Car-following model of multispecies systems of road traffic

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A speed-adjustment car-following model is extended to systems of traffic where there is a variety of vehicle response times and speed-headway relationships. This is proposed as a model of the interactions between cars and trucks on single-lane roads where there is no overtaking, and some of its properties are derived. First, we make a distinction between temporal stability on a circular road and spatial stability on a straight road and go on to derive criteria for linear stability in each case. The propagation and dispersion of a linear disturbance wave is studied, and we also compare the nonlinear evolution of both single- and multiple-species systems on circuitous and straight roads. When the speed-headway relationship of all vehicles is given by the nonlinear law proposed by Bando *et al.* [Phys. Rev. E **51**, 2 (1995)], we find that for models of car-truck systems, as for systems consisting of one type of vehicle only, there is a range of equilibrium headways for which the system is linearly unstable. The size of this range increases with the proportion of the more unreactive vehicle type, trucks, in the population of vehicles. Computer simulations verify the analytical results and show the nonlinear development of disturbances when the system is linearly unstable. It is demonstrated that slow vehicles in a platoon moving at close to their top speed can damp nonlinear congestion waves. [S1063-651X(97)06002-9]

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I. INTRODUCTION

With roads becoming more congested, the need to implement effective traffic control measures is becoming greater all the time. Such measures include on-ramp metering, speed-limit control, and imposing tolls. For these to be effective, we require accurate models of the flow of traffic. Much work in the past has centered on models in which all vehicles have the same characteristics, such as response time and top speed. However, it is seen that that variety of vehicle characteristics is an important factor in determining, among other things, the stability of steady traffic flow and the propagation of congestion waves.

There are several approaches to the problem of modeling traffic flow. On open roads, particularly motorways, most work has concentrated on (a) *continuum* models, where individual vehicles are smoothed out into continuous velocity and density fields, and (b) *car-following* models, where the behavior of each vehicle is linked to that of the vehicle in front by a mathematical rule (a ‘‘car-following law’’).

The understanding of certain features of traffic flows has improved with the advent of faster computers. An example of such a feature is the phenomenon of congestion waves. These waves occur when flow in the laminar regime (all vehicles traveling at a constant speed) exceeds the capacity of the driver-vehicle-road system, and successive overbraking of following vehicles leads to an instability. Congestion waves are generally nonlinear, which makes their exact analysis difficult.

Increased computational power has also made possible the recent development of *cellular automaton* models (closely related to car-following models) in which the road is discretized into two-state cells and vehicles progress in random jumps. Like the other two models, cellular automata can predict phenomena analogous to those seen on real roads.

From the point of view of consistency of the different types of models, it is desirable that all three mentioned are equivalent, at least in suitable limits. It is shown in the Appendix that car-following and continuum models are equivalent in the limit of slowly varying traffic density along the road. In what follows, we deal only with *single-lane* flows with no overtaking, such as might be found when lanes on a motorway close due to roadwork. Lane changing tends to spread and disperse congestion waves, an effect that has been studied in [1].

We wish to model the effect of having variety in the behavioral characteristics of vehicles in a traffic stream. For ease of analysis, we adapt a simple velocity-adjustment car-following model. Whereas continuum models implicitly take into account a degree of variation in vehicle type, since the length scale of variations in the density and velocity field is assumed to be large compared to vehicle spacing, multispecies systems have not been widely studied in the context of car-following models. In the latter model, each vehicle has associated with it a sensitivity parameter and a relationship describing its desired speed as a function of its headway. When we come to specify this relationship, for the purposes of computer simulation in Sec. V, we use the function introduced in [2], namely,

$$U(b)=\tanh(b-2)+\tanh 2, \tag{1}$$

where b is the headway. This speed-headway relationship is plausible in that the desired speed is increasing with headway, from zero, when the headway is zero, to a finite positive value when the headway becomes large.

First, we present the results on linear stability derived by Bando *et al.* in [2], for the case of a single-species system on a circular road. The model is then extended in a natural way

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to *multispecies* systems, where the sensitivity parameters and the equilibrium speed-headway relationships are specific to each vehicle. In studying the latter system, we assess the differences between circular road and straight road systems to establish whether or not periodic boundary conditions affect stability artificially. In the language of Herman *et al.* [3], we consider the *asymptotic stability* of the straight road system, where the growth or decay of perturbations takes place in *space* rather than in *time*, which is the case for the circular system.

Having established a framework for studying the behavior of general multispecies systems, we go on to discuss in greater depth the behavior of a particular two-species system as a model of cars and trucks. We find that increasing the population of unreactive trucks in a car-truck system tends to make the system less stable and cause traffic jams.

We then study numerically the evolution of nonlinear congestion waves, first in the single-species case, and the influence of periodic boundary conditions on it. We go on to investigate the effect of trucks on their propagation. Of particular interest in this respect is the way in which vehicles moving at close to their top speed in a platoon can damp nonlinear congestion waves, and a demonstration of this phenomenon is shown. Finally, as a step towards a more realistic view of road traffic, we perform some numerical simulations of systems where there is a random variation of sensitivity parameters and/or top speeds among the vehicles and we make qualitative remarks concerning the progress of congestion waves in each case.

II. DESCRIPTION OF THE MODEL

In the velocity-adjustment model proposed by Bando *et al.* [2], the acceleration of each vehicle is defined to be directly proportional to the difference between its actual speed and its ''desired speed,'' a function of its headway:

$$\ddot{x}_i = a(U(x_{i+1} - x_i) - \dot{x}_i). \tag{2}$$

x_i is the position of car i, with the vehicles numbered so that vehicle i follows vehicle $i+1$. The setting for their model is a circuit of length L, with n vehicles on it, so that vehicle N follows vehicle 1. In addition, the dual limit $L,n \to \infty$ is taken, with the vehicle density n/L finite. The function $U(b_i)$ of the headway $b_i = x_{i+1} - x_i$ defines the desired speed of vehicle i. The constant of proportionality a is called the *sensitivity* parameter, with an associated equilibration time of order $1/a$, so that a can be thought of as a reciprocal reaction time of the vehicles.

Bando *et al.* showed that the constant-speed state of a single-species system on a circuit is linearly stable if and only if

$$\frac{2}{a} U'(b) \leqslant 1, \tag{3}$$

where $b = L/n$, so that systems of vehicles for which a is small enough are unstable and that groups for which a is large enough are stable. Clearly, the stability depends on traffic density also. For their choice of speed-headway relationship $U(b) = \tanh(b-2) + \tanh 2$, they found a ''window of instability'' $b_1 \leqslant b \leqslant b_2$, in which the constant-speed state is unstable to small perturbations. In fact, the attractor in this parameter range is a nonlinear traveling-wave solution.

We now consider an extension of the model to *multispecies* systems, in which the vehicles are nonidentical:

$$\ddot{x}_i = a_i(U_i(x_{i+1} - x_i) - \dot{x}_i), \tag{4}$$

where the subscript i on the parameter a and the function $U(b)$ renders them specific to vehicle i. In what follows, we will refer to vehicles with relatively high values of a as cars and to those with low values of a as trucks. We expect to find that trucks in a group of cars will act to make the system more unstable and that cars in a group of trucks will be slaved to the trucks.

