

Universal cubic eigenvalue repulsion for random normal matrices

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Random matrix models consisting of normal matrices, defined by the sole constraint $[\mathbf{N}^\dagger, \mathbf{N}] = 0$, will be explored. It is shown that cubic eigenvalue repulsion in the complex plane is universal with respect to the probability distribution of matrices. The density of eigenvalues, all correlation functions, and level spacing statistics are calculated. Normal matrix models offer more probability distributions amenable to analytical analysis than complex matrix models where only a model with a Gaussian distribution is solvable. The statistics of numerically generated eigenvalues from Gaussian distributed normal matrices are compared to the analytic results obtained and agreement is seen. [S1063-651X(97)05801-7]

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I. INTRODUCTION

Random matrix theory (RMT) [1,2] has found much success in phenomenological models describing a wide variety of physical systems, from discretization of moduli space [3] to the statistics of the cells in the skin of a cucumber [4]. RMT is the study of eigenvalues derived from random ensembles of matrices with stochastic elements specified by a probability density $P(\mathbf{M})d\mathbf{M}$ in the space of matrices. Most interest is in examining the properties of the eigenvalues induced from the transformation to the eigenvalue basis. Early work consisted of using real symmetric, Hermitian, unitary, and real quaternion matrices [5], the eigenvalues of which are either real or unimodular (and termed one-dimensional eigenvalues here).

Recently several groups have begun to consider physical applications of matrix models composed of complex matrices. Introduced in the early 1960s by Ginibre [6], it took decades for others to consider applications. As the probability distribution is not invariant under similarity transformations, the diagonalizing parameters must be integrated out by brute force. It was found that only a distribution with a Gaussian weight (Ginibre ensemble) could be solved. The Ginibre ensemble was later found to exhibit cubic eigenvalue repulsion in the complex plane [7,2]. Such models are of interest in characterizations of quantum chaos. Cubic quasienergy level repulsion is a key signature of classical chaos within a quantum dissipative system, as defined by Haake [7].

Normal matrices are discussed in most matrix theory texts (see [8,9]). Defined by the sole constraint that they commute with their adjoint, they have the property of being the most general matrix that can be diagonalized by a unitary transformation. The normal matrix model was first introduced in showing how the Laughlin wave function could be modeled by it and offered some generalizations to inhomogeneous fields [10]. In [11] the statistics of eigenvalues of random normal matrices were first explored.

In this paper the universality of cubic eigenvalue repul-

sion in the complex plane for ensembles of random normal matrices will be shown. In Sec. II the probability distribution in the space of matrices, all correlation functions, and the eigenvalue densities will be obtained. In Sec. III the level spacing statistics of the eigenvalues obtained will be derived. The term level spacing is borrowed from the Wigner-Dyson ensembles and will represent here the spacing in the complex plane. In Sec. IV eigenvalues of numerical generated normal matrices will be studied and compared to the analytical results.

II. PROBABILITY DISTRIBUTIONS, CORRELATION FUNCTIONS, AND EIGENVALUE DENSITIES

We begin by defining the joint probability distribution (jpd) within the space of normal matrices,

$$P_N(\mathbf{M}) = C \exp[-\text{Tr}(V(\mathbf{M}, \mathbf{M}^\dagger))]. \quad (1)$$

Only potentials which are Hermitian will be considered, $V(\mathbf{M}, \mathbf{M}^\dagger) = V(\mathbf{M}, \mathbf{M}^\dagger)^\dagger$. As a result, the joint probability distribution for the eigenvalues will be rotationally symmetric in the complex plane. Normal matrices are the most general matrices which can be diagonalized by a unitary transformation, $\mathbf{M} = \mathbf{U}\mathbf{Z}\mathbf{U}^\dagger$, so that the measure and the weight are invariant. Thus it is simple to derive the Jacobian for the transformation to the eigenvalues.

