Self-organized criticality in a two-dimensional rotating drum model

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For a model of a slowly rotating two-dimensional drum the statistics of the potential energy dissipation in avalanches is studied. Self-organized criticality occurs only for the small disks in bidisperse fillings. The large disks in bidisperse fillings as well as monodisperse fillings show a stretched exponential distribution. This is due to the absence of niches on the surface which are able to trap these disks.
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The concept of self-organized criticality (SOC) was introduced by Bak, Tang, and Wiesenfeld in the context of energy dissipation in avalanches in a sandpile model [1]. The idea is that spatially extended, in general diffusively coupled systems in the limit of weak driving evolve into a critical state where no characteristic length scale is observed.

We developed a minimal model called bottom-to-top restructuring (BTR) model which simulates a slowly rotating two-dimensional drum with a granular filling in the limits of high static friction and low restitution coefficient [2–5]. The single grains are disks which relax in order of increasing height to the nearest local minimum after a rigid rotation of the whole configuration. The rigid rotation angle $\Delta\Phi$ defines a ratio between the angular velocity and the frequency of a weak superimposed vibration of the drum. The vibration forces the pile to start an avalanche after every rotation with $\Delta\Phi$. In this sense one can view $\Delta\Phi$ also as a triggered difference between the angle of repose α_r and the angle of marginal stability of the pile. For more information about the model and the related physical properties see Refs. [2–5].

In a recent paper [4] we investigated the statistics of avalanches in this model with a bidisperse filling. We measured the flow of the disks within the avalanche zone for nearly half filled drums of different radius (R) and constant $\Delta\Phi=(\sqrt{99}/10)^\circ$. The avalanche zone will be explained later. We found that the probability distribution of the flow of the large disks shows a trivial finite size scaling of the form

$$P(s,R)=R^{-\nu}\mathcal{F}\left(\frac{s}{R^{\beta}}\right), \tag{1}$$

with $\nu=\beta=3$. The scaling function $\mathcal{F}$ showed no power law behavior. By contrast, no scaling was obtained for the small disks.

In order to investigate finite size scaling in the context of self-organized criticality, it is crucial to keep the driving strength at a fixed small value as one varies the system size. This means, e.g., that the injection rate of grains onto the top of a sandpile has to be the same for all sizes of the pile. In that sense one has to be sure that the averaged mass of the avalanches per time step is constant. This is not true in rotating drums with constant $\Delta\Phi$. A mass proportional to R^2 (mass conservation) is transported in the avalanche zone if one uses or has got a constant difference between the angle of repose and the angle of marginal stability. Therefore we choose in this paper an angle increment,

$$\Delta\Phi\propto\frac{1}{R^2}. \tag{2}$$

The number of disks is proportional to R^2. We used a bidisperse filling of the drum with large disks of radius $r_l=1.0$ and small disks of radius $r_s=0.5$. All length scales were counted in units of r_l, and the number of large disks was equal to the number of small ones. $\Delta\Phi$ was choosen to be $(\sqrt{99}/10)^\circ$ for the smallest radius of the drum $R=60$.

To define the avalanche zone, we measured the averaged angle of repose, which depends on the radius of the drum, the rotation angle, and the ratio of the disk radii [5]. We drew a line parallel to the average surface at a distance $0.05R+w_{az}$ from the center of the drum. All drums were nearly half filled up to a distance $0.05R$, and w_{az} is a parameter specifying the width of the avalanche zone. All disks above this boundary have been taken into account in the calculation. The parameter w_{az} has been varied ($w_{az}=8$, 12, or $0.15R$) in order to check that the results do not depend on it, provided w_{az} is larger than the unknown width of the avalanches. In Fig. 1 we show this line in a small drum for $w_{az}=0.15R$. One can also clearly recognize the segregation effect of small and large disks.

