate the phase and the frequency in a simple way. Nevertheless, as we discuss below in Sec. VI, some synchronization effects in the lattice with funnel attractors can be observed in this case as well. In Secs. III–V we consider the phase-coherent Rössler attractor ($a=0.15$) only.

III. MUTUAL SYNCHRONIZATION OF TWO RÖSSLER OSCILLATORS

As the first step of our analysis of cooperative behavior in a long lattice, we describe briefly the phase synchronization of two chaotic elements, i.e., consider Eqs. (1) for $N=2$ (see also [34]). The phase diagram of different regimes observed in the system for varying the coupling ε and frequency mismatch $\delta=\omega_2-\omega_1$ exhibits three regions of qualitatively different behavior (see Fig. 2).

(I) The synchronization region is where the frequencies are locked $\Omega_1=\Omega_2$. It is important to note that there is no threshold of synchronization; this is a particular feature of the phase coherent Rössler attractor (see [34]).

(II) The region of nonsynchronized oscillations is where $|\Omega_1-\Omega_2|=|\Omega_b|>0$. In analogy to the case of periodic oscillators, this frequency Ω_b can be considered as a "beat frequency."

(III) In this region, due to the interaction, oscillations in both systems disappear. This effect is known for periodic systems as an oscillation quenching phenomenon [46–49]. We note that the boundaries between different regimes are slightly dispersed and windows of periodic behavior are present, although for a large domain of parameters in regimes I and II the amplitudes of the oscillations are chaotic.

Next, we study the behavior of the beat frequency Ω_b as the frequency mismatch changes (Fig. 3). We see that for weak coupling (small ε) the beat frequency Ω_b smoothly depends on δ. This means that the state where the frequencies of interacting oscillators differ by a rather small value is possible. For sufficiently strong coupling the transition from the synchronous to the asynchronous state appears to be rather sharp: a virtual jump in the dependence Ω_b on δ is observed. This means that the frequencies of interacting oscillators cannot be close: they either coincide or differ by a finite value. These two types of transitions to (or from) synchronization are similar to those well known for periodic oscillators [49]. We will see below that this difference results in two distinct types of behavior in a lattice of Rössler systems.

Hence we can conclude that the dynamics of the phase in the chaotic Rössler oscillator is similar to that in well-studied classical periodic systems, e.g., the van der Pol oscillator. In particular, we expect the synchronization effects known for lattices of periodic oscillators [21–23] to be observed in the chaotic case as well.

IV. PHASE SYNCHRONIZATION IN THE LATTICE WITH A LINEAR DISTRIBUTION OF NATURAL FREQUENCIES

We have performed numerical simulations with lattices of 20–50 oscillators, for different values of the parameters $\delta,\omega_1,\varepsilon$. The main quantities we calculated were the observed frequencies Ω_j. Generally, as the coupling increases, all the frequencies Ω_j become equal; we call this the onset of global synchronization. We have found that the regime of global synchronization in the chain [Eq. (1)] can appear in two ways, depending on the relative frequency mismatch δ/ω_1. Below we describe these two scenarios, referred to as the soft and the hard appearance of global synchronization.

A. Small frequency mismatch: Soft transition to a synchronous state

We first consider the case of relatively small frequency mismatches $\delta/\omega_1 \ll 1$. With the increase of coupling, a mu-

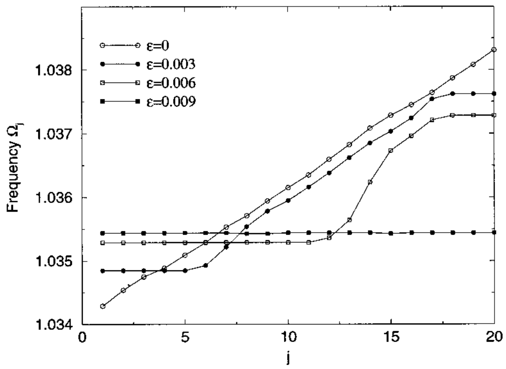

FIG. 4. Mean frequencies Ω_j for different values of coupling. The number of elements $N=20$, the frequency mismatch $\delta=2\times10^{-4}$ and $\omega_1=1$.