Proceeding analogously to the Hermitian matrix case [12], the metric in the space of normal matrices is defined as

$$\begin{aligned} (ds)^2 &= \text{Tr}[d\mathbf{M}^\dagger d\mathbf{M}] = \text{Tr}(\mathbf{U}d(\mathbf{U}^\dagger \mathbf{Z} \mathbf{U})\mathbf{U}^\dagger \mathbf{U}d(\mathbf{U}^\dagger \mathbf{Z}^* \mathbf{U})\mathbf{U}^\dagger) \\ &= \text{Tr}([d\mathbf{S}, \mathbf{Z}][d\mathbf{S}, \mathbf{Z}^*] + ([d\mathbf{S}, \mathbf{Z}]d\mathbf{Z}^* + d\mathbf{Z}[d\mathbf{S}, \mathbf{Z}^*]) \\ &\quad + |d\mathbf{Z}|^2) = c \sum_{i < k} |dS_{ik}(z_k - z_i)|^2 + \sum_i |dz_i|^2, \quad (2) \end{aligned}$$

where $d\mathbf{S} = \mathbf{U}d\mathbf{U}^\dagger = -d\mathbf{U}\mathbf{U}^\dagger$ is anti-Hermitian and c is an overall constant.

Using the standard Riemannian volume form with this metric, we get

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$$d\mu(\mathbf{M}) \rightarrow d\mu(z) = (d\Omega) |\Delta(z)|^2 dz_1^{(0)} \wedge dz_2^{(0)} \wedge \dots \wedge dz_N^{(0)} \\ \wedge dz_1^{(1)} \wedge \dots \wedge dz_N^{(1)}, \quad (3)$$

where $d\Omega$ is the volume of the unitary group $U(N)$ and $\Delta(z)$ is the well known Van der Monde determinant,

$$|\Delta(z)|^2 = \prod_{i < j}^N |z_j - z_i|^2. \quad (4)$$

If we consider weights which are invariant under unitary transformations we can factor $d\Omega$ out,

$$P_N(z) = C' |\Delta(z)|^2 \exp \left[- \sum_{i=1}^N V(z_i, z_i^*) \right], \quad (5)$$

where C' is a normalization constant.

Considering a Gaussian weight $V = |z|^2$ we notice that the jpd is identical to Ginibre's ensemble of complex matrices [6]. The model with this weight function is termed the normal Gaussian ensemble (NGE). It is clear that Gaussian ensembles of complex matrices and the NGE are identical. Any result derived here for the NGE will also be valid for Ginibre's ensemble of matrices. This is of little surprise since $\text{Tr}[\mathbf{M}^\dagger \mathbf{M}] = \text{Tr}[\mathbf{M} \mathbf{M}^\dagger]$.

The difference occurs when considering other ensembles. For complex matrices the parameter β (defined as the power of the Van der Monde determinant appearing in the jpd) is a function of the weight. In fact, only models involving Gaussian weights have ever been studied as this is the only non-trivial model which allows a separation of radial and angular parameters. For normal matrices $\beta=2$ regardless of the weight and it is possible to study a wide variety of ensembles.

For matrix models with one-dimensional eigenvalues the traditional method of analysis employs a basis of orthogonal polynomials. These allow the reduction of the determinant upon integration of a number of eigenvalues [2]. For complex eigenvalues we can also introduce an orthogonal basis. If a polar basis is chosen [and $P_N(z)$ is rotationally symmetric], it is simple to verify that the z 's themselves form an orthogonal basis. A basis of *orthogonal monomials*, $\{\phi_l\}$, is defined,

$$\phi_l(z_i) \equiv \frac{z_i^l}{N_l^{1/2}}, \quad (6)$$

where $N_l = \int_0^\infty dr^2 r^{2l} \exp[-V(r)]$ is a normalization constant.

The jpd can then be expressed as

$$P_N(z) = C' e^{-\sum_{i=1}^N V(z_i, z_i^*)} \det K_N(z_i, z_j) |_{i,j=1,N}, \quad (7)$$

where the "kernel" is

$$K_N(z_i, z_j) = \sum_{l=0}^{N-1} \phi_l(z_i) \phi_l^*(z_j). \quad (8)$$

The weight function is deliberately factored out of the kernel since all of the angular dependence is contained within K .