We measured the dissipation of the potential energy ΔE_{pot} by the avalanches in our model. ΔE_{pot} is calculated separately for large and small disks: In every update the sum of the difference between the vertical coordinates after the rigid rotation and after the relaxation is taken over all large or small disks in the avalanche zone. Each of the two sums is multiplied by the square of the respective disk radius. We took for every system size the histogram of ΔE_{pot} over ten different systems and five total revolutions of the drum, omitting three initial revolutions. The obtained distribution functions $P(\Delta E_{pot},R)$ are normalized in the usual way,

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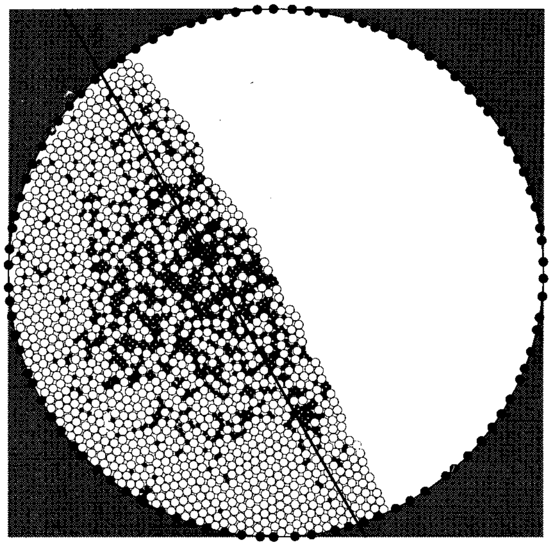

FIG. 1. A drum of radius $R=60$ after three total revolutions of the drum with small black and large white disks. The large black disks on the wall of the drum are randomly spaced fixed ones in order to increase the disorder in the structure of the filling. The solid line at a distance $0.2R$ from the center of the drum indicates the avalanche zone. Its angle to the horizontal is the averaged angle of repose $\alpha_r=60.44°$ in this system.

$$\int P(\Delta E_{\rm pot},R)d\Delta E_{\rm pot}=1, \tag{3}$$

and rescaled by the finite size scaling ansatz [Eq. (1)]. The normalization implies $\nu=\beta$, provided the scaling function falls off rapidly enough.

We found that for bidisperse and monodisperse systems the distribution functions scale with the system size: $\nu=\beta=1$. This is in agreement with the results of Frette *et al.* [6], who measured the dissipation of energy in a rice pile. Amaral and Lauritsen [7] observed linear finite size scaling also in a rice pile model, provided they had a kind of stroboscopic measurement of the potential energy difference by evaluating the different surface configurations of the pile before and after the deposition of about five rice grains. This uncertainty of the added potential energy is present in the experiment of Frette *et al.* and in our model as well. The number of grains which are transported into and out of the avalanche zone and the potential energy which is introduced into the system by the rigid rotation is not constant but depends on the configuration of the grains in the previous time step.

Rescaling the probability distribution of the dissipation of potential energy for the small disks in bidisperse systems with $\nu=\beta=1$, we observe a power law behavior for the distributions of the large avalanches. We noticed that one is able to achieve a much better data collapse for the power law part of the distributions when one shifts the distributions along the power law direction with an additional exponent μ. In that case the finite size scaling no longer conserves the normalization [8] and has the form

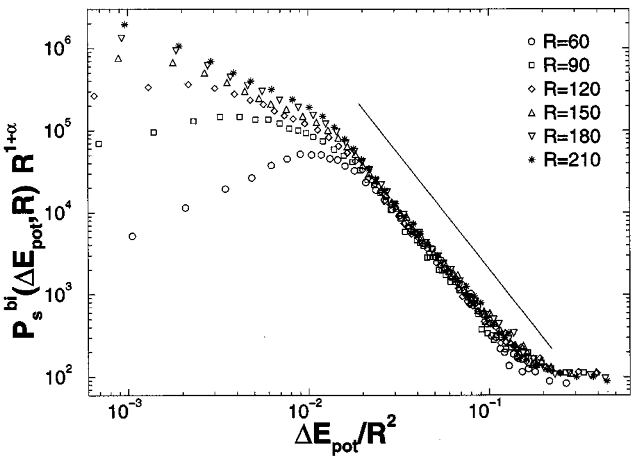

FIG. 2. Rescaled probability distribution of the dissipation of potential energy for the small disks in bidisperse systems. The width of the avalanche zone was $w_{\rm az}=0.15R$. The solid line indicates the obtained slope of the power law area $\alpha=2.8$ for all system sizes and avalanche zone widths.

