Driven granular media in one dimension: Correlations and equation of state

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We study a one-dimensional granular system in which each particle is excited by white noise, with inelastic interactions between the particles. When the coefficient of restitution, η, is one, the particles are uncorrelated. As η decreases, long-range correlations between the particles develop. A computer simulation of the system shows a steady-state, power-law particle-particle correlation function, which depends strongly on η. We give simple analytic arguments for the correlations. We also present an ''equation of state'' for the system of particles, which relates the noise amplitude to the particle density and the average particle speed. [S1063-651X(96)51007-X]

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With a few notable exceptions, our understanding of systems not in equilibrium is far from complete. Certainly, there is no general framework comparable to equilibrium statistical mechanics with which to study the behavior of nonequilibrium systems. Models of driven dissipative systems including granular media have been the subject of considerable recent interest, in part because of insights that such systems can provide into nonequilibrium behavior [1,2]. Granular materials are also of interest because they demonstrate a state of matter with properties reminiscent of both solids and liquids [3]. Other properties of granular materials such as heaping and the appearance of avalanches, however, are in marked contrast with the behavior of both macroscopic liquids and solids. Very recent studies of the effects of inelastic collisions in granular media and dissipative gases have shown surprising dynamics, including clustering [4–8], inelastic collapse [6,8], and the breakdown of macroscopic hydrodynamics [9]. Here, we study a continuously driven or heated one-dimensional dissipative gas and show that inelastic collisions lead to pronounced, long-range spatial correlations in the absence of any long-range forces between the particles (Fig. 1). Despite these correlations, we find that aspects of this nonequilibrium system can be described by a simple equation of state based on a global energy balance.

Clustering is often observed when granular media are sheared [5,10,11]. In general, clustering is driven by inelastic collisions. When two particles collide inelastically, they dissipate energy, slow down, and hence remain close to one another. Here we investigate numerically the effect of such inelastic collisions in a one-dimensional system of independent, pointlike particles that are excited by a thermal reservoir. We show that clustering, as evidenced for instance by the two-point correlation function, occurs even in the absence of any other forces between the particles. There have been several studies of dissipative systems that are started in a ''hot'' state and then slowly cool [4–8], since there is no energy input. In one and two dimensions, these can show ''inelastic collapse'' [6]. The novel feature of the collapse is that for coefficients of restitution η below a critical value η_c, the kinetic energy is dissipated in a finite time. For $\eta > \eta_c$, the kinetic energy dissipates gradually.

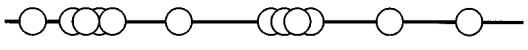

FIG. 1. Snapshot of a system of particles that are uniformly and individually heated. Periodic boundary conditions are used, and the particles are pointlike. The system forms cool liquidlike clusters surrounded by hot gaslike regions. As the coefficient of restitution is reduced, these clusters become more pronounced.

In our model, we consider a continuous input of energy locally to each particle, as well as dissipative collisions. Thus, for a coefficient of restitution $\eta < 1$, the system eventually settles down to a ''steady state.'' For $\eta \cong 1$, this steady state looks, at least superficially, like an ideal gas, in which there are no significant spatial correlations between the particles. However, we show that as η is reduced, even in the absence of any long-range interactions, the system develops a structure factor that is characteristic of an equilibrium system with long-range interactions. In particular, the dissipative interactions lead to a correlation function $g(x)$ that is no longer a constant as it would be for an ideal gas, but shows a peak near the origin. Furthermore, this enhancement of $g(x)$ near $x=0$ is of a power-law form. Thus, the system behaves as if there were long-range attractive interactions between the particles. Only in the limit $\eta \to 1$ does the correlation function become uniform. As η decreases, $g(x)$ becomes more and more sharply peaked about $x=0$.

Our system has two advantages over the more traditional problem where energy flows into the system from the boundaries, either by shear [5,10,11] or by other means [8,7,9]. The first is that the energy input to each particle is well controlled, which avoids complications due to boundary effects and spatial gradients across the system. The second is that since each particle is heated independently there is no inelastic collapse.