tual synchronization of the oscillators occurs at the ends of the lattice, i.e., only elements with small and large j are synchronized (see Fig. 4). With a further increase of ε, an increasing number of oscillators join the synchronous regions, so they become more extended (Fig. 4). Finally, for ε exceeding a critical value ε^*, all oscillators are synchronized, i.e., all the mean frequencies Ω_j coincide. With the transition to a global phase synchronization the amplitudes of the oscillators remain chaotic, which is clearly marked by the Lyapunov spectrum reported in Fig. 5. The number of positive Lyapunov exponents remains N even in the globally synchronous regime at $\varepsilon>\varepsilon^*$. As has been shown in [34], the appearance of phase synchronization in the system of two coupled oscillators manifests itself in the Lyapunov spectrum, namely, one of the zero exponents becomes negative, while the two largest exponents remain positive. For the lattice, the N largest Lyapunov exponents remain positive, while from the next N exponents only one zero exponent survives, and $N-1$ become negative.

The regime of full phase synchronization is rather sensitive to external noise. We have simulated the dynamics of the lattice (1) with noisy Gaussian terms being added to the right-hand side of the equations for x_j and y_j. Even small noise leads to a nonconstant distribution of observed frequencies; so, strictly speaking, global synchronization is not

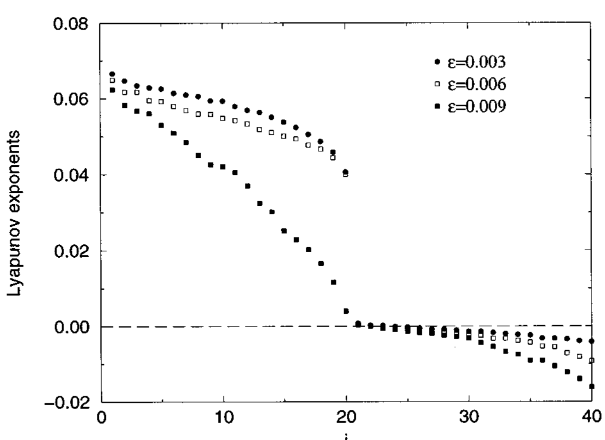

FIG. 5. Lyapunov spectra λ_i for the regimes reported in Fig. 4.

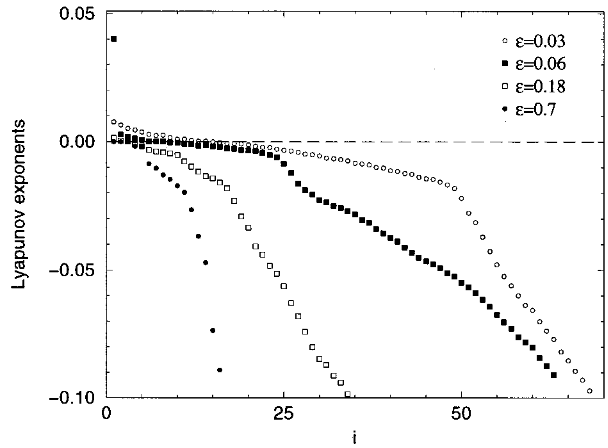

FIG. 6. Lyapunov spectra λ_i for different values of coupling for relatively large frequency mismatch. The number of elements $N=50$, the frequency mismatch $\delta=9\times10^{-3}$ and $\omega_1=1$.

observed. Noise destroys full phase synchronization in the lattice in the same way as it destroys it in the system of two coupled oscillators [50]: due to noise phase slips become possible.

B. Large frequency mismatch: Chaos suppression and clustering

For relatively large frequency mismatch δ/ω_1 the first effect of the interaction is the suppression of chaos. As the coupling ε grows, the number of positive Lyapunov exponents decreases. Before any synchronization effects are seen, only a few Lyapunov exponents are positive (see Fig. 6). Synchronization occurs in the form of clusters: the oscillators are divided into groups having the same frequency, with a relatively large frequency difference between groups. The dependence of the observed frequency on the oscillator position has a characteristic staircase form (see Fig. 7). With the increase of coupling the number of clusters decreases.