Similarly to Hermitian matrix models, when $N-n$ eigenvalues are integrated out of the jpd, the determinant shrinks by $N-n$ columns and rows [11]. This property allows the n -point correlation function to be derived easily,

$$R_n(z_1, \dots, z_n) \equiv \frac{N!}{(N-n)!} \int \prod_{i=n+1}^N \frac{1}{2} d\theta_i dr_i^2 P_N(z) \\ = e^{-\sum_{i=1}^n V(r_i)} \det K_N(z_i, z_j) |_{i,j=1,n}. \quad (9)$$

The prefactor $N!/(N-n)!$ is due to the ordering of eigenvalues. The two-point correlation function and eigenvalue density are

$$R_2(z, w) = \frac{e^{-V(z)-V(w)}}{\pi^2} [\rho(z)\rho(w) - K(z, w)K(w, z)], \\ \rho(z) = e^{-V(z)} \sum_{l=0}^{N-1} \frac{|z|^2}{N_l}. \quad (10)$$

In deriving expressions for the correlation functions, orthogonal monomials provide a more economical method. However, there are no simple recursion relations among the monomials as there are for the orthogonal polynomials. No recursion relation implies no Christoffel-Darboux formula and no asymptotic analysis. However, we can find analytical results for finite N . It is not necessary to examine the asymptotic form of the monomials to get large N forms.

In Table I various ensembles are defined and their corresponding eigenvalue density and two-point correlation function are given. It should be noticed that several of these ensembles allow *global* closed forms in the limit $N \rightarrow \infty$.

III. LEVEL SPACING STATISTICS

In order to study the properties of random matrices, the statistical properties of the spacings between eigenvalues (or level spacings) are examined. These statistics can be used as a definition of quantum chaos. The distinction of classically integrable and chaotic systems can be seen in a semiclassically quantized system as the transition from Poisson to Wigner-Dyson statistics of the nearest neighbor distribution of the energy levels [13]. It has been shown [7] that if there is dissipation in the system the level spacings undergo a similar transition from a plane Poisson distribution (random points in a plane) to Ginibre's distribution [6] generated by an ensemble of random complex matrices.

The distributions for eigenvalues in the complex plane will require the definition of statistical quantities analagous to those defined for Hermitian matrix models [2].

(i) The probability that no eigenvalue lies within a circle of radius s centered upon the point w , the *gap distribution*, is denoted as $E(s, w)$. A related quantity $E(s, w, \epsilon)$ is defined as the probability that no eigenvalues lie within an annulus of outer radius s , inner radius ϵ , and centered on the point w .

(ii) The probability that no eigenvalues lie within this same circle but one or more eigenvalues lie on the perimeter is denoted as $F(s, w)$.

(iii) For an eigenvalue lying at the center of the circle and

TABLE I. Ensembles of random normal matrices defined by their probability distribution. Also shown are the corresponding eigenvalue density and two point function. For the NGE, Laguerre $a=0$, and Legendre ensembles the closed form large N result has been shown.