We consider N point particles of unit mass, $m=1$, confined to a line of length $L=1$ (Fig. 1). We use periodic boundary conditions, so that the particles lie on a circle of unit circumference. When two particles i and j collide in this one-dimensional system, the final (primed) velocities are given in terms of the initial (unprimed) velocities by

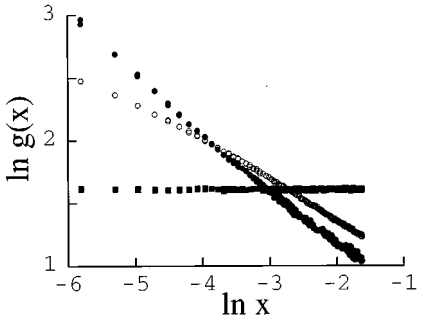

FIG. 2. A log-log plot of the two-point correlation function versus distance for three values of the coefficient of restitution for $N=10$ particles. For $\eta=0.99$ (lowest curve) the correlation function is almost a constant. For $\eta=0.5$ (middle curve) strong correlations have developed, and for $\eta=0.01$ (upper curve) the function is clearly a power law with exponent $-\frac{1}{2}$. The correlations arise because of the effect of inelasticity, i.e., when two particles collide they move more slowly and hence stay near each other, thus causing a correlation. The correlation function is independent of the heating rate Ω. The curve, at small η is also a power law for other values of N, with a slope of $-\frac{1}{2}$. Here we have superimposed data from two heating rates that differ by a factor of 100, $\Omega=0.0017$ and $\Omega=0.17$, and each point represents an average over 2×10^5 collisions.

$$v_i'=\frac{1}{2}(1-\eta)v_i+\frac{1}{2}(1+\eta)v_j. \tag{1}$$

The main difference between this and previous studies of dissipative gases is that each individual particle is ''heated'' at a constant rate. This is done by adding a random amount to the velocity of each particle during a time step Δt. Thus we write the usual Langevin equation

$$v_i(t+\Delta t)=v_i(t)+\sqrt{r}\sqrt{\Delta t}f(t), \tag{2}$$

where $f(t)$ is a random number chosen uniformly between $-\frac{1}{2}$ and $\frac{1}{2}$ and r is a number proportional to the heating rate. After the velocities are adjusted the system is transferred to the center-of-mass frame, so that $v_i\rightarrow v_i-\overline{v}$, where $\overline{v}$ is the average velocity of all the particles in the system. The algorithm (2) ensures that the velocities undergo a random walk, while the transfer to the center-of-mass frame ensures that the particle speeds do not increase indefinitely. It should be emphasized that this transfer step is for convenience only, since the properties of collisions do not depend on the absolute speeds, only on the relative speeds. Before the heating step, the kinetic energy of the system is $K=\frac{1}{2}\Sigma_{i=1}^{N}v_i^2$, and after heating, it is

$$K'=\frac{1}{2}\sum_{i=1}^{N}(v_i+\delta v_i)^2=K+\sum_{i=1}^{N}v_i\delta v_i+\frac{1}{2}\sum_{i=1}^{N}(\delta v_i)^2, \tag{3}$$

where $\delta v_i=\sqrt{r}\sqrt{\Delta t}f(t)$. The term linear in δv_i vanishes on average, by symmetry, and the term $\frac{1}{2}\Sigma_{i=1}^{N}(\delta v_i)^2$ is, on average $\frac{1}{2}rtN\langle f^2\rangle=\frac{1}{24}rtN$, hence our identification of r with the heating rate. It is convenient to introduce a quantity $\Omega\equiv\frac{1}{24}r$, which is the energy input per unit time per particle. Provided $\eta<1$, this system, started with some initial random speeds, eventually reaches a steady-state configuration.