The difference in the lattice dynamics for small and large frequency mismatches directly corresponds to the properties of two interacting systems discussed in Sec. III. First, let us mention that a larger frequency mismatch requires a larger coupling for synchronization to occur. We have shown in Sec. III that for small couplings the frequency difference can be arbitrarily small; therefore, with an increase of coupling a smooth transition to synchronization is observed in the lattice. Contrary to this, for large couplings the frequency difference is either zero or finite; therefore, synchronous clusters are formed with ''jumps'' between them.

In the space-time dynamics the boundaries between clusters correspond to the positions where the phase slips occur. We illustrate this with space-time plots in Fig. 7. In all plots the gray scale is used, with minimal values being represented by white and maximal by black. The left panel shows the quantity $\sin(\phi_j)=y_j/A_j$ [see Eqs. (6) and (7)], so that the white stripes correspond to the phase $\approx3\pi/2$ and the black stripes to the phase $\approx\pi/2$. The right panel shows the amplitudes of the oscillators. To characterize the instantaneous phase difference between neighboring oscillators, we plot in the center panel the quantity

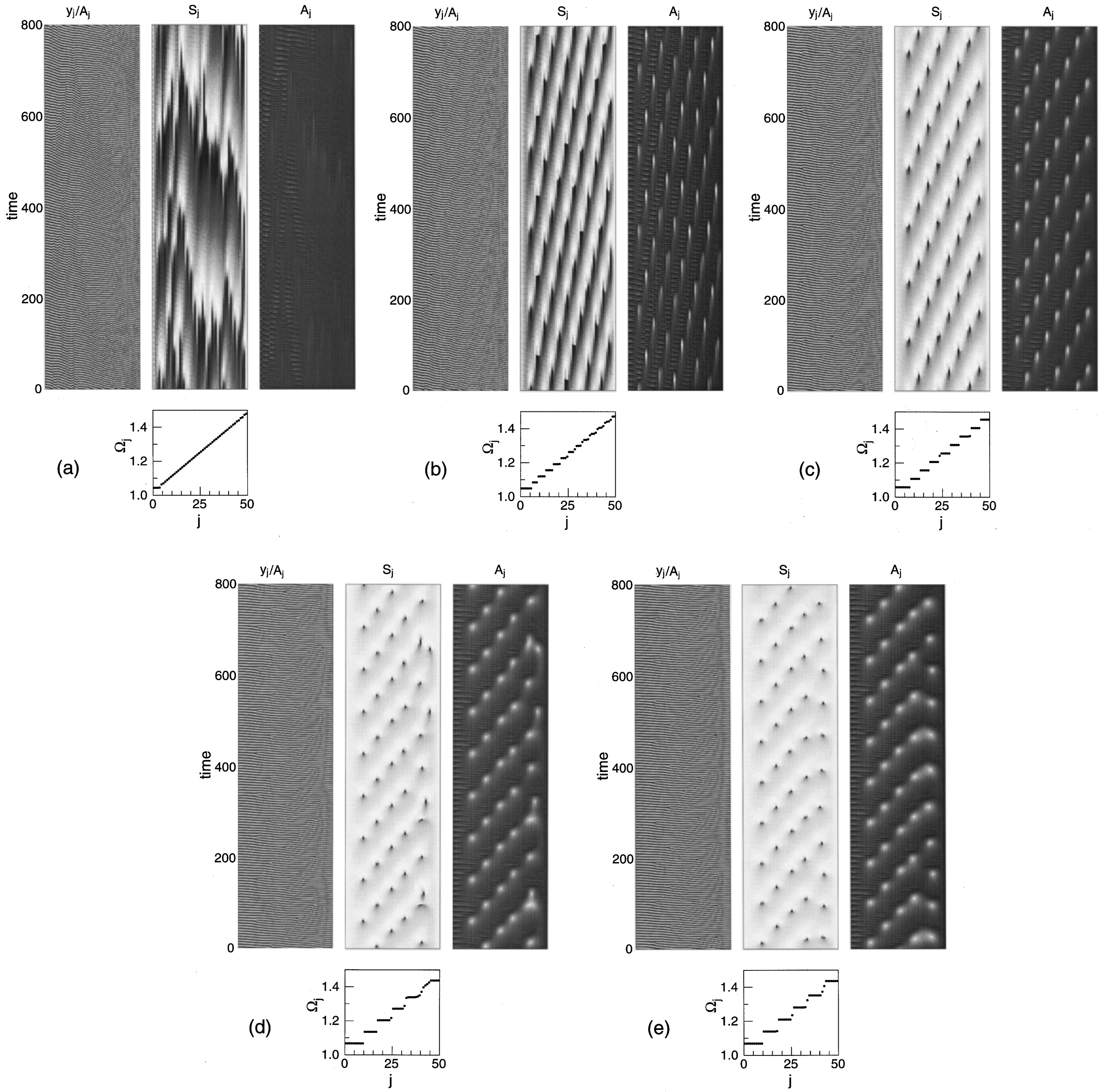

FIG. 7. Mean frequencies Ω_j and space-time structures in the lattice of 50 coupled oscillators with $\delta = 9\times 10^{-3}$ and different couplings. All plots show a gray-scale representation of the corresponding quantities. (a) $\varepsilon = 0.03$: no clusters are observed, although relatively large regions of phase coherence are seen. (b) $\varepsilon = 0.06$: first clusters appear, but the defects are extended in time. (c) $\varepsilon = 0.18$: a regular train of defects is observed. (d) $\varepsilon = 0.6$: at the transition from seven to six clusters an irregular sequence of defects is seen near the right edge of the lattice. (e) $\varepsilon = 0.7$: a regular train of well localized defects.

$$s_j = \sin^2\left(\frac{\phi_j - \phi_{j+1}}{2}\right), \tag{10}$$