Ensemble	Probability distribution	Density, ρ	$\rho(r_1)\rho(r_2) - R_2$
NGE	$e^{-\text{Tr}\mathbf{M}^\dagger\mathbf{M}}$	$\frac{1}{\pi}$	$\frac{1}{\pi^2}\exp[- z-w ^2]$
NQE	$e^{-\text{Tr}(\mathbf{M}^\dagger\mathbf{M} - g\mathbf{M}^\dagger\mathbf{M}\mathbf{M})}$	$\frac{e^{-r^2-gr^4}}{\pi} \sum_{l=0}^{N-1} \frac{r^{2l}}{N_l(g)}$	$\frac{e^{-r_1^2-r_2^2-gr_1^4r_2^4}}{\pi^2} K_N(z_1, z_2) K_N(z_2, z_1)$
Laguerre	$e^{-\text{Tr}[(\mathbf{M}^\dagger\mathbf{M})^{1/2} + (a/2)\ln(\mathbf{M}^\dagger\mathbf{M})]}$	$\frac{e^{-r}r^{a+1}}{2\pi} \sum_{l=0}^{N-1} \frac{r^{2l}}{\Gamma[2l+a+2]}$	$\sum_{l=0}^{N-1} \frac{(r_1r_2)^{a+1} e^{-r_1-r_2} z_1^{l*} z_2^{l*} z_1^{*m} z_2^{*m}}{4\pi^2 \Gamma[2l+a+2] \Gamma[2l+a+2]}$
Laguerre, $a=0$	$e^{-\text{Tr}(\mathbf{M}^\dagger\mathbf{M})^{1/2}}$	$\frac{e^{-r}}{2\pi r} \sinh(r)$	$\frac{e^{-r_1-r_2}}{4\pi^2 r_1 r_2} \sinh(\sqrt{z_1 z_2^*}) \sinh(\sqrt{z_1^* z_2})$
Legendre	1, (evals \in unit circle)	$\frac{1}{\pi(1-r^2)^2}$	$\frac{1}{\pi^2} \left[\frac{1}{(1-z_1 z_2^*)^2 (1-z_1^* z_2)^2} \right]$
Jacobi	$\det[1 - (\mathbf{M}^\dagger\mathbf{M})^{1/2}]^a$	$\frac{(1-r)^a}{2\pi} \sum_{l=0}^{N-1} \frac{r^{2l}}{B[a+1, 2l+2]}$	$\frac{1}{4\pi^2} \sum_{l=0}^{N-1} \frac{z_1^{l*} z_2^{l*} z_1^{*m} z_2^{*m}}{B[a+1, 2l+2] B[a+1, 2m+2]}$
Gen. Gaussian	$e^{-\text{Tr}[\mathbf{M}^\dagger\mathbf{M} - (a/2)\ln(\mathbf{M}^\dagger\mathbf{M})]}$	$\frac{e^{-r^2} r^a}{\pi} \sum_{l=0}^{N-1} \frac{r^{2l}}{\Gamma[l + \frac{a}{2} + 1]}$	$\sum_{l=0}^{N-1} \frac{(r_1 r_2)^a e^{-r_1^2 r_2^2} z_1^{l*} z_2^{l*} z_1^{*m} z_2^{*m}}{\pi^2 \Gamma[l + \frac{a}{2} + 1] \Gamma[m + \frac{a}{2} + 1]}$

no eigenvalues within the circle nor on the edge, this probability is denoted by $H(s, w)$.

(iv) The level spacing probability $p(s, w)$ is the most quoted quantity and is defined as the probability that one eigenvalue lies at the center and one or more lies on the perimeter.

By taking differential areas at the center and at the perimeter of this circle, we can obtain all of the probabilities from the gap distribution. The argument is analogous to Mehta's for one-dimensional eigenvalues and can be found in [11].

$$E(s, w) = \int \{dz\} P_N(\{z\}) \prod_{i=1}^N [1 - \chi(z_i, s, w)], \quad (11)$$

$$F(s, w) = -\frac{d}{ds} E(s, w), \quad (12)$$

$$H(s, w) = -\frac{d}{d\epsilon^2} E(s, w, \epsilon) \Big|_{\epsilon=0}, \quad (13)$$

$$\begin{aligned} p(s, w) &= -\frac{d}{ds} H(s, w) = -\frac{d}{d\epsilon^2} F(s, w, \epsilon) \Big|_{\epsilon=0} \\ &= \frac{d}{ds} \frac{d}{d\epsilon} E(s, w, \epsilon) \Big|_{\epsilon=0}, \end{aligned} \quad (14)$$

where χ is the characteristic function and ϵ is taken as the radius of an infinitesimal area centered at the center of the circle. These relations allow an elegant derivation of the level spacing distribution.

All of the probability distributions considered here are symmetric in the z_i , hence the derivation of the gap distribution will be the same as in [2] (see also [11]). For the circle centered at the origin we have the following:

$$E(s, 0) = \prod_{i=0}^{N-1} \left[\frac{\int_{s_2}^{\infty} dx \exp(-V(x)) x^i}{\int_0^{\infty} dx \exp(-V(x)) x^j} \right], \quad (15)$$

where $x = |z|^2$.