When we come to perform linear stability analysis, we will require a dynamically stationary state about which we can linearize, in which all vehicles are moving at the same speed. We first define the ''least top speed'' of the set of vehicles:

$$V = \inf_i \sup_b U_i(b). \tag{5}$$

For n cars on a circular road of length L, we have the condition that $\Sigma_{i=1}^{n} b_i = L$, where L is finite. We make the assumption that $U_i(b) = 0$ when $b = 0$ and that $U_i(b)$ is strictly increasing in b. This is consistent with the speed-headway relationship $U(b) = \tanh(b-2) + \tanh 2$ proposed by Bando *et al.* in [2] and discussed in Sec. V. Suppose we fix the speed of all vehicles to be v, satisfying Eq. (6). As v is increased from 0 to the least top speed V, the sum of the headways increases from 0 to an arbitrarily large value, so that there must exist a speed v such that the headways sum to L.

We will consider later the effect of removing periodic boundary conditions by analyzing the linear stability of a multispecies system on a straight road, where the lead vehicle in a platoon can be controlled. There is then a family of dynamically stationary states

$$U_i(b_i) = v, \tag{6}$$

where v can take any value between 0 and V. First, we analyze the linear stability of the multispecies system on a circuit in order to see how the stability properties found by Bando *et al.* change when we have two or more types of vehicles interacting on the circuit.

III. ANALYSIS OF A MULTISPECIES SYSTEM ON A CIRCUITOUS ROAD

The governing system of differential equations is second order in time, so we must introduce perturbations to both the position and speed of each vehicle. The perturbed quantities are $x_i = x_i^0 + vt + \xi_i(t)$ and $v_i = v + \eta_i(t)$, where x_i^0 are the vehicles' initial positions and $\xi_i(t)$ and $\eta_i(t)$ are small time-dependent perturbations. Linearizing Eq. (4), we obtain

$$\begin{pmatrix} \dot{\xi} \\ \dot{\eta} \end{pmatrix} = \begin{pmatrix} \mathbf{0} & \mathbf{I} \\ \mathbf{A} & \mathbf{D} \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix} \equiv \mathbf{M} \begin{pmatrix} \xi \\ \eta \end{pmatrix}, \tag{7}$$

where $\mathbf{I}$ is the $n \times n$ identity matrix, $\mathbf{D} = \mathrm{diag}\{a_1, a_2, \ldots, a_n\}$, and

$$\mathbf{A}=\begin{pmatrix} -a_1U_1'(\overline{b_1}) & a_1U_1'(\overline{b_1}) & 0 & & \\ 0 & -a_2U_2'(\overline{b_2}) & a_2U_2'(\overline{b_2}) & 0 & \\ \cdots & \cdots & \cdots & & \\ \cdots & & \cdots & \cdots & \\ \cdots & & & & \\ 0 & & & & \\ a_nU_n'(\overline{b_n}) & 0 & \cdots & \cdots & -a_nU_n'(\overline{b_n}) \end{pmatrix}. \tag{8}$$

$\overline{b_i}$ are the unperturbed headways. The problem is now a set of coupled first-order equations, and the solution is

$$\begin{pmatrix} \xi \\ \eta \end{pmatrix}=\sum_{j=1}^{2n} c_j e^{z_j t}\begin{pmatrix} \xi^j \\ \eta^j \end{pmatrix} \tag{9}$$

for an arbitrary initial condition

$$\begin{pmatrix} \xi_0 \\ \eta_0 \end{pmatrix}=\sum_{j=1}^{2n} c_j \begin{pmatrix} \xi^j \\ \eta^j \end{pmatrix}, \tag{10}$$

$(\xi^j,\eta^j)^T$ are the eigenvectors of $\mathbf{M}$, and z_j are the corresponding eigenvalues. For linear stability to a general perturbation, we require that the real part of z_j is negative, for all j. Note that each eigenvector will have an associated frequency of oscillation (if z_j is complex, which it is in general), so that a *finite number* of modes is set up.

We require an equation for the eigenvalues z. From Eq. (7) we get the pair of equations $z\xi=\eta$ and $z\eta=\mathbf{A}\xi+\mathbf{D}\eta$. These equations combine to form $z^2\eta=\mathbf{A}\eta+z\mathbf{D}\eta$, so that we are looking for solutions to $\det(\mathbf{A}+z\mathbf{D}+z^2\mathbf{I})=0$. Define $\beta_i=a_iU_i'(\overline{b_i})$. The characteristic equation is then

$$f(z)\equiv\prod_{i=1}^{n}(\beta_i+a_iz+z^2)-\prod_{i=1}^{n}\beta_i=0. \tag{11}$$

Commutativity of the products means that stability of system is *independent of the ordering of vehicles around the circuit* if we assume that the behavioral parameters of each vehicle depend only on the vehicle in question.

The number of roots in $\mathrm{Re}z<0$ is given by the integral [4]

$$n_-=\frac{1}{2\pi i}\int_\Gamma \frac{dz\,f'(z)}{f(z)}, \tag{12}$$

where the contour Γ runs up the imaginary axis from $-i\infty$ to $+i\infty$ and is closed by an counterclockwise semicircle at infinity in the left half plane.

It can be shown that the contribution from the semicircle is n (this is true for all polynomials f of degree $2n$) and that the contribution from the imaginary axis is the winding number of the contour γ around the origin, where γ is defined as

$$\gamma=\{z=f(i\omega):\omega\in\mathrm{Re}\}. \tag{13}$$

Consider the contour γ':

$$\gamma'=\{z=g(i\omega):\omega\in\mathrm{Re}\}, \tag{14}$$

where $g(z)\equiv\Pi_{i=1}^n(\beta_i+a_iz+z^2)\equiv f(z)+\Pi_{i=1}^n\beta_i$. The contour γ' is precisely the contour γ, shifted to the right through a distance $\Pi_{i=1}^n\beta_i$. Their winding numbers will be the same, unless γ' has any intersections with the positive real axis in the interval $(0,\Pi_{i=1}^n\beta_i)$.

We consider the magnitude of $g(i\omega)$ as ω varies, using the equation

$$|g(i\omega)|=\sqrt{\prod_{i=1}^{n}[(\beta_i-\omega^2)^2+a_i^2\omega^2]}. \tag{15}$$

This function is not necessarily monotonically increasing with ω^2, so it is possible that there exist values of ω for which $|g(i\omega)|<|g(0)|$ and $\arg g(i\omega)=2m\pi$ for some integer m (a necessary and sufficient condition for instability).