which is zero if the phases are equal and one if they differ by π.

The defects, which are clearly seen as maxima (black regions) of s_j and minima (white regions) of the local amplitude, can appear regularly at certain positions on the lattice; in this case the border between the clusters is sharp [Figs. 7(c)–7(e)]. Obviously, the frequency difference between the clusters is equal to the frequency of the defects' appearance. Near the transition at which the number of clusters changes, the defects appear irregularly in both space and time [see Fig. 7(d)], the border between the clusters is smeared.

C. Oscillations quenching

If the coupling between elements is not very small, the interaction can lead not only to a synchronization, but also to a suppression of oscillations. This effect, known as ''oscillation quenching,'' is observed for both pairs and chains of periodic oscillators [46–48]. The loss of self-excitation of

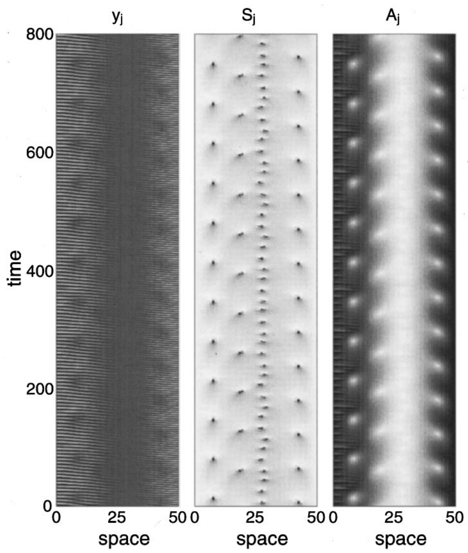

FIG. 8. Space-time diagrams of evolution of (a) $x_j(t)$ and (b) $A_j(t)$. In the middle of the chain the oscillations are suppressed due to the interaction, i.e., oscillation quenching is observed. The parameters are $N=50$, the frequency mismatch $\delta=15\times10^{-3}$, $\omega_1=1$, and $\varepsilon=0.75$.

two chaotic oscillators due to the interaction has been discussed in Sec. III. Here we demonstrate this effect for the chain of coupled oscillators.

To explain this effect, let us rewrite Eqs. (1) as

$$\dot{x}_j=-\omega_j y_j - z_j,$$

$$\dot{y}_j=\omega_j x_j+(a-2\varepsilon)y_j+\varepsilon(y_{j+1}+y_{j-1}), \tag{11}$$

$$\dot{z}_j=0.4+(x_j-8.5)z_j.$$

It is clear that the influence of the coupling can be considered as some additional damping introduced into the system [compare with Eq. (3)]. For large enough frequency mismatch the force from the neighboring oscillators is not resonant and does not compensate for the increased losses. As a result, if $2\varepsilon>a$ the oscillator can go out of the self-excited regime and oscillations decay, or ''die out.''