As a check on our results, we find the spacing distribution for the NGE and compare with previously known results. In this case the probability distribution is translationally invariant, hence the result (15) is valid for all w . The result for small s is

$$p(s) \approx 2s^3 - s^5 + \frac{1}{7} s^7 - \frac{11}{12} s^9 \dots \quad (16)$$

This result is the same as for complex matrices derived by Haake [7]. The level spacing statistics at the origin for other ensembles of normal matrices can be found in [11]. Here we are concerned with obtaining a universal result for the spacing distribution.

For ensembles other than Gaussian, the distributions are not translationally invariant and hence it will be necessary to resort to asymptotic approximations. Shifting the eigenvalue coordinate $z = z - w$ does not affect the measure or Van der Monde determinant. The gap distribution can then be expressed as,

$$\begin{aligned}
E(s, w) &= \prod_{i=1}^N \int_0^\infty dr_i^2 d\theta_i [1 - \chi_i(s^2)] \tilde{P}_N(z) \\
&= \int_0^\infty \{d^2 z\} \left\{ 1 - \sum_{i=1}^N \chi_i(s^2) \right. \\
&\quad \left. + \sum_{i \neq j=1}^N \chi_i(s^2) \chi_j(s^2) - \dots \right\} \tilde{P}_N(z) \\
&= 1 - \sum_i^N \int d^2 z_i \chi_i \tilde{\rho}(z_i) \\
&\quad + \sum_{i \leq j}^N \int d^2 z_i d^2 z_j \chi_i \chi_j \tilde{R}_2(z_i, z_j) - \dots, \quad (17)
\end{aligned}$$

where $\tilde{f}(z) = f(z-w)$ and χ_i is the characteristic function for the i th eigenvalue. In the limit $s \rightarrow 0$, and keeping only the first two terms, we obtain

$$\lim_{s \rightarrow 0} E(s, w) \approx 1 - N\pi s^2 \rho(w). \quad (18)$$

The level spacing distribution $p(s, w)$ can be found via Eq. (14). First the gap distribution $E(s, w, \epsilon)$ needs to be derived. The derivation is the same as before except for a modification to the characteristic function. Starting from Eq. (17), we have

$$\begin{aligned}
E(s, w, \epsilon) &= 1 - \sum_i^N \int d^2 z_i \chi_i \tilde{\rho}(z_i) \\
&\quad + \sum_{i \leq j}^N \int d^2 z_i d^2 z_j \chi_i \chi_j \tilde{R}_2(z_i, z_j) - \dots, \quad (19)
\end{aligned}$$

where now $\chi_i = \theta(s^2 - r_i^2) \theta(r_i^2 - \epsilon^2)$. As an intermediate step we find

$$\begin{aligned}
H(s, w) &= -\frac{d}{d\epsilon^2} E(s, w, \epsilon) \Big|_{\epsilon=0} \\
&= N \frac{d}{d\epsilon^2} \int_{\epsilon^2}^{s^2} dr^2 \int d\theta \rho(z+w) - \frac{N^2 + N}{2} \frac{d}{d\epsilon^2} \\
&\quad \times \int_{\epsilon^2}^{s^2} \int_{\epsilon^2}^{s^2} \frac{dr_1^2 dr_2^2}{4} \int \int d\theta_1 d\theta_2 R_2 \\
&\quad \times (z_1 + w, z_2 + w) + \dots \Big|_{\epsilon=0} \\
&= -N\pi \rho(w) + \pi(N^2 + N) \\
&\quad \times \int_0^{s^2} \frac{dr_2^2}{2} \int d\theta_2 R_2(w, z_2 + w) - \dots. \quad (20)
\end{aligned}$$

The last term exploits the symmetry in R_2 . For the level spacing we have

$$p(s, w) = -\pi(N^2 + N) s \int d\theta_2 R_2(w, s e^{i\theta_2} + w) \dots \quad (21)$$

In order to find the lowest order contribution to $p(s, w)$ the lowest order of R_2 will need to be found. For a model with weight $e^{-V(z)}$ we have

$$R_2(w, z+w) = e^{-V(w)} e^{-V(z+w)} \sum_{l=1}^N \sum_{m=0}^N \left[\frac{r_w^{2l} (r^2 + zw^* + wz^* + r_w^2)^m}{N_l N_m} - \frac{(zw^* + r_w^2)^l (wz^* + r_w^2)^m}{N_l N_m} \right]. \quad (22)$$

The normalization factor N_l is model dependent. It is easily verified that terms with $l=0$ and $l=m$ vanish. All higher order functions will have the same property.