Now the function $\arg g(i\omega)$ is given by

$$\arg g(i\omega)=\sum_{i=1}^{n}\arg(\beta_i-\omega^2+ia_i\omega). \tag{16}$$

Each term in the sum is strictly increasing from $-\pi$ to π and takes the value 0 at $\omega=0$. If we let n become large, then for $\omega=O(1)$, $\arg g(i\omega)$ increases by 2π for a $O(1/n)$ change in ω. We now consider $\ln|g(i\omega)|$, which is just the logarithm of the product in Eq. (15):

$$\ln|g(i\omega)|=\frac{1}{2}\sum_{i=1}^{n}\ln[(\beta_i-\omega^2)^2+a_i^2\omega^2]. \tag{17}$$

Around a quadratic minimum of $\ln|g(i\omega)|$, an $O(1/n)$ change in ω (one turn around the origin) leads to an $O(n/n^2=1/n)$ change in $\ln|g(i\omega)|$. So if n is large enough, and if for some $\omega_{\min}$, $\ln|g(i\omega_{\min})|$ is strictly less than $\ln|g(0)|$, then $\ln|g(i\omega)|$ will be less than $\ln|g(0)|$ for all ω on the same turn around the origin. This guarantees an intersection of $g(i\omega)$ with the positive real axis to the left of $g(0)$.

Therefore, if n is large and there exists a real value of ω for which $|g(i\omega)|<|g(0)|$, the system is linearly unstable. We have effectively approximated the discrete spectrum of excited modes by a continuum, in the limit $n\to\infty$.

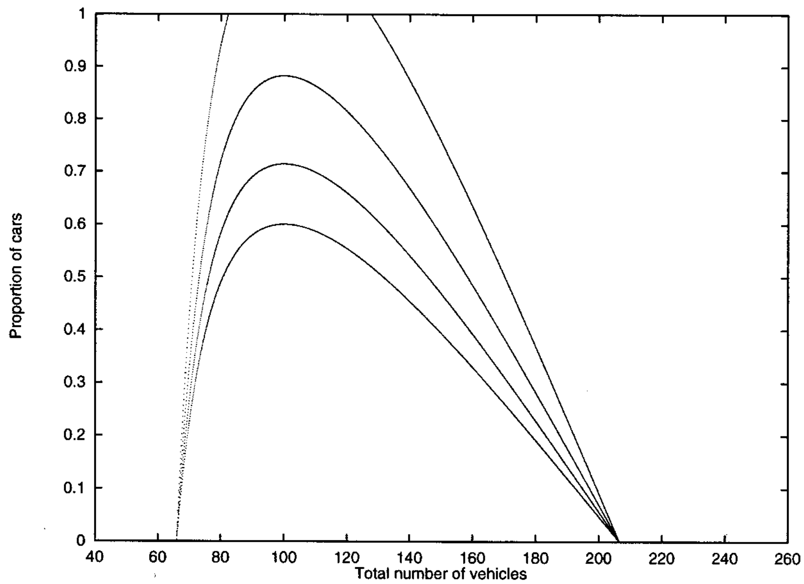

FIG. 1. Proportion of cars required to stabilize a car-truck system. $a_{\mathrm{truck}}=0.8$ and $L=200.0$. The curves correspond to $a_{\mathrm{car}}=1.67$, 2.5, 5, and ∞ reading from top to bottom.

Unfortunately, the equation $|g(i\omega)|=|g(0)|$ is not solvable analytically (except for the trivial solution $\omega=0$), so we resort to a numerical solution, in the case of a two species system, where each type has a different acceleration parameter but the same equilibrium speed-headway function $U(b)$. We can compute the derivative $(d/d\omega)\ln|g(i\omega)|$ easily, in order to find the minimum values of $\ln|g(i\omega)|$. The result is

$$\frac{d}{d\omega}\ln|g(i\omega)|=\omega\left(\frac{n_C[(a_C^2-2\beta_C)+2\omega^2]}{(\beta_C-\omega^2)^2+a_C^2\omega^2}+\frac{n_T[(a_T^2-2\beta_T)+2\omega^2]}{(\beta_T-\omega^2)^2+a_T^2\omega^2}\right), \quad (18)$$

where n_C and n_T are the number of vehicles of types C (cars) and T (trucks), respectively. To find the minimum of $\ln|g(i\omega)|$ we look at its turning points: the appropriate values of ω are given by the roots of Eq. (18). When expanded, the numerator is of the form $\omega p(\omega^2)$, where p is a cubic polynomial. The largest real root ω^* of this cubic can be found numerically to any desired accuracy using the Newton-Raphson method, and the stability of the system is determined accordingly as $\ln|g(i\omega^*)|\lessgtr\ln|g(0)|$ (require greater for stability). Hence, for this type of two-species system we can decide with certainty whether or not a given system is stable.

In fact, we can find a *sufficient* condition for stability or instability by analytical means. We note that $(d/d\omega)\ln|g(i\omega)|=0$ at $\omega=0$. Therefore, if $(d^2/d\omega^2)\ln|g(i\omega)|<0$ at $\omega=0$, there must be a minimum of $\ln|g(i\omega)|$ with value less than $\ln|g(0)|$, which implies instability.

This condition can be translated into parameter space, using Eq. (17): if $\Sigma_{i=1}^{n}(a_i^2-2\beta_i)/\beta^2<0$, then the system is linearly unstable. Using the definition of β_i, a sufficient condition for instability is

$$\sum_{i=1}^{n}\frac{1}{U_i'(\overline{b_i})^2}\left(1-\frac{2}{a_i}U_i'(\overline{b_i})\right)<0; \quad (19)$$

N.B. This condition guarantees instability, but even if inequality (19) does not hold, there may be other points on the graph of $|g(i\omega)|$ at which instability can arise.

For the case of only one species of vehicle, the above analysis is exact; that is, we do not need to assume n is large. The condition for instability is necessary and sufficient and can be written

$$1-\frac{2}{a}U'(\overline{b})<0, \quad (20)$$

which is just that obtained in [2]. In some sense, the overall stability of the mixed system is governed by a weighted mean of ''stability parameters'' corresponding to the vehicles in that system.

Car-truck system: Semianalytic results

We use the numerical scheme detailed above to find the stability region in $n_{C,T}$ space. For all values of $a_{C,T}$ tested, this region was the same as that derived from Eq. (19). We present graphs (Fig. 1) depicting the proportion of cars $\rho_C=n_C/n$ required to stabilize a population of cars and trucks ($a=0.8$) for various values of the car sensitivity parameter. The road length L is 200, and we give all vehicles the same desired speed-headway function $U(b)$, so that the vehicles are evenly spaced around the track with $U'(b)=1$.

For $n<66$, the system is always stable, independently of the ratio ρ_C: all vehicles travel at their free-flow speed, but with large headways. The same is true for $n>206$, but this time, the system is stable because every vehicle is traveling slowly enough to offset the effect of small headways. In the range of n for which a population of trucks alone would be unstable ($66\leqslant n\leqslant 206$), the replacement of some trucks by the same number of cars often, but not always, stabilizes the system.

Cars can be thought of as being more responsive and have a damping influence on perturbations. Note that even in the limit of large sensitivity parameter, we require a fairly large proportion of cars before the system stabilizes (up to 50% cars). In the limit $a_{\rm cars} \to \infty$ with $a_{\rm trucks}$ fixed at 0.8, the equation of motion $\dot{x}_i = U_i(b_i)$ must hold to avoid singularities in Eq. (4). However, such ultrareactive cars do not ''absorb'' disturbances, as might be expected. The cars are effectively ''slaved'' to the trucks and behave passively. We now move onto a straight road to see the effects, if any, of removing the periodic boundary conditions associated with the circuit.

