Dark solitons in weakly saturable nonlinear media

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Exact explicit analytical solutions of dark (gray and black) solitons to the quintic nonlinear Schrödinger equation that includes the first high order nonlinear saturable term are presented. The dark solitons in the weakly saturable self-defocusing media are shown to be stable to a small perturbation (noncollision type) and, as in the case of a Kerr nonlinearity, the two black solitons launched in parallel repel with propagation distance. However, when the two dark solitons are launched towards each other in tilted angles heading for a collision, the solitons in the saturable nonlinear media could not survive through a collision above a critical background intensity (although they survive below that critical intensity value). This is in contrast to the Kerr law nonlinearity. [S1063-651X(96)00812-4]

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The propagation of a soliton in an optical fiber or in a uniform nonlinear medium is governed by the nonlinear Schrödinger equation. For Kerr law nonlinearity, this governing equation of the cubic nonlinear Schrödinger equation is completely integrable and its soliton solutions could be derived by the inverse scattering method [1,2]. In most nonlinear media, there usually exists a certain amount of nonlinear saturation [3–7]. In solid state materials, such as an optical fiber, the nonlinear saturation involved is weak. A pulse or wave evolution in such weakly saturable nonlinear media can well be described by the quintic nonlinear Schrödinger equation where the first higher order nonlinear saturation term is included to account for the effect [7–14]. With inclusion of this nonlinear saturation, the integrability of the system is then lost and the governing model of the nonlinear Schrödinger equation becomes nonintegrable, in the sense of the inverse scattering transform [1].

Although being nonintegrable, we will show that there exist the exact analytical dark (black and gray) soliton solutions to the quintic nonlinear Schrödinger equation which explicitly reveal the effect of nonlinear saturation on the propagation of the dark solitons in a practical nonlinear medium. This is in contrast to highly saturable nonlinearities where the exact soliton solutions to nonlinear Schrödinger equations require a numerical approach [15–17]. Furthermore, the dark solitons in the presence of the weak nonlinear saturation are found to be stable to small (noncollision type) perturbations and the two black solitons launched in parallel will repel with propagation distance, similar to those in the Kerr law nonlinear medium [1,18]. When a collision is involved, however, the difference between the Kerr nonlinearity and the saturable nonlinearity emerges. The two dark solitons in the weakly saturable nonlinear media, launched in tilted angles heading for collision, cannot survive through a collision for the background intensity above a critical value although they recover themselves after the collision when the background intensity is below the critical value. These propagation and collision characteristics of the dark solitons in the saturable nonlinear medium should be useful in providing a guideline for the system design in device applications using dark solitons [19–22].

The quintic nonlinear Schrödinger equation governing a pulse or wave evolution in an optical fiber operating in the normal dispersion regime or in a self-defocusing uniform medium for weakly saturable nonlinearity can be written in a dimensionless form

$$i\,\frac{\partial u}{\partial \xi}-\frac{1}{2}\,\frac{\partial^2 u}{\partial T^2}+|u|^2u-n_4|u|^4u=0, \tag{1}$$

where in the context of a pulse [2] u is the normalized envelope function of the electrical field, T is the normalized time related to the real time t by $T=(t-z/v_g)/t_0$ with v_g the group velocity, and t_0 a measurement of the pulse width, ξ is the normalized distance and $n_4>0$ measures nonlinear saturation.

To derive solutions of Eq. (1), we write $u(T,\xi)$ in the form of $u(T,\xi)=\sqrt{\sigma(T-\eta\xi)}\exp[i\theta(T-\eta\xi)+i\beta\xi]$ which upon substitution into Eq. (1) leads to the two coupled equations governing the real variables $\sigma(\tau)$ and $\theta(\tau)$ with $\tau=T-\eta\xi$

$$\eta\theta'\sigma^2-\beta\sigma^2-\sigma''\sigma/4+\sigma'^2/8+(\theta'\sigma)^2/2+\sigma^3=0, \tag{2a}$$

$$\eta\sigma'+\theta'\sigma'+\theta''\sigma=0, \tag{2b}$$

where the prime indicates derivative with respect to τ. In general, the solutions to Eq. (2) are periodic functions expressible in terms of Jacobian elliptical functions [9]. Here, we are interested in the soliton solutions, which can be derived by integrating Eq. (2) with boundary conditions $\sigma'(\infty)=0$ and $\sigma(\infty)=\sigma_0$ to lead to

$$u(T,\xi)=\left[\frac{\alpha\sigma_0}{1-4n_4(\sigma_1+\sigma_0)/3}\right]^{1/2} \times\frac{\nu\,\tanh[\sqrt{\sigma_0}\nu(t-\sqrt{\sigma_0}\mu\xi)]-i\mu/\alpha}{\sqrt{1+\alpha_3\,\mathrm{sech}^2[\sqrt{\sigma_0}\nu(t-\sqrt{\sigma_0}\mu\xi)]}} \times\exp[i\sigma_0(1-n_4\sigma_0)\xi], \tag{3}$$