The remaining angular integral over $d\theta_2$ in Eq. (21) insures that no terms linear in z or z^* appear. It is also apparent that the leading term in r_w cancels (terms to order z^0). This yields the lowest order of R_2 to be s^2 . Combined with Eq. (14), it is found that the nearest level spacing is cubic in s as $s \rightarrow 0$. A few exceptions occur when the weight function vanishes as z vanishes. Specifically the Jacobi, Laguerre ($a > 1$), and generalized Gaussian ($a > 1$) will have a higher order repulsion. Our final result is as follows. Normal matrix models with normalizable Hermitian weight functions have eigenvalues experiencing a minimum of cubic repulsion in the complex plane.

It is also worthwhile to examine the case of real normal matrices. Here the constraint is that the matrix commutes with its transpose. Such a matrix is diagonalizable by a unitary transformation and will in general have complex eigenvalues. Retracing the steps above, it is easy to see that the eigenvalues will also exhibit a minimum of cubic repulsion in the complex plane.

IV. NUMERICAL RESULTS

Whenever discussing random matrix theory, it is always of interest to include numerical simulation of large random matrices. These are used to verify the analytical properties and supply support for asymptotic forms, conjectures, etc. When dealing with normal matrices, however, we are in a

predicament. Consider the following problem: Write down an arbitrary 2×2 normal matrix that is neither Hermitian, unitary, nor real. It is not so simple. If one was asked to write down a Hermitian matrix, it could be done without thinking twice. For a normal matrix there is no simple parametrization. It would most likely require the commutator to be calculated and a solution found by trial and error. Likewise, if you want a computer to generate normal matrices you must give it a parametrization. For a 2×2 you can parametrize it as follows, given that $n_{11}^* \neq n_{22}^*$ (remember there are $N^2 + N = 6$ independent real elements within a normal matrix):

$$\mathbf{N} = \begin{pmatrix} n_{11} & n_{12} \\ n_{12}^* \left(\frac{n_{11} - n_{22}}{n_{11}^* - n_{22}^*} \right) & n_{22} \end{pmatrix}. \quad (23)$$

For arbitrary N the constraint equations become nonlinear and highly coupled. We will choose the following parametrization of normal matrices and define the independent elements as n_{ij} (those along the diagonal and above the diagonal). Define the elements that are constrained as f_{ji} (those which lie below the diagonal). For the j th element the constraint equation is

$$\begin{aligned} f_{ji} = & \frac{1}{(n_{jj}^* - n_{ii}^*)} \left[n_{ij}^* (n_{jj} - n_{ii}) + \sum_{k=1}^{i-1} n_{kj}^* n_{ki} - \sum_{m=j+1}^N n_{im}^* n_{jm} \right] \\ & - \frac{1}{(n_{jj}^* - n_{ii}^*)} \left[\sum_{k=1}^{i-1} f_{jk} f_{ik}^* - \sum_{m=j+1}^N f_{mi} f_{mj}^* \right. \\ & \left. - \sum_{l=i+1}^{j-1} (n_{lj}^* f_{li} - n_{il}^* f_{jl}) \right]. \quad (24) \end{aligned}$$

Notice that the term in the first bracket contains only independent elements. With this parametrization the probability of obtaining normal matrices which are Hermitian, unitary, orthogonal, or real symmetric matrices becomes infinitesimal for large N . The subspaces for these types of matrices within the space of normal matrices are of much lower dimensionality. There are many solutions to this nonlinear equation. Another method must be devised to obtain a normal matrix. It is also found that due to the nonlinearity, only normal Gaussian matrices can be generated. (Others not respecting the Hermiticity and normalizability could be analyzed, but this is not of interest here.)