Qualitatively, the system appears to form liquidlike clusters of high density surrounded by a gaslike ''phase'' of lower density (Fig. 1). In order to study this effect quantitatively, we introduce the two-particle correlation function $g(x)$. We select a particle and ask what is the density of particles at distance x from it. Since our system is translationally invariant, the choice of origin is arbitrary. This, suitably normalized, is $g(x)$. For an ideal gas of point particles, the answer is $g(x)=$ a constant. For a gas that has some attractive potential $U(x)$ acting between the particles, the answer is more complicated but, in general, $g(x)$ will be peaked about $x=0$ and will decay to a constant as $x\rightarrow\infty$. For the dissipative gas discussed here, we find $g(x)$ shows a peak at the origin, even though we have no potential acting between the particles. This peak is caused by the dissipation effect discussed above and can be thought of as a steady-state version of the collapse and clustering seen in dissipative cooling gases [4,8,5–7]. As $\eta\rightarrow1$ the structure becomes less pronounced and $g(x)$ approaches a constant. However, as η becomes small, $g(x)$ becomes very sharply peaked. Some characteristic results are shown in Fig. 2. We find that the correlation function depends only on the density and on η and is independent of the heating rate. At least for small x, where the finite size of the system has little effect, $g(x)$ can be approximated by a power law $g(x)\sim x^{-\alpha(\eta)}$. Here $\alpha(\eta)$ is a monotonically increasing function of η. In the limit of a perfectly elastic system $\eta\rightarrow1$ and $\alpha\rightarrow0$. However, for perfectly inelastic systems where $\eta\rightarrow0$ we find $\alpha\rightarrow\frac{1}{2}$. For the problem of interactions between two *isolated* particles, a scaling argument predicts a limiting exponent of $\frac{1}{3}$. This result can be understood as follows. When η is small the collisions between particles are almost perfectly inelastic and the particles lose almost all of their energy. The particles only move away from each other because of the random kicks induced by the heating. Thus we can consider a single particle moving away from a wall and, subject to its velocity, undergoing a random walk. We thus have $\langle|v(t)|^2\rangle\sim t$ and

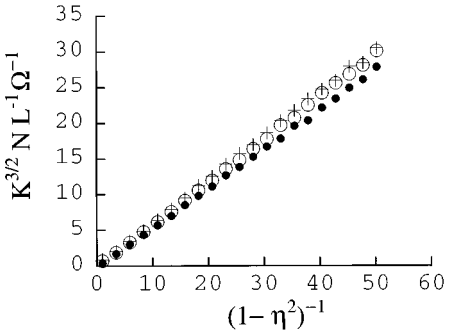

FIG. 3. A test of the η dependence in the equation of state (5). Here we have plotted $K^{3/2}NL^{-1}\Omega^{-1}$ versus $(1-\eta^2)^{-1}$. As predicted by theory (5), the relation is close to linear. Note that because the scaling with N in (5) is not precise, the lines for different N are not exactly superimposed. This effect is more pronounced in Fig. 4. The data are points from the computer simulation and each point represents an average over 10^6 collisions. The following pairs of particle number and heating rate (N,Ω) were used: ●, (10,0.0017); ○, (20,0.0017); +, (40,0.0017).

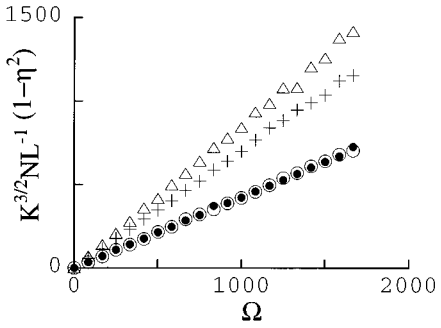

FIG. 4. A test of the Ω dependence in the equation of state (5). Here we have plotted $K^{3/2}NL^{-1}(1-\eta^2)$ versus Ω. As predicted by theory (5), the relation is close to linear. Note that because the scaling with N in (5) is not precise, the lines for different N are not exactly superimposed. The circles are points from the computer simulation and each point represents an average over 5×10^6 collisions. The following pairs of the particle number and coefficient of restitution (N,η) were used: ●, (10,0.25); ○, (10,0.55); +, (20,0.55); △, (30,0.35).