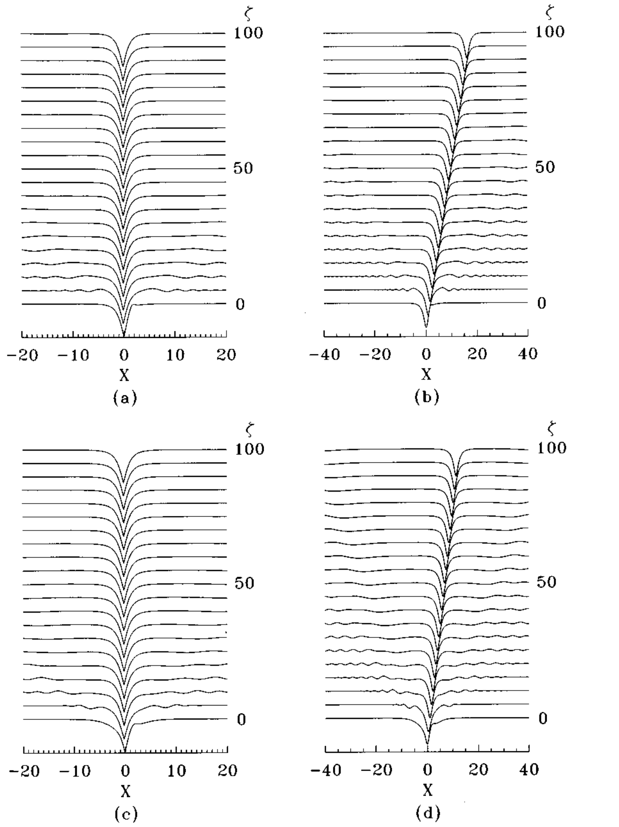

FIG. 1. Demonstration of stable evolution of dark solitons at $n_4\sigma_0=0.4$ for (a) the black soliton and (b) gray soliton of $(\sigma_1/\sigma_0)^{1/2}=0.3$, and at $n_4\sigma_0=0.45$ for (c) the black soliton and (d) gray soliton of $(\sigma_1/\sigma_0)^{1/2}=0.3$ in quintic nonlinear media, where $X=(\sigma_0)^{1/2}T$ and $\zeta=\sigma_0\xi$.

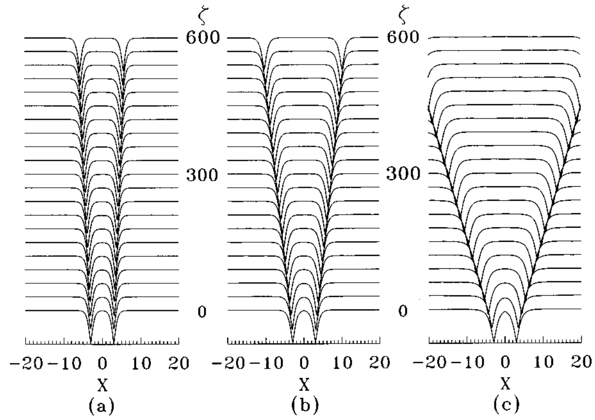

FIG. 2. Demonstration of repelling of the two black solitons in quintic nonlinear media launched in parallel with propagation distance for (a) $n_4\sigma_0=0$, (b) $n_4\sigma_0=0.2$, and (c) $n_4\sigma_0=0.4$.

where $\alpha=[1-2n_4(\sigma_1+2\sigma_0)/3]/[1-2n_4(\sigma_1/3+\sigma_0)]$, $\alpha_3=n_4(\sigma_0-\sigma_1)/(2n_4\sigma_0+2n_4\sigma_1-3/2)$, σ_0 is the maximum or background intensity, σ_1 is the minimum intensity, $\mu=\sqrt{(\sigma_1/\sigma_0)[1-2n_4(\sigma_1+2\sigma_0)/3]}$, and $\nu=\sqrt{(1-\sigma_1/\sigma_0)[1-2n_4(\sigma_1/3+\sigma_0)]}$ measures the grayness of gray solitons with $\sigma_1=0$ corresponding to the black soliton and $\sigma_1=\sigma_0$ to the plane wave solution. These dark soliton solutions of (3) in the saturable nonlinear media reduce to their counterpart for the Kerr law nonlinearity derived by the inverse scattering method [1] when setting $n_4=0$ with no nonlinear saturation. For a nonzero n_4, Eq. (3) indicates that the soliton parameters, including the (normalized) width $1/(\sqrt{\sigma_0}\nu)$ and velocity (or steering angle) $1/\eta=1/(\sqrt{\sigma_0}\mu)=1/\sqrt{\sigma_1[1-2n_4(\sigma_1+2\sigma_0)/3]}$, are all dependent on nonlinear saturation. The soliton width $1/(\sqrt{\sigma_0}\nu)$ increases with increasing n_4 and is larger than the corresponding one of the Kerr nonlinearity for a fixed minimum intensity σ_1. This means nonlinear saturation tends to expand the soliton width. Similarly, the velocity of the gray solitons increases with increasing saturation and it is larger than that of Kerr law nonlinearity as $1/(\sqrt{\sigma_0}\mu)=1/\sqrt{\sigma_1[1-2n_4(\sigma_1+2\sigma_0)/3]}>1/\sqrt{\sigma_1}$. Note here that $\mu^2+\nu^2\neq 1$ in contrast to that in the Kerr law nonlinearity for which $\mu^2+\nu^2=1$. Also it is clear from Eq. (3) that the dark soliton solutions exist in the saturable medium only for intensities $2n_4(2\sigma_0+\sigma_1)/3<1$ below the value $2n_4\sigma_0=1$ at which the self-phase modulation or the induced refractive index change reaches the maximum.