Instead of directly attempting to produce random normal matrices, we begin with a random complex matrix and use an optimization algorithm to obtain a normal matrix. The following algorithm is the simplest to implement but by no means the most efficient. (i) Parametrize the normal matrix as before: Elements along the diagonal and above the diagonal are independent parameters. Elements below the diagonal will be optimized.

(ii) Using the Box-Muller technique, generate random Gaussian variables and construct a 200×200 complex Gaussian matrix. This is the initial condition.

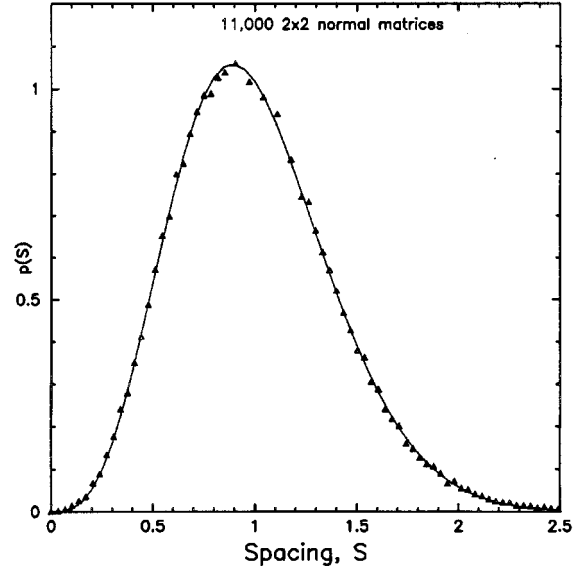


FIG. 1. The level spacing distributions $p(S)$ for $N=2$, analytical form, and 11 000 numerically generated normal matrices.

(iii) Calculate the commutator and define the error function $\text{tr}[\mathbf{N}, \mathbf{N}^\dagger]^2 = E$. The error function will be minimized below a chosen threshold (a value of 0.2 is used).

(iv) Use gradient descent to adjust the constrained elements. If the value of E is below the chosen threshold, then pass the matrix to the diagonalization routine. If E has not converged after 500 iterations, then reset the matrix and go to step 2.

The results of a preliminary numerical study based on this algorithm are now presented. A full discussion of the algorithm used is to be found elsewhere [11]. Due to limited computational resources and time, only 50 matrices of size 200×200 have been analyzed. The level spacing statistics for the generated matrices is compared to the analytical result.

Before attempting to generate large normal matrices, the algorithm was checked with 2×2 matrices. As these were easily produced, 11 000 were generated and compared to the exact analytical result,

$$p_{2 \times 2}(s) = \frac{\pi^2 s^3}{10.4683} e^{-(1/4)s^2} I_0\left(\frac{s^2}{4}\right). \quad (25)$$

The eigenvalues are unfolded so that the mean spacing is unity. This is achieved by rescaling the spacing and renormalizing the spacing distribution so that the mean spacing between eigenvalues is unity,

$$S \equiv 1.6622s, \quad p(S) \equiv p(s)/10.4883. \quad (26)$$

Plots of the analytical form and the unfolded eigenvalues are displayed in Fig. 1.

For normal matrices of size 200×200 the exact analytical form for the spacing distribution can easily be found from Eqs. (11)–(14),

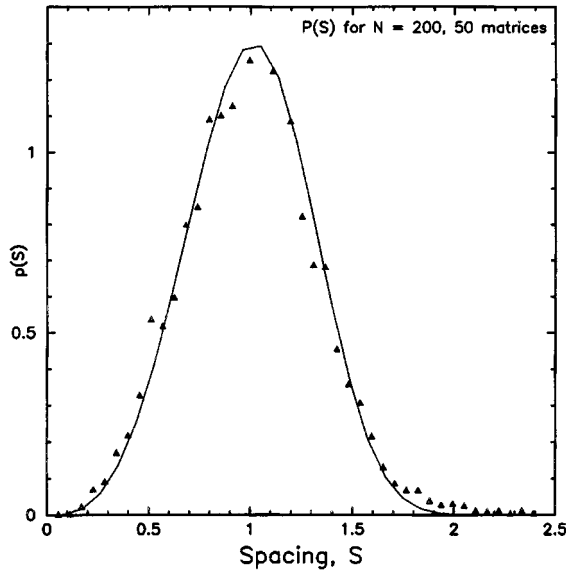


FIG. 2. The level spacing distributions $p(S)$ for $N=200$, analytical result (with eigenvalues unfolded), and 50 numerically generated 200×200 normal matrices.