IV. ANALYSIS OF A MULTISPECIES SYSTEM ON A STRAIGHT ROAD

On a circuitous road, only a finite number of oscillatory modes can be set up. However, this spectrum may be approximated as a continuum in the limit of a large number of vehicles. On a straight road, we analyze the stability of a platoon of cars. We assume that the behavior of the leading vehicle can be controlled for all time, and consider the response of the system to small, pure harmonic disturbances of *arbitrary* frequency ω. Since we are considering the linear stability of the system, the response to each frequency component of a general time-dependent disturbance can be considered separately.

We consider small perturbations about the dynamically stationary state. Vehicle i is following vehicle $i+1$, with the equation of motion

$$\ddot{x}_i = a_i(U_i(x_{i+1} - x_i) - \dot{x}_i). \tag{21}$$

Linearizing, so that $x_i = x_i^0 + vt + \epsilon_i$, we obtain

$$\frac{1}{a_i}\ddot{\epsilon}_i + \dot{\epsilon}_i + U_i'(\overline{b_i})\epsilon_i = U_i'(\overline{b}_i)\epsilon_{i+1}, \tag{22}$$

where, as for the circuitous road, $\overline{b_i}$ is the unperturbed headway between vehicles i and $i+1$. Now we decompose the perturbations into components of different frequencies. Writing $\underline{\epsilon_i} = \delta_i e^{i\omega t}$ and using the shorthand U_i' to represent $U_i'(b_i)$, we see that

$$\delta_i = \frac{\delta_{i+1}}{\left(1 - \frac{\omega^2}{a_i U_i'}\right) + \frac{i\omega}{U_i'}}. \tag{23}$$

In passing from vehicle $i+1$ to the following vehicle i, the perturbation is amplified by a factor

$$\gamma_i = \frac{1}{\sqrt{\left(1 - \frac{\omega^2}{a_i U_i'}\right)^2 + \left(\frac{\omega}{U_i'}\right)^2}}, \tag{24}$$

so the disturbance grows if and only if $\gamma_i > 1$. In that case, we say that frequency ω disturbances are *asymptotically unstable* [3] in passing from vehicle $i+1$ back to vehicle i. From Eq. (24), it can be shown that only frequencies in the range $0 < \omega < (1/U_i')\sqrt{(2/a_i)U_i' - 1}$ are amplified. In the case $(2/a_i)U_i' - 1 \leqslant 0$, all frequencies are attenuated.

The overall amplification γ of frequency ω from the leading vehicle to vehicle 1 is computed by taking the product $\Pi_{i=1}^n \gamma_i$. We say that a group of n of vehicles is asymptotically stable to disturbances of frequency ω if and only if

$$\gamma(\omega) = \prod_{i=1}^{n} \gamma_i < 1 \tag{25}$$

since if a line of traffic is made up of repeating units of permutations of this group, disturbances of frequency ω will grow down the line of vehicles.

The parallels with the circular road system can now be seen. In that case (assuming a large enough number of vehicles to be able to approximate the spectrum of excited frequencies as a continuum), the system is stable if and only if there exists a solution for positive, real ω of the inequality $|g(i\omega)| \leqslant |g(0)|$ [see Eq. (15)]. This is exactly the same inequality as Eq. (25), but with one crucial difference in statement. In the case of the circular road, we ask if there are *any* solutions to the inequality, but for a straight road, we ask if the inequality is satisfied for a *prespecified* ω. The problem set on a circular road is thus about temporal stability to arbitrary initial spatial disturbances, whereas the problem set on a straight road is about spatial stability to temporal (ongoing) disturbances.

It is perhaps surprising at first sight that the interaction of disturbance waves on a circuit (something that cannot happen on a straight road) appears to have no effect on the stability of the system. However, the analysis for a circuit assumes a large number of vehicles, so that decaying disturbance waves will have practically disappeared before they complete one circuit and growing waves will have become nonlinear. It turns out that wave interactions are important to the evolution of the system in the nonlinear regime, and in that setting, the system's behavior depends strongly on whether we have a circuitous or a straight road.

In addition to growth or decay in its amplitude, the disturbance suffers a phase lag of

$$\phi_i = \arctan\frac{\omega}{U_i' - \frac{\omega^2}{a_i}} \tag{26}$$

in going from vehicle $i+1$ to vehicle i, from which the time lag is derived by dividing by ω,

$$\tau_i = \frac{1}{w}\arctan\frac{\omega}{U_i' - \frac{\omega^2}{a_i}}. \tag{27}$$

In this discrete setting, we define the wave speed at position i as

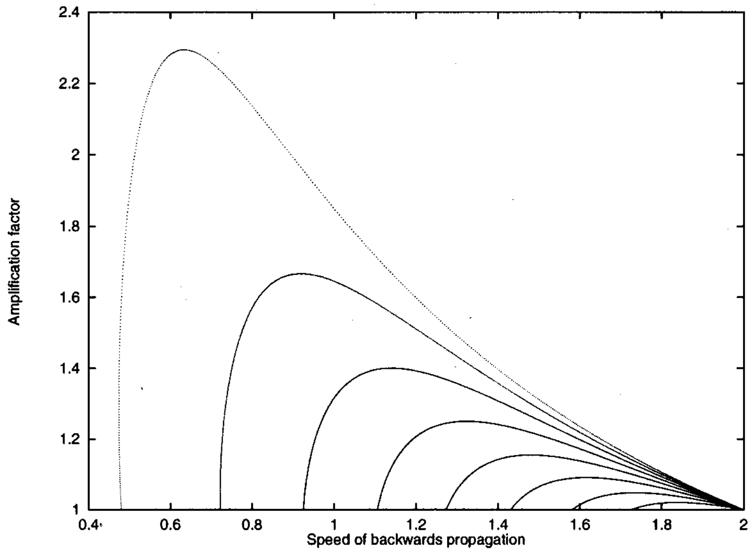

FIG. 2. Amplification factor against backward propagation speed. $b=2$ and $U'(b)=1$. The curves are parametrized by disturbance frequency ω and only growing disturbances are represented. From left to right, a increases from 0.2 to 1.6 in increments of 0.2.

$$c_i := -\frac{b_i}{\tau_i}. \tag{28}$$

We see from this equation that different frequencies are transmitted backward relative to the traffic at different speeds. For large ω, this speed asymptotes to $-b_i\omega/\pi$. However, this occurs in the region of attenuated ω, so that such fast waves are never observed over more than several vehicles. For small ω the asymptotic result is $c_i = -b_i U_i'(b_i) + o(\omega^2)$, so that low-frequency disturbances travel at a speed independent of the sensitivity parameter. This is because the period of oscillations is much greater than the time scale for equilibration $(1/a_i)$, so the vehicles are in quasiequilibrium.