This effect can occur locally in the chains of chaotic oscillators simultaneously with the synchronization. This is illustrated in Fig. 8, where the state with two synchronous clusters near the ends of the lattice separated by the nonoscillating elements is shown.

V. SYNCHRONIZATION IN THE LATTICE WITH RANDOMLY DISTRIBUTED NATURAL FREQUENCIES

Here we describe the effect of coupling on a lattice with randomly distributed natural frequencies (see also [51,6]). As in the case of a linear distribution of frequencies, the regime of global synchronization arises via the formation of clusters (Fig 9).

The essential difference is that for the same mismatch between the largest and the smallest partial frequencies ω_j, global synchronization appears for considerably lower values

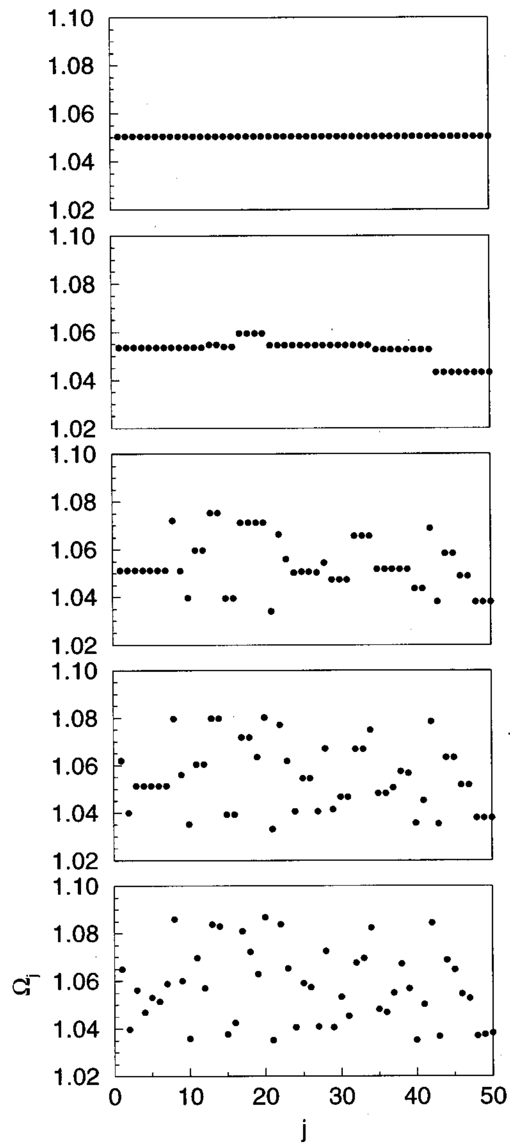

FIG. 9. Distribution of observed frequencies in a lattice with natural frequencies uniformly distributed in the interval $1<\omega<1.05$. From bottom to top $\varepsilon=0,0.01,0.02,0.05,0.2$.

of coupling than in the case of linear distribution. Qualitatively, this can be explained as follows. For the case of a linear distribution of frequencies, the left neighbor of some element is on average behind in phase and the right neighbor is respectively ahead. Hence they ''pull'' the oscillator in different directions, and in this sense their actions are compensated. For the random case it is possible that both neighbors are behind (ahead) in phase and both respectively slow (speed) the element down (up). As a result, their frequencies tend to each other, and these elements form a synchronous cluster. Such clusters can arise at arbitrary places in the chain and can coexist with oscillators that belong to no cluster. With the increase of coupling the clusters are first observed at the location of elements with smaller frequency mismatch. We note that distributions of mean frequencies do not depend on initial conditions, i.e., for each random distribution of partial frequencies in Eqs. (1) there exists only one attractor.