So far as the stationary solutions are concerned, the dark solitons in weakly saturable nonlinear media exhibit qualitatively the same characters as those in the Kerr law nonlinear media (i.e., no extra soliton solutions are introduced due to the nonlinear saturation), although quantitatively they differ in width, amplitude, and velocity for fixed intensities. Then do they still share similar characters when stability is involved? To address this question, we conduct the stability analysis on the dark soliton solutions of Eq. (3). The stability analyses (employed in Ref. [23] for analyzing the stability of $(3+1)D$ and $(2+1)D$ dark solitons in the Kerr nonlinearity) applied to the dark soliton solutions of Eq. (3) reveal that these dark solitons are stable to small perturbations (noncollision type). Some stable evolutions of the dark solitons of Eq. (3) at $n_4\sigma_0=0.4$ and 0.45 from numerical simulations are illustrated in Fig. 1 where perturbations $\delta u=u-u_{\text{stationary}}=0.2\sqrt{\sigma_0}\,\text{sech}[4\sqrt{\sigma_0}\nu(t-1.5)]$ initially implanted are radiated, leading to the stationary propagations.

It is known that in Kerr law nonlinear media the two black solitons launched in parallel will repel [Fig. 2(a)]. Our numerical simulations show that the two black solitons in the saturable nonlinear media launched in parallel will repel too for all the values of $n_4\sigma_0$ [<0.5, below which the solutions of Eq. (3) exist]. This is shown in Figs. 2(b) and 2(c) for $n_4\sigma_0=0.2$ and 0.4. With increasing $n_4\sigma_0$, the repelling force between the solitons increases and the separation between the two solitons with the propagation distance becomes fast for a fixed initial separation, i.e., the two solitons in the saturable media separate at a rate faster than their counterpart in Kerr nonlinear media. The origin of this increasing force of repelling with increasing $n_4\sigma_0$ for a fixed initial separation arises from a widened width with increasing $n_4\sigma_0$. Increasing in width effectively reduces the ratio of initial separation to the soliton width, leading to an augmentation of repelling force which increases with a decreasing ratio of initial separation to the soliton width.

Having discussed the effect of small perturbations on the propagation of the dark solitons, we now consider the stabil-

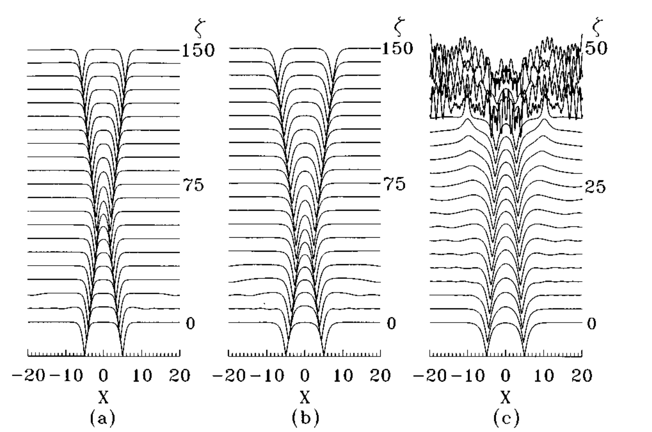

FIG. 3. Collision of the two black solitons launched in tilted angles heading for collision in quintic nonlinear media where (a) $n_4\sigma_0=0$, (b) $n_4\sigma_0=0.3$, and (c) $n_4\sigma_0=0.4$.