$$P(\Delta E_{\rm pot},R)R^{\nu+\mu}=\mathcal{F}\left(\frac{\Delta E_{\rm pot}}{R^{\beta+\mu/\alpha}}\right) \quad \text{with} \quad \mathcal{F}(x)\approx x^{-\alpha}. \tag{4}$$

In Fig. 2 this is plotted for $w_{\rm az}=0.15R$. Similar plots for the other choices of the avalanche zone are showing the same result. We obtained the power law exponent α from fits to consecutive slopes of the scaling parts of the data and found $\alpha=2.8\pm0.2$. This value of α takes systematical errors into consideration. It is larger than the one obtained by Frette *et al.* ($\alpha\approx2.04$). This is understandable from the fact that in the BTR model the possibility of finding a large avalanche relative to the one of finding a small avalanche has to be much smaller than in experiments. Avalanches in experiments can grow due to collisions with disks of the bulk. We achieved the best collapse of the data with $\mu=\alpha$. This shows that there is no additional independent exponent present in the system. The resulting scaling of the large avalanches with R^2 is based on the fact that a large avalanche consists of

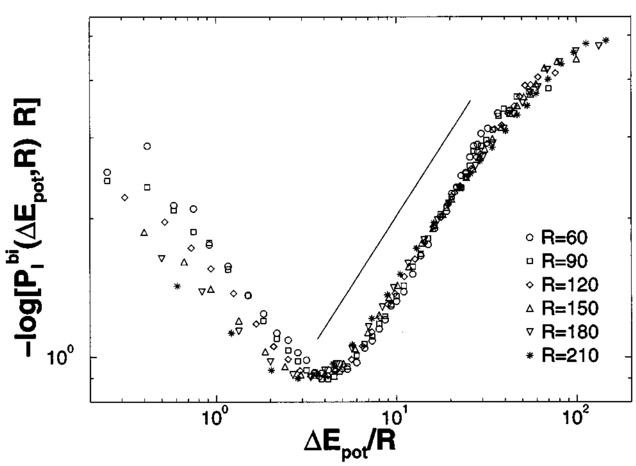

FIG. 3. Logarithm of the rescaled probability distribution of the dissipation of potential energy for the large disks in bidisperse systems. The width of the avalanche zone was $w_{\rm az}=0.15R$. The slope of the solid line is the mean value of the exponent $\gamma_{\rm bi}$ for all system sizes and avalanche zone widths.

nearly all segregated small disks at the surface ($\propto R$) which flow down a distance proportional to R. Similar to the segregation effect the behavior of the small disks in the avalanches is affected by the large grains but not vice versa. Therefore the scaling of the large and small disks in bidisperse systems is different.

It is interesting to see that the scaling function for the large disks in Fig. 3 does not show the power law behavior of the small disks. Instead we found that the scaling function corresponds better to a stretched exponential function of the following form:

$$P(\Delta E_{\rm pot},R)R=\frac{\gamma}{\xi^{1/\gamma}\Gamma\left(\frac{1}{\gamma}\right)}\exp\left[-\xi^{-1}\left(\frac{\Delta E_{\rm pot}}{R}\right)^{\gamma}\right]. \quad (5)$$

Here ξ is a characteristic value of the dissipated potential energy per system size, which is obtainable from the average value $\langle \Delta E_{\rm pot}/R\rangle=\xi^{1/\gamma}\Gamma(2/\gamma)/\Gamma(1/\gamma)$. The exponent γ in the bidisperse system can be obtained from the slope of the linear part of the data (see Fig. 3). We found $\gamma_{\rm bi}=0.61\pm0.05$ and $\xi_{\rm bi}\approx1.0$. In monodisperse systems we found similar results (see Fig. 4). Here the stretched exponential is given for the whole distributions over three orders of magnitude of the dissipated potential energy. We found $\gamma_{\rm mono}=0.31\pm0.02$ and $\xi_{\rm mono}\approx0.9$.

