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Gaussian unitary ensemble eigenvalues and Riemann ζ function zeros: A nonlinear equation for a new statistic

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For infinite Gaussian unitary ensemble random matrices the probability density function $S_{nn}(t)$ for the nearest neighbor eignenvalue spacing (as distinct from the spacing between consecutive eigenvalues) is computed in terms of the solution of a certain nonlinear equation, which generalizes the σ form of the Painlevé V equation. Comparison is made with the empirical value of $S_{nn}(t)$ for the zeros of the Riemann ζ function on the critical line, including data from 10^6consecutive zeros near zero number 10^{20}. [S1063-651X(96)50111-X]

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Random matrix theory successfully predicts many features of the statistical properties of the energy levels of classically chaotic quantum systems (see, e.g., Refs. [1,2]). One such statistical property is the probability density function (PDF), $p(s)$ say, for the spacing between consecutive energy levels. Jimbo *et al.* [3] (see Ref. [4] for subsequent derivations) proved that for the Gaussian unitary ensemble (GUE) of infinite dimensional random matrices, scaled so that the mean eigenvalue spacing is $1/\rho$, $p(s)$ is given by

$$p(s)=\frac{1}{\rho}\frac{d^2}{ds^2}\exp\int_0^{\pi\rho s}\frac{\sigma(s')}{s'}ds', \tag{1}$$

where $\sigma(s)$ satisfies the σ form of the Painlevé V equation

$$(s\sigma'')^2+4(s\sigma'-\sigma)[s\sigma'-\sigma+(\sigma')^2]=0, \tag{2}$$

subject to the boundary condition $\sigma(s)\sim -s/\pi-(s/\pi)^2$ as $s\to 0$.

The GUE is applicable to chaotic quantum systems with broken time reversal symmetry. The zeros of the Riemann ζ function $\zeta(z)$ with large imaginary part on the critical line Re $(z)=\frac{1}{2}$ are known to possess characteristics of such a system [5], and according to the so-called GUE hypothesis (see, e.g., Ref [6]) in the infinite imaginary part limit, the joint distribution of the zeros is locally equal to the joint distribution of the eigenvalues of the GUE. The eigenvalues and zeros must be scaled so that their mean spacing takes on the same fixed value, $1/\rho$ say. In a large-scale numerical computation by one of the present authors [6], involving over 10^7 zeros $\frac{1}{2}+i\gamma_n$ about $n=10^{20}$ (here n labels the zeros along the critical line), the PDF $p(s)$ has been determined empirically and compared with $p(s)$ for the GUE. Excellent agreement is found.

In this Rapid Communication a statistic for the infinite GUE, which is very similar to the spacing between consecutive levels, is calculated exactly, and compared to that obtained empirically from the data of [6] for $\{\gamma_n\}$. This statistic is the PDF $S_{nn}(t)$ for the spacing between nearest neighbor levels (note that each eigenvalue has two neighbors but only one nearest neighbor).

Below the following results are established. The PDF $S_{nn}(t)$ for the infinite GUE with mean eigenvalue spacing π is given in terms of a Fredholm determinant by

$$S_{nn}(t)=-\frac{d}{dt}\det(1-K_1), \tag{3}$$

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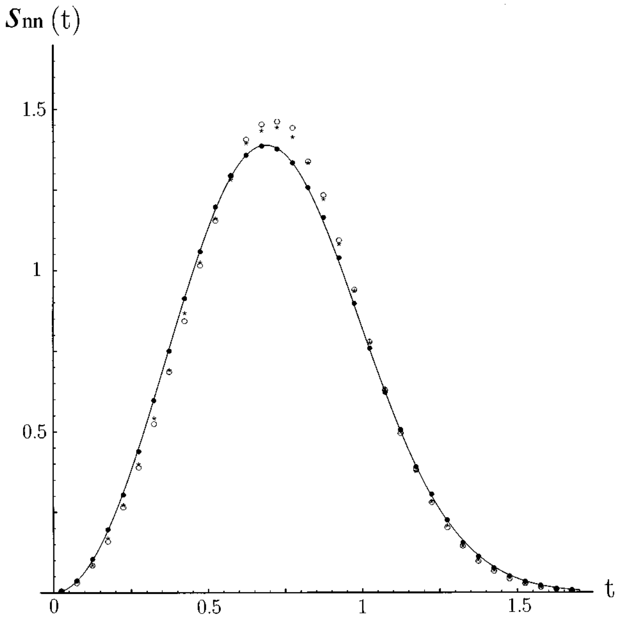

FIG. 1. Comparison of $S_{nn}(t)$ for the GUE (solid line) and for 10^6 consecutive zeros of the Riemann ζ function on the critical line, starting near zero number 1 (open circles), 10^6 (asterisks), and 10^{20} (filled circles), respectively. The mean spacing between consecutive (eigenvalues) zeros has been normalized to unity.

where K_1 is the integral operator on $(-t,t)$ with kernel

$$K_1(x,y) := \frac{\sqrt{xy}}{2(x-y)}[J_{b+1/2}(x)J_{b-1/2}(y) - J_{b+1/2}(y)J_{b-1/2}(x)] \tag{4}$$