The most unstable mode $\omega^* = \sqrt{a_i U_i'(b_i) - a_i^2/2}$, defined only when $0 \leqslant a_i/U_i' \leqslant 2$, travels backward with speed

$$c^* = -\frac{b\omega^*}{\arctan(2\omega^*/a_i)} \tag{29}$$

and has an associated amplification factor

$$\gamma_i^* = \frac{1}{\sqrt{\frac{a_i}{U_i'} - \frac{1}{4}\left(\frac{a_i}{U_i'}\right)^2}}. \tag{30}$$

The plots presented (Fig. 2) are of amplification per vehicle (γ) against backward wave speed, for various values of a in the range $(0.2,1.6)$, for a single-species system where $b_i = 2.0$ and $U_i'(b_i) = 1.0$. ω is used as the parameter for each curve, and only growing modes $(\gamma_i \geqslant 1)$ are displayed. There is clearly a range of wave speeds, so disturbances spread as a ''wedge'' in the space-time plane and the amplitude of the disturbance experienced by a given vehicle increases and then decreases smoothly in time. These properties have been confirmed numerically. We now present the results of some numerical simulations that verify the results on stability and propagation of disturbances presented above, both for circuitous and for straight roads.

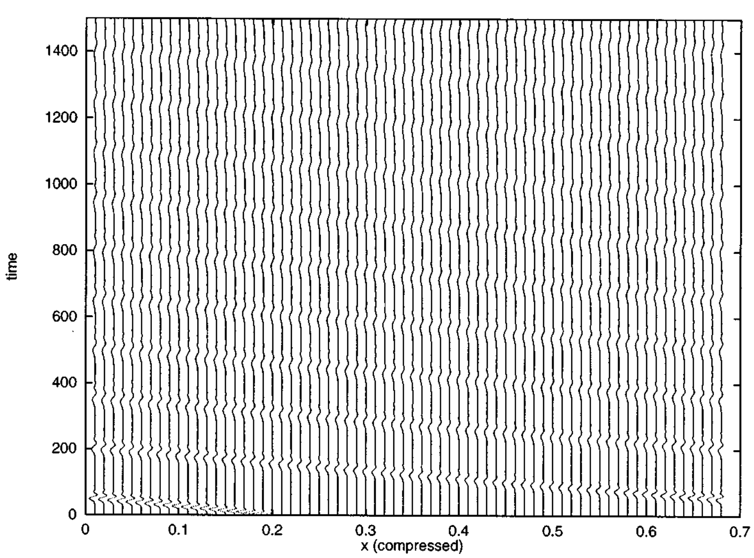

FIG. 3. Perturbations relative to trajectories of vehicles in the x-t plane. The arrangement of vehicles (left to right) is $TTT \cdots TCCC \cdots C$. $L=200$, $n_C=20$, $n_T=48$, $a_C=1.5$, and $a_T=0.8$.

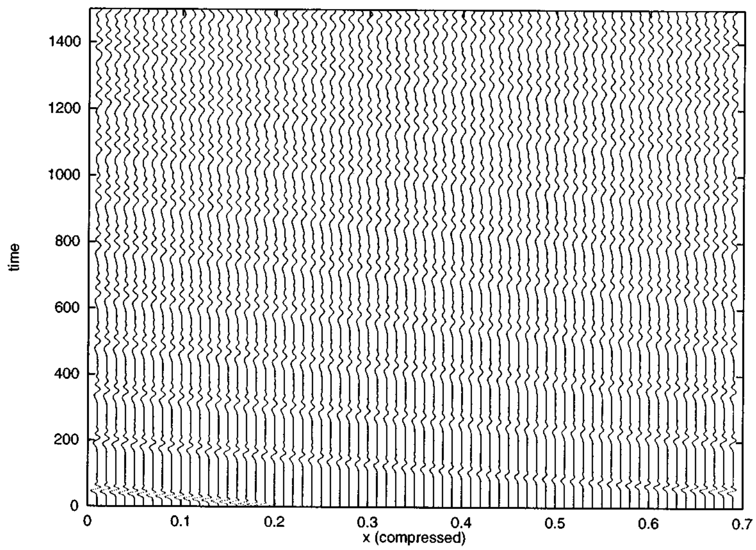

FIG. 4. Perturbations relative to trajectories of vehicles in the x-t plane. The arrangement of vehicles (left to right) is $TTT\cdots TCCC\cdots C$. $L=200$, $n_C=20$, $n_T=49$, $a_C=1.5$, and $a_T=0.8$.

V. NUMERICAL VERIFICATION OF STABILITY ANALYSIS

We employ the same speed-headway relationship as in [2], namely,

$$U_i(b)=\tanh(b-2)+\tanh 2. \tag{31}$$

The numerical scheme used is the fourth-order Runge-Kutta one, with a step size of 0.01 time units. It is found that the results are correct to five decimal places after 1000 time steps.

A. Circular road

The n vehicles on the track (length 200) are spaced evenly with headway b and are all given an initial speed of $v=U(b)$, except one vehicle, which has an initial speed of $0.99v$. It is found for both stable and unstable systems that, before nonlinear effects take over, disturbances are localized in time and resemble a wave packet propagating backward around the circuit, so that vehicles ''drive through'' it. Each vehicle has long periods of motion at (very nearly) constant speed and short periods of oscillatory disturbance. The system's qualitative behavior is very similar to the straight road system to be discussed in Sec. V B.

We restrict our attention to two-species systems of cars and trucks. Two space-time diagrams are presented (Figs. 3 and 4), showing perturbations to the trajectories of individual vehicles on a circuit. In both diagrams $a_C=1.5$ and $a_T=0.8$ on a circuit of length 200. The spacing of the vehicles is compressed on the graphs so that the small perturbations show up. The vehicles were arranged so that a single group n_T of trucks was following a single group n_C of cars.

In Fig. 3, which depicts a stable system ($n_C=20$ and $n_T=48$), the perturbation starting at $x=0.04$, which is localized to begin with, remains localized as it travels as a wave around the circuit. It decays in the region of cars, but is amplified through the line of trucks. However, at the end of each loop of the circuit, this amplification is not enough to bring it back to the same amplitude it had at the start, and it dies away.

In the second diagram (Fig. 4), for which $n_C=20$ and $n_T=49$, the opposite occurs. The trucks are able to effect a net amplification of the wave per circuit, and the disturbance grows (instability). This change in stability, from stable to unstable, has occurred with the addition of just one truck: we have crossed the boundary of the stability region in parameter space.

In the first case, the localized wave propagation would be very similar if we were to ''unroll'' the system onto a straight road: the influence of one pass through the disturbance wave almost completely dies away before the wave comes round again. In the second case, phase interactions between successive passes are likely to be having a pronounced effect, since the disturbance wave has become completely delocalized by time $t=1500$, and nonlinear effects are becoming important.

B. Straight road

The vehicles are spaced with all headways b_i equal to $b=2$ units. The lead car travels at all times at a constant speed $v=U(b)$. The other cars all start at speed v, except for the second-to-leading car $(n-1)$, which starts off with speed $0.99v$. The disturbance to this car is localized in that its speed equilibrates to v over a time scale of order $1/a$, where a is its sensitivity parameter.

Again, the x scale is compressed for diagrammatic purposes. The disturbance wave packet is seen to travel backward down the line of traffic and to disperse in time, i.e., become less localized. As with the circuitous system, the perturbation increases in size when passing through clusters of trucks and decreases in size when passing through clusters of cars (if the spacing is set up as for the circular road). This is shown in Fig. 5, where the wave first passes through a region where $a=1.5$ (destabilizing), then through a region where $a=3.0$ (stabilizing), and finally through a region where $a=1.0$ (destabilizing). The wave grows, then decays, and then grows again.

