Analysis of minimal pinning density for controlling spatiotemporal chaos of a coupled map lattice

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Feedback pinning is a useful method in controlling spatiotemporal chaos of a coupled map lattice (CML). We analytically derive the minimal pinning density for controlling spatiotemporal chaos in a general one-dimensional CML. The results give a general condition for controlling a CML to the desired states. The results are verified by numerical simulations of the coupled logist map. [S1063-651X(97)08302-5]

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Chaos is common and has been extensively studied in the last decade. Chaos is a beneficial feature in some cases, however, it is an undesirable phenomenon in many situations. The problem of controlling chaos, that is, to convert the chaotic behavior to a periodic one, has attracted much interest [1–4]. However, the majority of these studies are concerned with the control of temporal chaos in finite- and low-dimensional systems. There are many practical systems where both spatial and temporal chaos exist. Controlling such spatiotemporal chaos is also important. But the study of controlling spatiotemporal chaos, in spite of its importance, is still in an early stage.

The high dimensionality of the spatially extended systems causes many difficulties both in analytical and numerical studies. The relatively simple coupled map lattice (CML) model is often taken as a convenient tool to study the characteristics of spatiotemporal systems [5]. The CML has been used to model various phenomena in hydrodynamics, optics, and solid-state physics [6,7]. Furthermore, some physical experiments have been carried out [8].

Some methods of controlling spatiotemporal chaos were suggested [9–11]. Among these studies, Gang and Zhilin suggested a method in which they put some local controls (pinnings) in space [9]. Doing so, they could control the total systems by modulating very few freedoms. Gang and Zhilin verified the minimal pinning density for controlling spatiotemporal chaos through numerical analysis of the coupled logist map. The result is only a numerical analysis and cannot be applied to other systems.

In this paper we approximate a CML with pinnings to a linear system, and analytically derive the minimal pinning density for controlling a CML to the desired state for a general one-dimensional CML. The results give a general condition for controlling a CML to the desired states.

Let us consider a well known one-dimensional CML model [5].

$$x_{n+1}(i)=(1-\epsilon)f[x_n(i)]+\tfrac{1}{2}\epsilon\{f[x_n(i-1)]+f[x_n(i+1)]\}, \tag{1}$$

where $i=1,2,\ldots,L$ are the lattice sites and L is the system size. $f[\ ]$ is a one-dimensional chaotic map. The parameter ϵ is coupling strength and constrained to $0<\epsilon<1$. In the case of $\epsilon=0$, the map lattice is called an uncoupled map lattice, and all sites in the lattice then become independent of one another. The periodic boundary condition, $x_n(i+L)=x_n(i)$, is assumed. To control this system, a control term (pinning) is added on the right hand side of Eq. (1).

$$x_{n+1}(i)=(1-\epsilon)f[x_n(i)]+\tfrac{1}{2}\epsilon\{f[x_n(i-1)]+f[x_n(i+1)]\}+\sum_{k=0}^{L/I}\delta(i-Ik-1)u_n(i), \tag{2}$$

where I is the distance between two neighboring pinnings and $1/I$ is called the pinning density. That is, pinning, $u_n(i)$, is applied to the $(Ik+1)$th site. We analyze the CML with pinnings through linear analysis, which gives the necessary condition for stabilizing the CML to the desired states $\overline{x_n}(i)$.

First, we consider the case where the desired state $\overline{x_n}(i)$ is a fixed point, i.e., $\overline{x_n}(i)=\overline{x}$ and $\overline{x}=f(\overline{x})$. It is assumed that the controlled sites converge to the desired state $\overline{x}$. That is, after transient time, the controlled sites can be approximated to $x_n(Ik+1)\cong\overline{x}$, $k=0,1,2,\ldots,L/I-1$. We expand the Taylor series of $f[\ ]$ at $\overline{x}$ and ignore the higher order terms. Then the CML with pinnings is approximated to $(I-1)$-dimensional linear systems.