How is it possible that only the small disks in a bidisperse system show a power law behavior and therefore SOC? We think that in the bidisperse systems the segregation effect is responsible for this fact. Looking at Fig. 1 one notices that the surface of the filling is relatively smooth. We showed in a previous paper [4] that a large number and a broad distribution of the sizes of the niches at the surface are only given for the small disks. There are only a few niches that are large enough to trap a large disk (see Fig. 1). Therefore, the large disks are forced to roll down the whole surface and smaller avalanches of large disks are suppressed. When nearly all avalanches flow down to the bottom of the pile, there is a characteristic length scale present in the system. The broad distribution of niches for the small disks is able to trap them at different heights of the surface before they reach the bottom. In monodisperse systems the situation is similar to the one of the large disks in bidisperse systems. It was necessary to introduce randomly spaced fixed disks at the wall of the drum in the monodisperse systems to prevent a highly ordered structure of the filling and a sharply peaked avalanche size distribution [2,4,5]. Nevertheless, it is observable in the simulations that the surface of the monodisperse fillings is very smooth with only a few niches. Therefore large avalanches are preferred.

We think that these observations are fundamental for the understanding of the SOC process in granular systems. One has to be sure that not only avalanches of a single typical size can be produced, but of all sizes up to the cutoff size, i.e., that avalanches can stop flowing at any height of the pile. Therefore it is necessary that the surface of the pile shows a broad distribution of niches; i.e., it is rough. This is an explanation for the fact that Frette *et al.* [6] found SOC only for a rice sort with elongated grains which

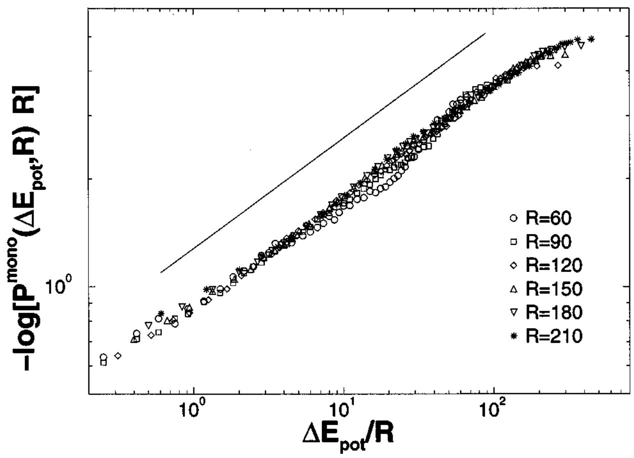

FIG. 4. Logarithm of the rescaled probability distribution of the dissipation of potential energy for monodisperse systems. The width of the avalanche zone was $w_{\rm az}=0.15R$. The slope of the solid line is the mean value of the exponent $\gamma_{\rm mono}$ for all system sizes and avalanche zone widths.

shows a much rougher surface than rice with more spherical grains. In our case, for large disks the surface looks smooth so that they flow all the way down to the bottom of the free surface.

We notice that a similar effect is observable in several experiments. A reanalysis of their data by Feder showed that the distribution functions also obeyed stretched exponentials [9]. The reason is that the internal avalanches had not been taken into account (see the original papers). Only the flows over the rim of the supports were measured. This defines the characteristic length scale in the flow distance of the avalanches. Sandpile models like [7] show SOC because every site that does not topple when a grain is added to it models a niche at the surface. Therefore this type of model is simulating sandpiles with rough surfaces.

In summary, we showed in this paper that the distribution function of the decrease of potential energy shows finite size scaling only when the system is driven with the same strength for all system sizes. In that sense, one is not able to find finite size scaling in rotating drums if there is a constant difference between angle of repose and angle of marginal stability. This results in a driving proportional to the square of the system size, which is inconsistent with SOC. In our model this is avoided due to the vibration superimposed on the rotation. It is also essential for SOC that no characteristic length scale is introduced into the system. Such length scales can possibly be introduced by the measuring process when only the flow over the rim is measured. It is necessary that the surface be rough. A smooth surface of the pile supports avalanches down to the bottom of the pile, which gives a characteristic length scale. When a characteristic length scale is present in a system it results in a stretched exponential distribution of the dissipated potential energy.

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