VI. NONLINEAR BEHAVIOR OF SINGLE-SPECIES SYSTEMS

We now return briefly to looking at single-species systems in order to investigate the effect of periodic boundary

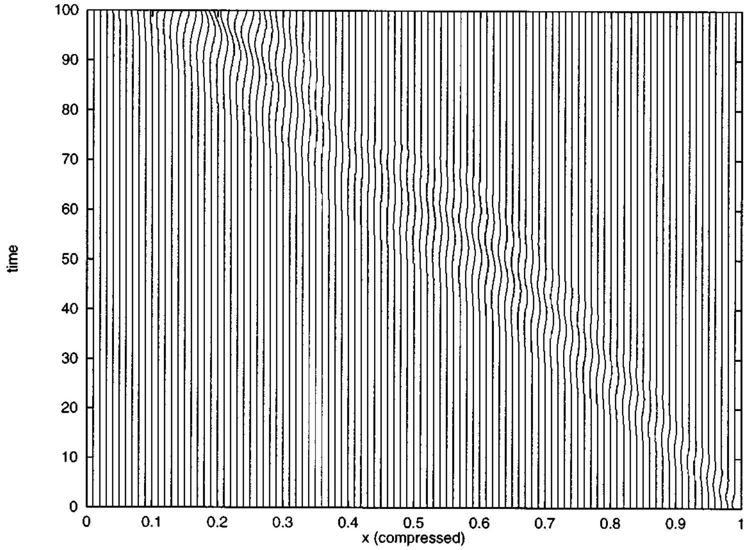

FIG. 5. Perturbations relative to trajectories of vehicles in the x-t plane, for a straight road system. Vehicles are arranged $PP\cdots PQQ\cdots QRR\cdots R$, where $a_P=1$, $a_Q=3$, and $a_R=1.5$. Headways are all $b=2$ units and $U'(b)=1$.

conditions on the propagation of nonlinear waves. We consider the straight road case first.

In a plot of headway against position, we see a fully developed nonlinear wave train (Fig. 6). Congestion waves are seen as plateaus in this diagram. The speeds of the backward-propagating shocks that join the plateaus are greater in magnitude towards the back of the wave train, so that the regions between the shocks increase in length with time. Between the front plateau and the front of the wave train are regions of high spatial frequency oscillation, which also increase in length with time. In these regions, there are typically four or five vehicles per oscillation.

On a circuit, the nonlinear wave train evolves in exactly the same way until it becomes longer than the circuit, at which time the back comes into contact with the front of the wave train. There follows a self-interaction of the wave train (Fig. 7), in which the high spatial frequency oscillations are consumed first as the plateaus move backward through them.

These plateaus then annihilate each other, as follows. Plateau A, moving just in front of plateau B, starts to move backward more quickly when its shock nearest B changes strength, as a result of vehicles not having time to attain top speed in transit from B to A. As the two plateaus get closer, this effect becomes more pronounced, until plateau A actually merges with B: an example on Fig. 7 is at $t=600$ and $x=75$.

This process continues to annihilate plateaus until we are left with a set of well-separated plateaus, all moving at the same speed backward along the road. The system thus organizes itself into a stable state. The selected amplitude of each of the plateaus, which are kink solitons [5], is observed as the largest-amplitude part of the wave train in the case of a straight road.

Periodic boundary conditions are therefore important when discussing the nonlinear evolution of single-species systems; this is also the case when we have a variety of vehicle types. In that case, the presence of trucks has no effect on the mechanism of plateau annihilation described above, and we are again left with a set of well-separated regions of congestion. We now investigate the formation and

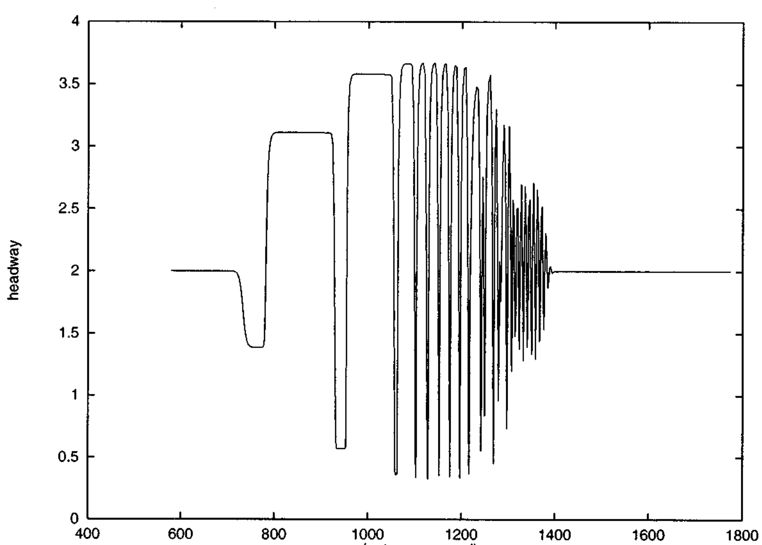

FIG. 6. Fully developed nonlinear wave train, for a straight road system (single species): headway is plotted against position. $a=1$ and $b=2$ for all vehicles initially. Snapshot at $t=600$, with 600 vehicles.

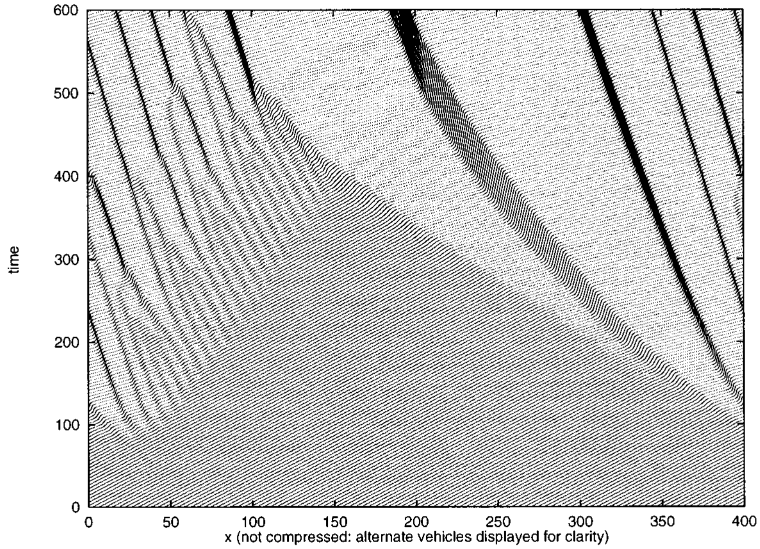

FIG. 7. Evolution of a set of congestion waves relative to the road, in the x-t plane, for a circuitous road system (single species). $a=1$, $b=2$, and $U'(b)=1$. 200 vehicles, up to time $t=600$.

behavior of multispecies congestion waves on a straight road.

VII. NONLINEAR BEHAVIOR OF MULTISPECIES SYSTEMS

We first introduce to the system a random spread of sensitivity parameters a, as a step towards a more realistic model of road traffic. We keep the same speed-headway relationship $U(b)$ for each vehicle, and the progress of the nonlinear waves is shown in Fig. 8.