$$\mathbf{E}_{n+1}(k)\cong\mathbf{A}\mathbf{E}_n(k),\quad k=0,1,2,\ldots,L/I-1$$

$$\mathbf{A}=\begin{bmatrix} (1-\epsilon)Df & \frac{1}{2}\epsilon Df & 0 & 0 & 0 & \cdots & 0 \\ \frac{1}{2}\epsilon Df & (1-\epsilon)Df & \frac{1}{2}\epsilon Df & 0 & 0 & \cdots & 0 \\ 0 & \frac{1}{2}\epsilon Df & (1-\epsilon)Df & \frac{1}{2}\epsilon Df & 0 & \cdots & 0 \\ \vdots & & & & & & \vdots \\ 0 & 0 & \cdots & 0 & \frac{1}{2}\epsilon Df & (1-\epsilon)Df & \frac{1}{2}\epsilon Df \\ 0 & 0 & 0 & \cdots & 0 & \frac{1}{2}\epsilon Df & (1-\epsilon)Df \end{bmatrix}, \tag{3}$$

where $\mathbf{E}_n(k)=[e_n(Ik+2)\ e_n(Ik+3)\ \cdots\ e_n(Ik+I)]^T$, $e_n(i)=x_n(i)-\overline{x_n}(i)$, and Df is the Jacobian of f at $\overline{x}$. Let an eigenvalue of $\mathbf{A}$ be λ_j, $j=1,2,\ldots,I-1$. Then λ_j must be $|\lambda_j|<1$ to control all sites to $\overline{x}$. We rewrite $\mathbf{A}$ as $\mathbf{A}=Df(\mathbf{I}-\epsilon(\mathbf{I}+\mathbf{A}'))$, where $\mathbf{I}$ is an $(I-1)$-dimensional identity matrix and

$$\mathbf{A}'=\begin{bmatrix} 0 & -1/2 & 0 & 0 & 0 & \cdots & 0 \\ -1/2 & 0 & -1/2 & 0 & 0 & \cdots & 0 \\ 0 & -1/2 & 0 & -1/2 & 0 & \cdots & 0 \\ \vdots & & & & & & \vdots \\ 0 & 0 & \cdots & 0 & -1/2 & 0 & -1/2 \\ 0 & 0 & 0 & \cdots & 0 & -1/2 & 0 \end{bmatrix}.$$

Then, λ_j can be rewritten as $\lambda_j=Df(1-\epsilon(1+\lambda_j'))$, where λ_j' is an eigenvalue of $\mathbf{A}'$ and $\lambda_j'=\cos(j\pi/I)$, $j=1,2,\ldots,I-1$ [12]. The condition $|Df(1-\epsilon(1+\lambda_j'))|<1$ must be satisfied to control the system.

To obtain the relationship among I, ϵ, and Df, we solve the above inequality. The results are (i) if $|Df|<1$, then the system is always stable, and (ii) if $|Df|>1$, then the system is stable if

$$|Df|<1/\cos(\pi/I), \tag{4}$$

$$\epsilon>[1-(1/|Df|)]\frac{1}{1-\cos(\pi/I)}. \tag{5}$$

Since $\cos(\pi/I)$ increases to 1 as I increases, more pinnings are necessary as $|Df|$ increases. From Eq. (5) we know that the larger $|Df|$ is or the larger I is, the larger ϵ is needed. In Fig. 1, we plot I_m in the $|Df|$-ϵ plane, where $1/I_m$ is the minimal pinning density. From the figure, it is clear that the minimal pinning density can be considerably reduced by decreasing $|Df|$ and increasing ϵ. When $|Df|$ approaches 1, the minimal pinning density $1/I_m$ approaches zero. In other words, if the local system $f[\ ]$ is stable, then the total system is stable for arbitrary ϵ and I. In [9], Gang and Zhilin showed the numerical result of I_m for the coupled logist map, which is much the same as Fig. 1.