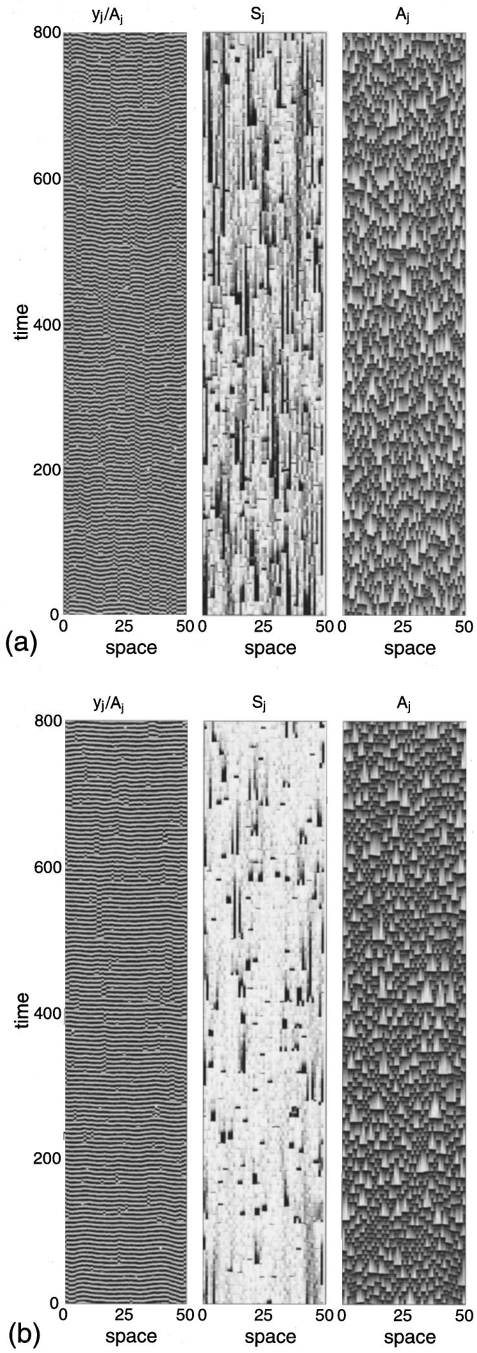

FIG. 10. Space-time evolution of the field and the phase difference [according to Eq. (10)] in the lattice of funnel Rössler attractors with $a=0.23$ and different couplings: (a) $\varepsilon=0.02$ and (b) $\varepsilon=0.05$. The values of $y_j(t)$ normalized on the amplitude are depicted in order to make the phase dynamics visible $[y_j/A_j=\sin(\phi_j)]$.

VI. PHASE SYNCHRONIZATION OF THE RÖSSLER SYSTEMS WITH THE FUNNEL ATTRACTOR

Above we have considered the case of the phase-coherent attractor in the Rössler system. This attractor is topologically simple and, correspondingly, the phase is well defined. If, however, the parameter a in Eq. (2) exceeds 0.21, the so-called funnel attractor [Fig. 1(b)] is observed. Here the pro-

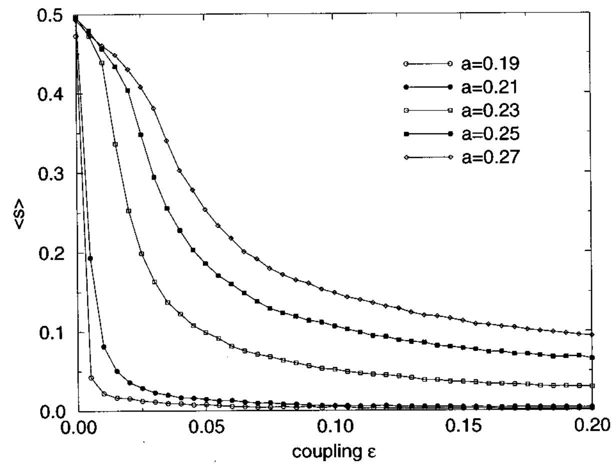

FIG. 11. Average value of the phase difference $\langle s_j(t)\rangle$ vs coupling ε for different values of the parameter a in the Rössler system (2).

jection on the (x,y) plane demonstrates not only full rotations around the origin, but also ''half'' rotations (small loops). In regard to the phase, these small loops produce phase slips of the value $\approx\pi$ and occur irregularly due to the chaotic nature of the process.

In the lattice of funnel Rössler attractors (1) (here we take all the oscillators to be identical) these phase slips, in contrast to the phase-coherent case, prevent a global synchronization. However, as the coupling increases, relatively large regions of coherent rotation of the oscillators appear, separated by defects (Fig. 10). The defects appear spontaneously due to local irregular phase slips. It is interesting that their lifetime is relatively large compared to the cluster regime of the phase-coherent oscillators. This is because after a slip the phase difference between neighboring oscillators is $\approx\pi$, which roughly corresponds to the unstable but stationary configuration of phases. The relaxation to the stable phase difference ≈ 0 is therefore slow.