Vehicles with different sensitivity parameters a take different lengths of time to move from low-density free flow into the congestion regime and back again, and we expect this to have a bearing on the backward propagation speed of nonlinear waves. Indeed, from Fig. 8, this appears to be verified: having a random spread of values of a among the vehicles makes the nonlinear wave speed nonconstant as it propagates. However, despite being distorted, the nonlinear waves maintain their overall structure as they propagate. Hence the single-species system is not simply a degenerate case, which happens to support these waves.

Following from this, we demonstrate the effect of varying the speed-headway relationships from vehicle to vehicle, while keeping a constant. The function $U(b)$ of vehicle i is now given by $U_i(b)=c_i[\tanh(b-2)+\tanh 2]$, where the ''top speed parameters'' c_i are defined independently for each vehicle. In the diagram presented, we see a single-species system of cars, with one truck. The truck is given a slightly lower value of c than the cars (0.8 compared to 1) and, as a result, a gap opens up between it and the car in front. The gap has the effect on nonlinear traveling waves of inducing a temporary phase shift as the gap passes through (Fig. 9). The disturbance to the congestion wave then decays away and the gap, which is compressed as it enters the wave, starts to expand again as it leaves.

If, however, the truck is given a *much* lower top speed than the cars (0.5 compared to 1), the gap that opens in front of the truck is wider than before (Fig. 10). In fact, the gap is large enough to destroy the nonlinear wave. This is a form of

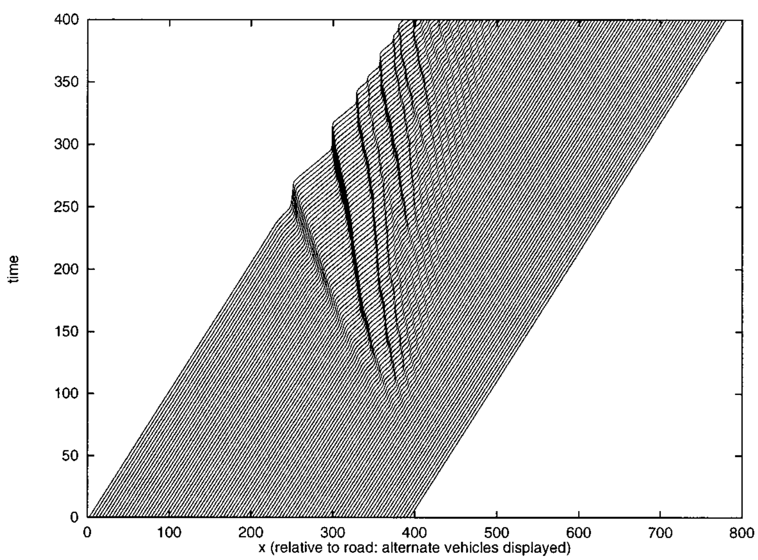

FIG. 8. Nonlinear waves in a system of vehicles with independent, randomly distributed sensitivity parameters $a \sim U$ [0.6,1.6]. Vehicles are evenly separated with $b=2$ and their trajectories are shown relative to the stationary road.

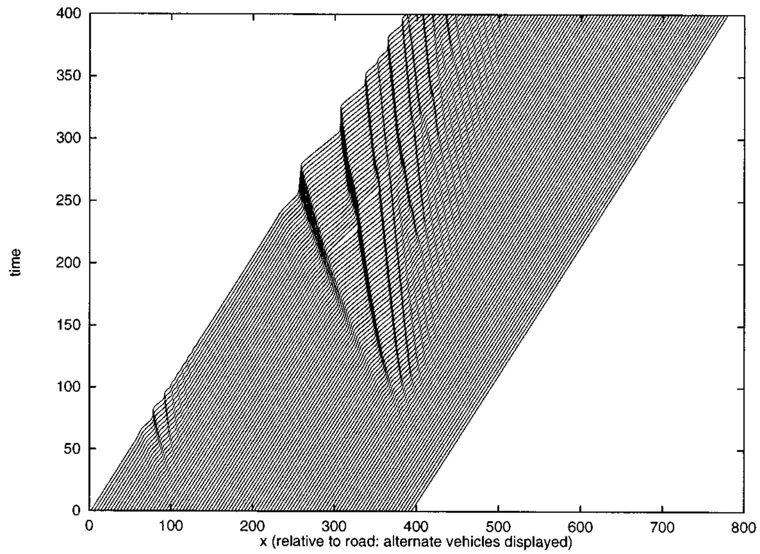

FIG. 9. Single truck with top speed parameter $c = 0.8$, observed within a system of 199 cars with top speed parameter $c = 1$. $a = 1.0$ for all vehicles, and they set off evenly spaced with $b = 2$. Trajectories are displayed relative to the road.

nonlinear damping and it depends on the fact that $U'(b)$ is small for large b: the car in front of the truck can make large deviations from its equilibrium position without significantly changing the truck's speed. This is a possible mechanism for stabilizing real traffic platoons.

However, because the system is so unstable in this case ($b = 2$ is the most unstable configuration, when cars are driving at their desired speed), a new wave is formed further upstream as a result of the very small perturbation that the truck experiences. The wave cannot be said to have experienced a temporary disturbance, as in Fig. 9, because it ceases to be nonlinear in its interaction with the truck and has to start its nonlinear evolution afresh. This scenario is an example of a truck both causing a traffic jam and breaking one up.

VIII. APPLICATION TO REAL TRAFFIC FLOWS

Before a disturbance can start to evolve *non*linearly to any significant degree, it must have attained a large enough amplitude. Linear theory gives us some indication of how long it would take for a small disturbance to grow sufficiently to cause ''harsh'' braking (deceleration greater than some preset value) further down the line of traffic, for example. However, the exact nature of the braking profile cannot be determined by this method.

The vehicle-following model discussed here makes many assumptions about the real world, not least that there is no ''global perception'' about the *general* state of the traffic. In reality, drivers learn that they are in a congested flow not just from the behavior of the car in front: much more information is taken into account. Also, there is always an element of randomness over time in, say, the sensitivity parameter of a given driver.