Now, we consider the case where the desired orbit is a time period 2 orbit, i.e., $\overline{x_1}=f(\overline{x_2})$, $\overline{x_2}=f(\overline{x_1})$, and $\overline{x_1}\neq\overline{x_2}$. Using a similar method, we obtain the $(I-1)$-dimensional linear equation,

$$\mathbf{E}_{n+1}(k)=Df_1Df_2\mathbf{A}''\mathbf{E}_n(k), \tag{6}$$

where

$$\mathbf{A}''=\begin{bmatrix} (1-\epsilon)^2+\frac{1}{4}\epsilon^2 & \epsilon(1-\epsilon) & \frac{1}{4}\epsilon^2 & 0 & 0 & 0 & 0 & \cdots & 0 \\ \epsilon(1-\epsilon) & (1-\epsilon)^2+\frac{1}{2}\epsilon^2 & \epsilon(1-\epsilon) & \frac{1}{4}\epsilon^2 & 0 & 0 & 0 & \cdots & 0 \\ \frac{1}{4}\epsilon^2 & \epsilon(1-\epsilon) & (1-\epsilon)^2+\frac{1}{2}\epsilon^2 & \epsilon(1-\epsilon) & \frac{1}{4}\epsilon^2 & 0 & 0 & \cdots & 0 \\ 0 & \frac{1}{4}\epsilon^2 & \epsilon(1-\epsilon) & (1-\epsilon)^2+\frac{1}{2}\epsilon^2 & \epsilon(1-\epsilon) & \frac{1}{4}\epsilon^2 & 0 & \cdots & 0 \\ \vdots & & & & & & & & \vdots \\ 0 & 0 & 0 & \cdots & 0 & \frac{1}{4}\epsilon^2 & \epsilon(1-\epsilon) & (1-\epsilon)^2+\frac{1}{2}\epsilon^2 & \epsilon(1-\epsilon) \\ 0 & 0 & 0 & 0 & \cdots & 0 & \frac{1}{4}\epsilon^2 & \epsilon(1-\epsilon) & (1-\epsilon)^2+\frac{1}{4}\epsilon^2 \end{bmatrix}$$

and Df_1 and Df_2 are the Jacobians of f at $\overline{x_1}$ and $\overline{x_2}$, respectively. In order to stabilize Eq. (6), the maximum absolute value of eigenvalues of the above system must be less than 1, i.e., $\max|Df_1Df_2\lambda_j''|<1$ where λ_j'' is an eigenvalue of $\mathbf{A}''$. In Fig. 2, we plot $\max|\lambda_j''|$ versus ϵ and I. as shown in Fig. 2, as ϵ decreases to 0 or I increases, $\max|\lambda_j''|$ converges to 1. Similarly to the case of a fixed point, coupling strength ϵ and pinning density $1/I$ must be large to stabilize the total

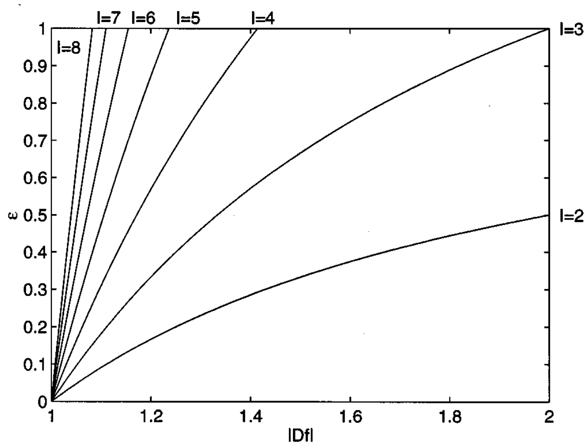

FIG. 1. The controllable regions of the fixed point. The line numbers indicate the minimal pinning I_m. Above each line the desired state $\overline{x}$ can be stabilized by the indicated pinnings.

system. In Fig. 3, I_m is shown in the $|Df_1 Df_2|$-ϵ plane. The results are very similar to those of Fig. 1.