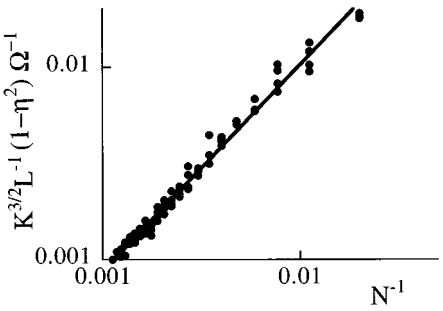

FIG. 5. A test of the N dependence in the equation of state (5). Shown is a log-log plot of $K^{3/2}\Omega L^{-1}(1-\eta^2)$ versus $1/N$. The theory (5) would predict a line of slope unity. The line of best fit (shown) has a slope of 1.07. The circles are points from the computer simulation and 10^5 collisions are averaged over for each point. The following pairs of heating rate and coefficient of restitution (Ω,η) were used: (0.0017,0.25),(0.20,0.85),(0.20,0.35),(0.74,0.35).

$v\sim t^{1/2}$. The distance traveled from the wall in time t is then $x\sim\int_0^t v(t')dt'\sim t^{3/2}\sim v^3$. The density of a particle at any point is just the inverse of the time it spends in that region, i.e., v^{-1}, thus $g(x)\sim v^{-1}\sim x^{-1/3}$, and $\alpha=\frac{1}{3}$. The fact that $\alpha(0)=\frac{1}{2}$ is presumably caused by interactions between many particles.

In spite of the nonequilibrium nature of the system, we can write down an equation of state by a simple energy argument [12,13]. By ''equation of state'' we mean the relation between the heating rate, the number of particles, the system size, and the kinetic energy of the particles. One possible approximation to the equation of state can be derived as follows. Let us ignore numerical prefactors and correlations between the particles. The average distance a particle must move between collisions is $L/(2N)$. Then, for particles that have some average speed v, the time between a collision (for each particle) is approximately $t=lL/(2N)]v^{-1}$. Since there are $N/2$ pairs of particles, the rate of collisions is $\approx(N/2)t^{-1}=vL^{-1}N^2$. During each collision, an amount of energy $(1-\eta^2)v^2$ is dissipated. Thus the rate of energy dissipation is approximately [13]

$$W=v^3(1-\eta^2)L^{-1}N^2. \tag{4}$$

However, the rate of energy input is ΩN. In the steady state these must be equal. Thus we find an equation of state

$$K^{3/2}(1-\eta^2)N=C\Omega L, \tag{5}$$

where K is the kinetic energy per particle and C is a numerical constant. The dependencies on K, Ω, and L are somewhat trivial since they can be derived independently by dimensional analysis. However, the dependence upon η and especially N are less trivial. We can test how accurately this equation describes the system by comparing it with the results of our computer simulation. The scaling of the kinetic energy with the heating rate and the coefficient of restitution are exactly as predicted by the theory (5) (Figs. 3 and 4). The scaling with N is satisfactory, (Fig. 5) but at a small number of particles there are significant variations from (5). These are clear in Fig. 4, where the data for different values of N have different slopes.

We can use the equation of state to obtain the force on a wall placed at the boundary of a finite system. We assume that the particles undergo perfectly elastic collisions with the wall. The time-averaged force F exerted by the particles is the ''pressure'' for a one-dimensional gas. The current of particles hitting the wall is $J=Nv/L$. Each particle imparts momentum $2v$ to the wall. Thus the time-averaged force is $F=\rho v^2$, where $\rho=1/L$ is the linear density. Thus the relation between the force and the density is

$$F=\Omega^{2/3}\rho^{1/3}(1-\eta^2)^{-2/3}. \tag{6}$$

Due to dissipation, this is very different from the result for an ideal gas where the pressure is proportional to the first power of the density.

In this paper we have discussed a simple one-dimensional model of an excited dissipative gas. The system shows steady-state clustering, or power-law correlations between particles, and can be described by a simple equation of state. The nontrivial correlation function implies that the particles move as if interacting via a potential. There have been several attempts to describe the average properties of dissipative gases [12,13]. However, their full statistical mechanics, including a theory of their correlation functions, analogous to that for simple liquids, remains a problem for future research.

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