$$p_{200}(s) = \left(199 - \sum_{q=1}^{199} \frac{e_{q-1}(s^2)}{e_q(s)} \right) 2s \prod_{i=1}^{199} [e^{-s^2} e_i(s^2)], \quad (27)$$

where $e_q(s^2) = \sum_{l=0}^q (s^{2l}/l!)$. Under the optimization algorithm it is found that many eigenvalues “leak” out into the complex plane. Adding a further constraint to keep the eigenvalues within a radius of $\sqrt{2\sigma N - 1} \approx 20$ lengthens the generation time dramatically. Therefore, for this preliminary study the eigenvalues were “pruned,” those falling outside $r = 18.1$ were neglected (the number was chosen so that edge effects would be eliminated as well). Overall, about one-third of the eigenvalues were pruned. The results are displayed in Fig. 2.

The slight deviation of the numerical and analytical results for large S is due to the leakage of eigenvalues. The small S behavior is not affected by the leakage. It is found that changing the initial conditions has little effect on $p(S)$. It is also observed that lowering the threshold for the error function by a factor of 10 has a negligible effect.

V. CONCLUSION

It has been shown that eigenvalues obtained from random normal matrices experience a minimum of cubic repulsion in the complex plane. Even though normal matrices are subsets of complex matrices we have demonstrated that Gaussian normal matrix models yield the same results as the Ginibre

ensemble of complex matrices. One can define ensembles of complex matrices with weights invariant under similarity transformations (see Appendix B of [11]). These ensembles will fall under $\beta=4$ and experience a minimum of quartic repulsion. (Getting analytical results for the statistics of the eigenvalues is difficult, however.) Ensembles of complex matrices with weight functions invariant under unitary transformations other than the Gaussian are not factorable and are not amenable to analytical analysis. For normal matrices we can define identical ensembles to complex models invariant under similarity transformations and as well define a wide range of ensembles invariant under unitary transformations (as demonstrated in this paper). The special case of real normal matrices was also shown to experience cubic repulsion in the complex plane. As well, there are even more significant differences between random normal matrices and more traditional random matrix models composed of Hermitian, unitary, and real symmetric matrices, themselves being subsets of normal matrices. Although numerical generation of normal matrices is not as straightforward as for other types of matrices, we have shown that analytical results are often much easier to obtain. What numerical information we could obtain does agree with the theoretical result.

It is a little disappointing that universality, in the sense of Brezin and Zee [14], is not found for normal matrix models. The simple scaling arguments that lead to universal kernels for other types of matrices do not work here. It may be possible to rescale S to obtain a universal spacing distribution $P(S)$. This allows the potential of analytically exploring the statistics of complex eigenvalues, which was not possible before. Another interesting feature is the closed form expressions for the two point correlation functions in NGE, Laguerre ($a=0$), and Legendre ensembles in the infinite matrix limit (see Table I).

In terms of physical applications it should be noted that results using the Ginibre ensemble of complex matrices can be duplicated with the normal Gaussian ensemble. Such work includes modeling cellular structures by Le Caer and Ho [4], two-dimensional quantum gravity by Morris [15], and Haake’s work on generators of quantum dynamics for systems with dissipation [7]. It has recently been shown [16] that Haake’s generator of dissipation, D , in [17,7], is properly represented by a normal matrix rather than a complex one. This result strengthens the idea that the eigenvalues of these generators universally exhibit cubic repulsion when dissipative dynamics is present.

The results obtained in this work give a more complete picture of the statistics of eigenvalues from random matrices. The normal matrix model, a link between the Dyson ensembles and the Ginibre ensemble, has a rich structure where interesting results can still be found.

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