To describe the synchronization of the funnel attractors quantitatively we have characterized the instantaneous phase difference with the quantity s_j [see Eq. (10)], which is depicted in Fig. 10. (Because all oscillators are identical, one cannot characterize synchronization as the frequency locking: the averaged frequencies are always equal.) With the increase of ε the number and length of defects decrease, indicating the tendency to synchronization. Small values of s_j correspond to the zero phase difference, while the phase difference $\approx\pi$ gives large values $s_j\approx 1$. In Fig. 11 the dependence of the average of $s_j(t)$ (for independent systems it is obviously 0.5) on the coupling is shown for different parameter values of the Rössler attractor. While for the phase-coherent case the full synchronization appears for very small couplings, in the funnel case a rather slow decay of $\langle s_j\rangle$ is observed.

VII. DISCUSSION

In this paper we have considered phase synchronization effects in a lattice of diffusively coupled Rössler oscillators. When the individual attractors are phase coherent, the phase is well defined and its dynamics is similar to that of regular oscillators. In the inhomogeneous lattice synchronization appears when the coupling exceeds some threshold. We have found two scenarios of the transition: in the first one a gradual adjustment of the frequencies is observed, while in the other one an intermediate clustered state occurs. The borders of the clusters appear in the space-time diagrams as positions where phase defects are observed. We have demonstrated that these defects can be both periodic and irregular. The two scenarios directly correspond to the synchronization properties of two interacting systems: for small couplings the frequencies are adjusted gradually, while for large couplings a virtual jump is observed. If the dynamics of the phase in the individual system is nontrivial, like for the funnel attractor in the Rössler model, in the homogeneous lattice a spontaneous appearance of defects is observed, leading to a complex phase dynamics of the lattice with space-time clusters.

Although we do not suggest a rigorous definition of the phase for chaotic systems, in many situations it can be defined at least approximately. Thus one can expect that the phase synchronization is a general property of chaotic systems. The effect is, however, greatly influenced by the phase-coherent properties of the attractor. The two regimes in the Rössler system exactly represent the cases of extreme phase coherence and the presence of phase slips. The study of the systems with moderate phase coherence is now in progress.