Next we consider the case where the desired orbit is a spatial period 2 orbit,

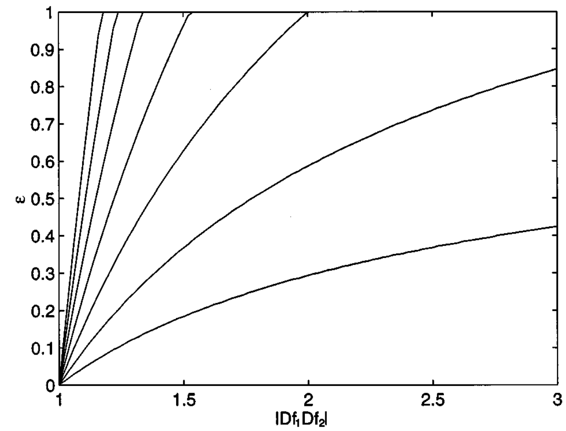

FIG. 3. The controllable regions of the time period 2 orbit. The line numbers indicate the minimal pinning I_m. Above each line the desired states can be stabilized by the indicated pinnings.

$$\overline{x_n}(2j-1)=\overline{x}(1), \quad \overline{x_n}(2j)=\overline{x}(2), \quad j=1,2,\ldots,L/2.$$

Using a similar method, we obtain the following $(I$-$1)$-dimensional linear equation:

$$\mathbf{E}_{n+1}(k)=\begin{bmatrix} (1-\epsilon)Df_1 & \frac{1}{2}\epsilon Df_2 & 0 & 0 & 0 & \cdots & 0 \\ \frac{1}{2}\epsilon Df_1 & (1-\epsilon)Df_2 & \frac{1}{2}\epsilon Df_1 & 0 & 0 & \cdots & 0 \\ 0 & \frac{1}{2}\epsilon Df_2 & (1-\epsilon)Df_1 & \frac{1}{2}\epsilon Df_2 & 0 & \cdots & 0 \\ \vdots & & & & & & \vdots \\ 0 & 0 & 0 & \cdots & 0 & \frac{1}{2}\epsilon Df_2 & (1-\epsilon)Df_1 \end{bmatrix}\mathbf{E}_n(k). \tag{7}$$

Equation (7) is obtained when the $(Ik+1)$th site is controlled to $\overline{x}(2)$. If the $(Ik+1)$th site is controlled to $\overline{x}(1)$, then Df_1 and Df_2 are interchanged with each other. The maximum absolute value of eigenvalues of the error system

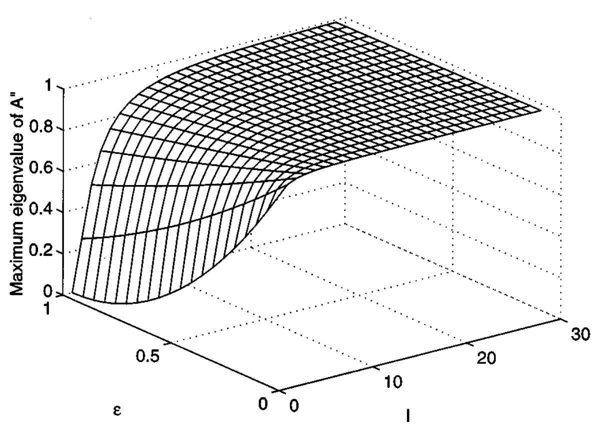

FIG. 2. The maximum absolute value of eigenvalues of $\mathbf{A}''$ for various ϵ and I.

(7) must be less than 1 to control the CML to the desired states. It is difficult to obtain I_m for the spatial period 2 orbit of the general CML. I_m for the spatial period 2 orbit of the coupled logist map is shown in Fig. 4.