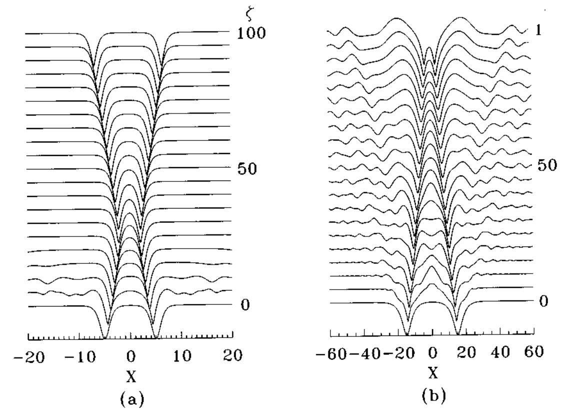

FIG. 4. Collision of the two black solitons launched in tilted angles heading for collision in fully saturable nonlinear media of Eq. (4), where (a) $n_4\sigma_0=0.3$ and (b) $n_4\sigma_0=3$.

ity of the dark solitons to big (collision type) perturbations. From numerical experiments we find that the two black solitons in the saturable nonlinear media, launched in tilted angles heading for collision, can emerge through collision when the background intensity is lower than $n_4\sigma_0<0.38$ [see Fig. 3(b) for $n_4\sigma_0=0.3$). This again is similar to the Kerr law nonlinearity as shown in Fig. 3(a). However, when $n_4\sigma_0\geqslant 0.38$ the two black solitons, launched in tilted angles heading for collision, could not survive through a collision [see Fig. 3(c) for $n_4\sigma_0=0.4$]. Collision of the two solitons results in radiation. This contrasts with the Kerr law nonlinearity [Fig. 3(a)]. Similar phenomena are observed for gray solitons. For gray solitons the critical values $n_4\sigma_0$, above which collision of the two solitons leads to radiation, vary with σ_1/σ_0 and these critical values are all higher than $n_4\sigma_0=0.38$ for the black soliton. The worst affected case by nonlinear saturation is the gray solitons around $\sqrt{\sigma_1/\sigma_0}=0.3$, for which the critical value for radiation resulting from collision has the smallest value $n_4\sigma_0=0.47$. This implies that gray solitons are less affected by nonlinear saturation than the black soliton. The reason for radiation or recovery after collision of the two dark solitons above or below a critical value $n_4\sigma_0$ results from integrability. With a small $n_4\sigma_0$, Eq. (1) is quasi-integrable and thus two solitons could survive through collision. When $n_4\sigma_0$ is greater than the critical value (which is 0.38 for the black soliton), the integrability is completely lost, leading to a breakdown of the soliton propagation after collision.

Here, it should be mentioned that for a large value of $n_4\sigma_0$ (with $\sigma_0=\max\{|u|^2\}$), the quintic nonlinearity employed in the analysis becomes less accurate in approximating a fully saturable nonlinearity since higher order saturation terms take effect. For example, expanding the fully saturable nonlinearity $|u|^2(1+n_4|u|^2/3)/(1+2n_4|u|^2/3)^2$, we have [24]

$$\frac{|u|^2(1+n_4|u|^2/3)}{(1+2n_4|u|^2/3)^2}=|u|^2-n_4|u|^4+(8/9)n_4^2|u|^6\dots \quad (4)$$

By keeping the first two terms in the expansion, the fully saturable nonlinearity reduces to the quintic nonlinearity. For a small $n_4\sigma_0$ (with σ_0 the peak intensity), the propagation characteristic of the solitons in quintic nonlinear media [e.g., Fig. 3(b)] is (nearly) identical to that in a fully saturable nonlinear medium as shown in Fig. 4(a) for propagation of two solitons in collision at $n_4\sigma_0=0.3$. For a larger $n_4\sigma_0$, a big quantitative difference occurs. For example, when $n_4\sigma_0>0.38$, collision of two solitons in quintic nonlinear media leads to radiation. For a fully saturable nonlinear medium, the collision of the two solitons also leads to radiation at a large $n_4\sigma_0$, but this does not occur around a critical value. Rather the amount of radiation from collision increases slowly with increasing $n_4\sigma_0$. At $n_4\sigma_0=3$, the radiation from collision reaches considerable amount as shown in Fig. 4(b). These indicate that the propagation and collision characteristics illustrated in the figures for a weak nonlinear saturation are expected in a full saturable nonlinear medium, although quantitatively different.

In summary, we present exact explicit analytical solutions of the dark solitons to the quintic nonlinear Schrödinger equation that includes the first higher order nonlinear saturable term. The dark solitons in the weakly saturable self-defocusing nonlinear media are shown to be stable to a small perturbation and as in the case of the Kerr nonlinearity the two black solitons launched in parallel repel with the propagation distance. On the other hand, when the two dark solitons are launched towards each other in tilted angles heading for a collision the solitons in the saturable nonlinear media could not survive through collision above a critical background intensity (although they survive below that critical intensity value). This is in contrast to Kerr law nonlinearity.

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