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[1] Y. Kuramoto, *Chemical Oscillations, Waves and Turbulence* (Springer, Berlin, 1984).
[2] P. Hedley, M.R. Beasley, and K. Wiesenfeld, Phys. Rev. B **38**, 8712 (1988).
[3] S.S. Wang and H.G. Winful, Appl. Phys. Lett. **52**, 1774 (1988).
[4] K. Wiesenfeld, C. Bracikowski, G. James, and R. Roy, Phys. Rev. Lett. **65**, 1749 (1990).
[5] J. Benford *et al.*, Phys. Rev. Lett. **62**, 969 (1989).
[6] K. Wiesenfeld, P. Colet, and S.H. Strogatz, Phys. Rev. Lett. **76**, 404 (1996).
[7] H.G. Winful and L. Rahman, Phys. Rev. Lett. **65**, 1575 (1990).
[8] S.K. Han, C. Kurrer, and Y. Kuramoto, Phys. Rev. Lett. **75**, 3190 (1995).
[9] L. Glass and M.C. Mackey, *From Clocks to Chaos* (Princeton University Press, Princeton, 1988).
[10] D.S. Levine, Math. Biosci. **66**, 1 (1983).
[11] G.B. Ermentrout and N. Kopell, SIAM J. Math. Anal. **15**, 215 (1984).
[12] A. Sherman, J. Rinzel, and J. Keizer, J. Biophys. **54**, 411 (1988).
[13] W. Freeman, Int. J. Bifur. Chaos **2**, 451 (1992).
[14] P. Tass and H. Haken, Z. Phys. B **100**, 303 (1996).
[15] J.J. Collins and I.N. Stewart, J. Nonlin. Sci. **3**, 349 (1993).
[16] H.D.I. Abarbanel *et al.*, Usp. Fiz. Nauk **166**, 3 (1996) [Phys. Usp. **39**, 337 (1996)].
[17] M.I. Rabinovich and M.M. Sushchik, Usp. Fiz. Nauk **160**, 3 (1990) [Sov. Phys. Usp. **33**, 1 (1990)].
[18] G.V. Osipov and M.M. Sushchik, Phys. Lett. A **201**, 205 (1995).
[19] L. Brunnet, H. Chaté, and P. Manneville, Physica D **78**, 141 (1994).
[20] A.T. Winfree, *The Geometry of Biological Time* (Springer, Berlin, 1980).
[21] Yu.M. Romanovsky, N.V. Stepanova, and D.S. Chernavsky, *Mathematical Biophysics* (Nauka, Moscow, 1984).
[22] N. Koppel and G.B. Ermentrout, Commun. Pure Appl. Math. **39**, 623 (1986).
[23] P.C. Matthews and S.H. Strogatz, Phys. Rev. Lett. **65**, 1701 (1990).
[24] M. De Sousa Vieira, A.J. Lichtenberg, and M.A. Lieberman, Int. J. Bifur. Chaos **4**, 1563 (1994).
[25] H. Fujisaka and T. Yamada, Prog. Theor. Phys. **69**, 32 (1983).
[26] A.S. Pikovsky, Z. Phys. B **55**, 149 (1984).
[27] V. Afraimovich, N. Verichev, and M. Rabinovich, Izv. Vyssh. Uchebon. Zaved. Radiofiz. **29**, 1050 (1986) [Sov. Radiophys. **29**, 795 (1986)].
[28] L.M. Pecora and T.L. Carroll, Phys. Rev. Lett. **64**, 821 (1990).
[29] P. Landa and S. Perminov, Electronics **28**, 285 (1985).
[30] I. Blekhman, P. Landa, and M. Rosenblum, Appl. Mech. Rev. **48**, 733 (1995).
[31] V. Anischenko, T. Vadivasova, D. Postnov, and M. Safonova, Int. J. Bifur. Chaos **2**, 633 (1992).
[32] P.S. Landa and M.G. Rosenblum, Dokl. Akad. Nauk SSSR **324**, 65 (1992) [Sov. Phys. Dokl. **37**, 237 (1992)].
[33] P.S. Landa and M.G. Rosenblum, Appl. Mech. Rev. **46**, 414 (1993).
[34] M. Rosenblum, A. Pikovsky, and J. Kurths, Phys. Rev. Lett. **76**, 1804 (1996).
[35] A. Pikovsky, M. Rosenblum, and J. Kurths, Europhys. Lett. **34**, 165 (1996).
[36] A. Pikovsky, M. Rosenblum, G. Osipov, and J. Kurths, Physica D (to be published).
[37] U. Parlitz, L. Junge, W. Lauterborn, and L. Kocarev, Phys. Rev. E **54**, 2115 (1996).
[38] G.B. Ermentrout, Physica D **41**, 219 (1990).
[39] D. Golomb, D. Hansel, B. Shraiman, and H. Sompolinsky, Phys. Rev. A **45**, 3516 (1992).
[40] A. Goryachev and R. Kapral, Phys. Rev. Lett. **76**, 1619 (1996).
[41] O.E. Rössler, Phys. Lett. A **57**, 397 (1976).
[42] D. Gabor, J. IEE London **93**, 429 (1946).
[43] P. Panter, *Modulation, Noise, and Spectral Analysis* (McGraw-Hill, New York, 1965).
[44] J.D. Farmer *et al.*, Ann. N. Y. Acad. Sci. **357**, 453 (1980).
[45] E.F. Stone, Phys. Lett. A **163**, 367 (1992).
[46] Y. Yamaguchi and H. Shimizu, Physica D **11**, 212 (1984).

[47] K. Bar-Eli, Physica D **14**, 242 (1985).
[48] D.G. Aronson, G.B. Ermentrout, and N. Koppel, Physica D **41**, 403 (1990).
[49] P.S. Landa, *Nonlinear Oscillations and Waves in Dynamical Systems* (Kluwer, Dordrecht, 1996).
[50] R.L. Stratonovich, *Topics in the Theory of Random Noise* (Gordon and Breach, New York, 1963).
[51] Y. Braiman, J.F. Lindner, and W.L. Ditto, Nature **378**, 465 (1995).