There is a large number of practically unmeasurable parameters associated with the traffic system, and the best we can do in the case of a linearly unstable system is to say that it is *likely* that harsh braking will take place during a given time interval. The likelihood of such an event occurring would be unacceptably high for a linearly unstable system: a

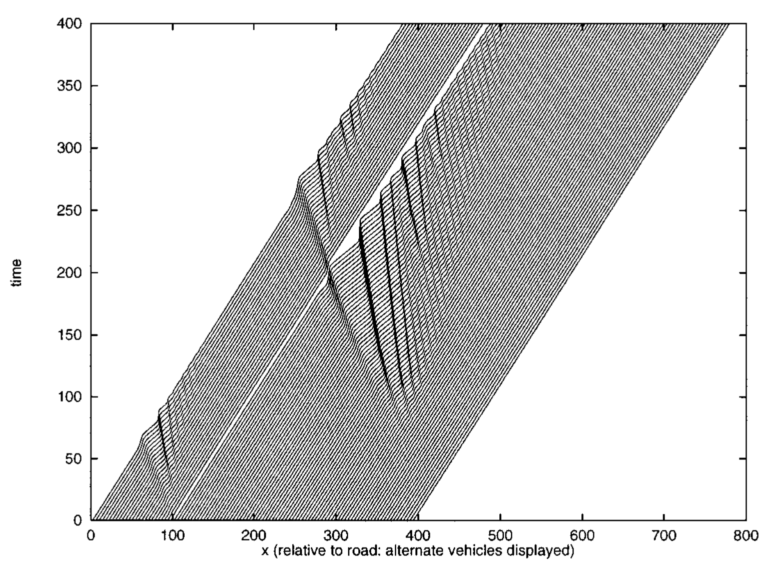

FIG. 10. Same as for previous diagram, but with $c_{\mathrm{truck}} = 0.5$.

suitable indicator of this instability is given by Eq. (19). Given fitted curves of $U(b)$ for a range of vehicle types, the quantity $\delta_i=[1/U_i'(b)^2][1-(2/a_i)U_i'(b)]$ could be evaluated for each vehicle automatically and in real time, using sensors under the road. Steps could be taken to stabilize the system (e.g., by reducing the speed limit) if the sum of this quantity over the vehicles in a platoon were to fall below zero.

IX. CONCLUSION

We have studied several aspects of the behavior of a multispecies car-following model both on circular and on straight roads. It was found, in the limit of long roads with a large number of cars, that the linear stability criteria for circular and straight roads were the same, given an appropriate definition of stability on a straight road. The definition used was that of *asymptotic stability*, a term used by Herman *et al.* [3] to describe the overall attenuation of a disturbance as it passes through a group of vehicles. Computer simulation verified the stability criterion for a two-species system on a circuitous road.

We also looked at the propagation of small disturbances, both on circular and straight roads. Again, computer simulations were used to verify that disturbances grow through asymptotically unstable groups of vehicles and decay through stable groups. It was found that by having a high enough concentration of trucks in a population of cars, the system as a whole could be made asymptotically unstable, as predicted: in this sense, trucks can cause traffic jams. Dispersion, i.e., dependence of propagation speed on the frequency of disturbance, was also analyzed, and seen in practice.

We then made some qualitative remarks concerning the development of small disturbances into nonlinear waves, including a description of the mechanisms involved in pattern selection when periodic boundary conditions are applied. We went on describe the effects of how, in the case of a predominantly ''all-car'' system, trucks interact with congestion waves, including the nonlinear damping effect of trucks moving at close to their maximum speed in a platoon of vehicles. Finally, some general remarks were made about how this work could be applied to real-life traffic control problems.

APPENDIX

We show, following Whitham in [6], that in the limit of slowly varying headway along the road, all car-following models are equivalent to a continuum model. We take as a starting point an equation true for all car-following models

$$\dot{b}_i=v_{i+1}-v_i, \tag{A1}$$

where b_i is the headway of car i (the distance to car $i+1$) and v_i and v_{i+1} are the speeds of cars i and $i+1$, respectively. The overdot denotes a total time derivative.

Letting x_i denote the position of car i, we can define a continuous headway function $b(x,t)$ such that $b(x_i,t)=b_i(t)$ for all i and we define $k(x,t)\equiv 1/b(x,t)$ to be the traffic density. We also require a continuous speed function. As for the headway, we define this function such that it interpolates the ''discrete speed'' v_i when evaluated at each car position, i.e., $v(x_i,t)=v_i(t)$ for all i. Clearly, $v_{i+1}=v(x_i+b(x_i),t)$, and substituting in Eq. (A1),

$$\frac{d}{dt}b(x_i,t)=v(x_i+b(x_i,t),t)-v(x_i,t). \tag{A2}$$

Expanding the total derivative, we obtain

$$\frac{\partial b}{\partial t}(x_i,t)+v(x_i,t)\frac{\partial b}{\partial x}(x_i,t)=v(x_i+b(x_i,t),t)-v(x_i,t). \tag{A3}$$

We now replace x_i by x throughout:

$$\frac{\partial b}{\partial t}(x,t)+v(x,t)\frac{\partial b}{\partial x}(x,t)=v(x+b(x,t),t)-v(x,t). \tag{A4}$$

If we now assume that all quantities $\Lambda(x,t)$ associated with the traffic flow vary on length scales much greater than the headway, i.e.,

$$\epsilon_\Lambda=\frac{b\Lambda_x}{\Lambda}\ll 1, \tag{A5}$$

we can expand Eq. (A4) as a Taylor series, obtaining

$$b_t+vb_x=bv_x+\frac{1}{2}b^2v_{xx}+\cdots, \tag{A6}$$

where each consecutive term on the right-hand side is much less than the one preceding it. Neglecting all but the leading-order terms, replacing b by $1/k$, and rearranging, we obtain

$$k_t+q_x=0, \tag{A7}$$

which is the equation of vehicle conservation common to all continuum models, with $q=kv$.

We also require a car-following law. In general, this can be expressed as

$$T(b_i,\dot{b}_i,\ddot{b}_i,\ldots,v_i,\dot{v}_i,\ldots)=0, \tag{A8}$$

which can be placed within the framework of continuum models simply by replacing b_i with $1/k$ and v_i with v. In the limit of slow variation, we can s*et al.*l the arguments involving a time derivative to zero, yielding $T(1/k,v)=0$: speed is a function of local density alone.

In the case of multispecies models, we define traffic densities $k^j(x,t)$ for each vehicle type j as the number of vehicles of type j per unit length of road. We replace the equation $kb=1$, true by definition of density for single-species systems, with the equation $\Sigma_{j=1}^n k^j(x,t)b^j(v)=1$, where n is the number of vehicle types and $v(x,t)$ is the local speed of traffic. The functions $b^j(v)$ give the headway for vehicle types j moving at speed v, in the limit of slow variation. Consider a stretch of roadway of length L, much longer than $1/k^j$ for each j, but much shorter than the length scale of density variation. Conservation of vehicles yields $\int_x^{x+L} k_t^j(x',t)dx' = Lk^j(x,t)v(x,t) - Lk^j(x+L,t)v(x+L,t)$, which after truncating Taylor expansions gives n separate conservation equations

$$k_t^j + (k^j v)_x = 0. \tag{A9}$$

For the continuum approximation to be valid for multispecies systems, we require longer length scales of variation than for single-species systems, as each vehicle type must have slowly varying density along the road and in general each density will be lower than in the single-species case.

[1] E. N. Holland, Trans. Res. B. (to be published).
[2] M. Bando, K. Hasebe, A. Nakayama, A. Shibata, and Y. Sugiyama, Phys. Rev. E **51**, 1035 (1995).
[3] R. Herman, E. W. Montroll, R. B. Potts, and R. W. Rothery, Opns. Res. **7**, 86 (1959).
[4] G. Arfken, *Mathematical Methods for Physicists*, 3rd ed. (Academic, London, 1985).
[5] T. S. Komatsu and Sasa, Phys. Rev. E **52**, 5574 (1995).
[6] G. B. Whitham, *Linear and Nonlinear Waves* (Wiley-Interscience, New York, 1973).