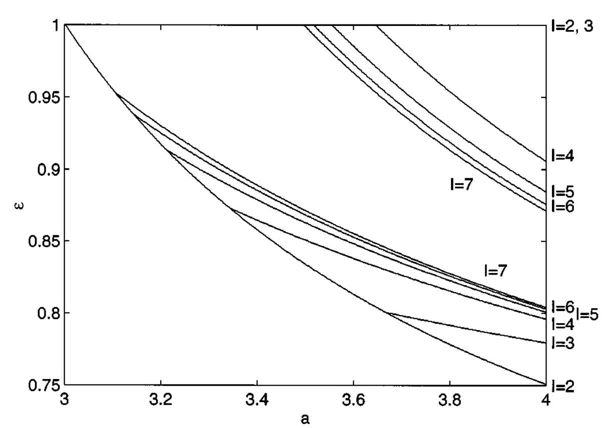

FIG. 4. The controllable regions of the spatial period 2 orbit. The line numbers indicate the minimal pinning I_m. Between the same numbered lines the desired orbit can be stabilized by the indicated pinnings.

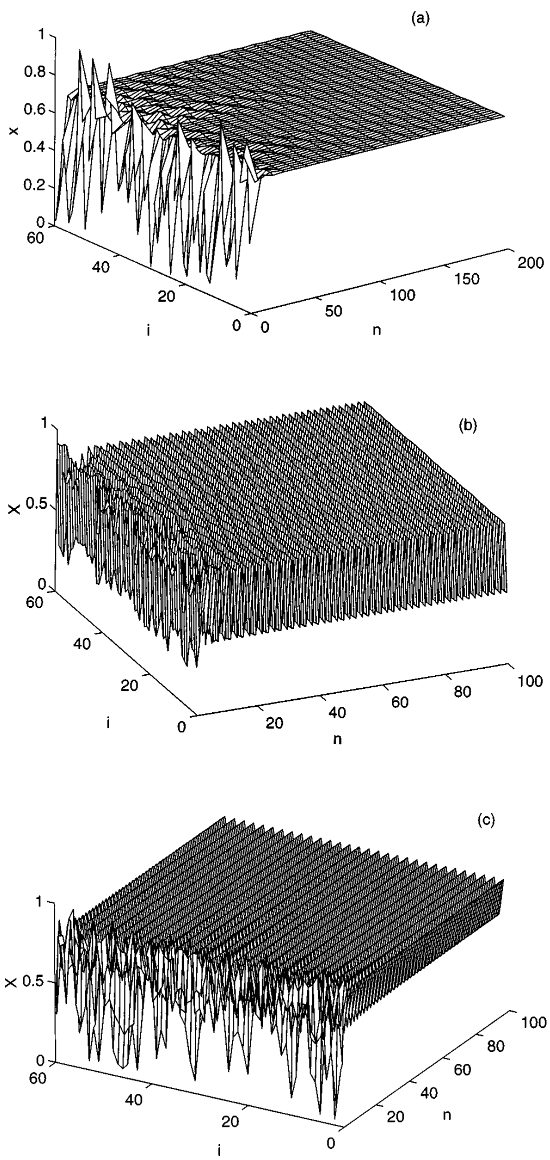

FIG. 5. (a) Space-time diagram of the CML controlled to the fixed point with $a=3.3$, $\epsilon=0.8$, $I=4$, $L=60$, and $p_n=3$. $\overline{x}=0.696\,969\ldots$. (b) Space-time diagram of the CML controlled to the time period 2 orbit with $a=3.5$, $\epsilon=0.4$, $I=4$, $L=60$, $p_{2m-1}=3$, and $p_{2m}=-1$, $m=1,2,\ldots$. $\overline{x_1}=0.857\,142\ldots$, and $\overline{x_2}=0.428\,571\ldots$. (c) Space-time diagram of the CML controlled to the spatial period 2 orbit with $a=3.6$, $\epsilon=0.84$, $I=4$, $L=60$, and $p_n=2.5$. $\overline{x(1)}=0.816\,756\ldots$ and $\overline{x(2)}=0.591\,740\ldots$.

To verify our analysis, we have applied the results to the coupled logist map. In the logist map, the first period-doubling bifurcation occurs at $a=3$ and it continues till $a=a_c=3.569\,945\,6\ldots$. Chaos can be found in the regime $a_c<a\leqslant 4$ [13]. In the simulation we set $L=60$ and $I=4$. ϵ and a are chosen to be close to the boundary value of stabilization. First we want to control the system to a fixed point, $\overline{x}=1\text{-}1/a$. We use feedback control $u_n(i)$ as Gang and Zhilin did [9].

$$u_n(i)=(1-\epsilon)p_n x_n(i)[x_n(i)-\overline{x_n}(i)] + \tfrac{1}{2}\epsilon\{p_n x_n(i-1)[x_n(i-1)-\overline{x_n}(i-1)] + p_n x_n(i+1)[x_n(i+1)-\overline{x_n}(i+1)]\} \quad (8)$$

The initial conditions are randomly chosen from 0 to 1 when the desired orbit is a fixed point or a spatial period 2 orbit. When the desired orbit is a time period 2 orbit, the initial condition is randomly chosen from $\overline{x_1}-0.1$ to $\overline{x_1}+0.1$ because the attracting basins of the time period 2 orbit are small. In Fig. 5(a) the space-time diagram is plotted with $\epsilon=0.8$, $a=3.3$, and $p_n=3$. As shown in Fig. 5(a), all sites converge to $\overline{x}$. For the case of time period 2 orbit, the desired states are $\overline{x_1}=[a+1+\sqrt{(a-3)(a+1)}]/2a$ and $\overline{x_2}=[a+1-\sqrt{(a-3)(a+1)}]/2a$. The simulation is performed with $\epsilon=0.4$ and $a=3.5$. The control parameter p_n is chosen as $p_{2m-1}=3$ and $p_{2m}=-1$, where $m=0,1,2,\ldots$. The space-time diagram is shown in Fig. 5(b). Similarly to the case of fixed point, all sites converge to the desired orbit after transient time. For the case of spatial period 2 orbit, the desired states are $\overline{x(1)}=(B+\sqrt{B^2-4C})/2a(2\epsilon-1)$ and $\overline{x(2)}=(B-\sqrt{B^2-4C})/2a(2\epsilon-1)$, where $B=1-a+2a\epsilon$ and $C=\epsilon-a\epsilon+2a\epsilon^2$. In Fig. 5(c) the space-time diagram is plotted with $\epsilon=0.84$, $a=3.6$, and $p_n=2.5$. Similarly to other cases, after transient time all sites are stabilized to the desired states.

In conclusion, we analyzed the minimal pinning density of the general one-dimensional CML and verified the results through numerical simulations of the coupled logist map. From the results, it is shown that the smaller coupling strength is or the larger the Jacobian of the local system is, the more pinnings are necessary. Too much coupling, however, has an adverse effect for the spatial period 2 orbit.

[1] M. J. Ogorzalek, IEEE Trans. Circuits Syst. **40**, 700 (1993).
[2] W. L. Ditto *et al.*, Physica D **86**, 198 (1995).
[3] E. Ott *et al.*, Phys. Rev. Lett. **64**, 1196 (1990).
[4] W. L. Ditto *et al.*, Phys. Rev. Lett. **65**, 3211 (1990).
[5] E. J. Ding and Y. N. Lu, J. Phys. A **25**, 2897 (1992); K. Kaneko, Physica D **23**, 436 (1986); **37**, 60 (1989); **34**, 1 (1989).
[6] H. Chate and P. Manneville, Physica D **32**, 409 (1988).
[7] I. S. Aranson *et al.*, Nonlinearity **3**, 639 (1990).
[8] R. Buskirk and C. Jeffries, Phys. Rev. A **31**, 3332 (1985).
[9] H. Gang and Q. Zhilin, Phys. Rev. Lett. **72**, 68 (1994).
[10] D. Auerbach, Phys. Rev. Lett. **72**, 1184 (1994).
[11] V. V. Astakhov *et al.*, IEEE Trans. Circuits Syst. **42**, 352 (1995).
[12] G. Strang, *Linear Algebra and Its Applications* (Academic Press, San Diego, 1980).
[13] J. M. T. Thompson and H. B. Stewart, *Nonlinear Dynamics and Chaos* (John Wiley and Sons, New York, 1987).