$[J_\alpha(x)$ denotes the Bessel function$]$ and $b=1$. Furthermore,

$$\det(1-K_1) = \exp \int_0^{\pi\rho t} \frac{\sigma_1(2t')}{t'}dt' \tag{5a}$$

and so

$$S_{nn}(t) = -\frac{\sigma_1(2\pi\rho t)}{t} \exp \int_0^{\pi\rho t} \frac{\sigma_1(2t')}{t'}dt' \tag{5b}$$

(here the mean eigenvalue spacing is $1/\rho$), where $\sigma_1(s)$ satisfies the nonlinear equation

$$(s\sigma_1'')^2 + 4(-b^2 + s\sigma_1' - \sigma_1)\{(\sigma_1')^2 + [b-(b^2 - s\sigma_1' + \sigma_1)^{1/2}]^2\} = 0, \tag{6}$$

with $b=1$, subject to the boundary condition

$$\sigma_1(s) \sim -\frac{(s/2)^{2b+1}}{\Gamma(\frac{1}{2}+b)\Gamma(\frac{3}{2}+b)}, \quad \text{as } s \to 0, \tag{7}$$

with $b=1$ (the parameter b is included for later convenience). Note that with $b=0$ Eq. (6) reduces to Eq. (2).

We have computed many terms of the power series expansion of Eq. (6) about $s=0$ with $b=1$ and subject to Eq. (7). Comparison with the analogous expansion of Eq. (1) (see, e.g., [1]) shows that $p(t) - (\frac{1}{2})S_{nn}(t) = O(t^7)$, which in qualitative terms, says that very small spacings between consecutive eigenvalues will most likely be nearest neighbor spacings (the factor of $\frac{1}{2}$ accounts for the fact that the nearest neighbor occurs with equal probability to the left or to the right). The solution of Eq. (6) with $b=1$ was computed numerically [the power series solution to $O(s^{11})$ was used to compute $\sigma_1(1)$ and $\sigma_1'(1)$ which were used as initial conditions] and substituted in Eq. (5b) with $\rho=1$ to give the theoretical prediction for $S_{nn}(t)$ in the infinite GUE, which was then compared with $S_{nn}(t)$ determined empirically from the data of [6] for $\{\gamma_n\}$. Three sets of 10^6 consecutive zeros $\frac{1}{2} + i\gamma_n$ were analyzed, the data sets starting at zero number $N_1 = 1$, $N_2 = 10^6 + 1$, and $N_3 = 10^{20} + 143, 782, 842$, respectively. The quantity $\delta_n' := \min(\delta_n, \delta_{n-1})$, where $\delta_n := (\gamma_{n+1} - \gamma_n)\rho_n$ with $\rho_n = (1/2\pi)\ln(\gamma_n/2\pi)$ denoting the smoothed local density of zeros at $\frac{1}{2} + i\gamma_n$, was calculated and a histogram constructed for the number of values out of the 10^6 tested that fell into the intervals $[(k-1)/20, k/20]$, $k = 1,2,\ldots$. In Fig. 1 the corresponding empirical values of $S_{nn}(t)$ at the points $(k-\frac{1}{2})/20$ are plotted and compared with the value of $S_{nn}(t)$ for the infinite GUE. The convergence towards the GUE value as the magnitude of the imaginary part increases is evident.

TABLE I. Comparison of the moments of $S_{nn}(t)$ for the GUE (second column) and for 10^6 consecutive zeros of the Riemann ζ function (subsequent columns) on the critical line, starting near zero number 1, 10^6 and 10^{20}respectively. The mean spacing between consecutive (eigenvalues) zeros has been normalized to unity.

p	$\langle t^p \rangle$	$\langle \delta_n'^p \rangle_1$	$\langle \delta_n'^p \rangle_2$	$\langle \delta_n'^p \rangle_3$
1	0.725 227	0.731 988	0.730 706	0.725 291
2	0.603 251	0.606 386	0.605 762	0.602 470
3	0.555 775	0.551 262	0.551 956	0.553 540
4	0.555 527	0.540 113	0.542 599	0.551 074
5	0.594 314	0.563 548	0.568 467	0.586 454
6	0.674 002	0.620 786	0.629 172	0.660 788
7	0.804 518	0.717 187	0.730 735	0.782 709
8	1.005 15	0.864 325	0.885 824	0.969 281
9	1.308 70	1.081 77	1.115 83	1.249 35
10	1.769 24	1.400 75	1.455 04	1.670 02

For further comparison the moments $\langle t^p \rangle := \int_0^\infty t^p S_{nn}(t)dt$ for $p=1,\ldots,10$, were calculated and compared with the empirical data according to the law of large numbers prediction $\langle t^p \rangle \approx \langle \delta_n'^p \rangle_a := 10^{-6}\sum_{n=N_a+1}^{N_a+10^6} \delta_n'^p$ $(a=1,2,3)$. The results are contained in Table I. Again the trend is towards convergence to the GUE value. Note in particular the four figure agreement between $\langle t \rangle$ and $\langle \delta_n' \rangle_3$. The PDF $S_{nn}(t)$ therefore provides quantitative statistical evidence supporting the validity of the GUE hypothesis, thus adding to the statistical evidence obtained in Ref. [6] and the analytic arguments of Ref. [7].

Our derivation of Eqs. (3)–(7) uses a recent result of Nagao and Slevin [8] to obtain Eq. (3), and the theory of Tracy and Widom [9] to obtain Eq. (6). Nagao and Slevin consider the random matrix ensemble with unitary symmetry defined by the eigenvalue PDF

$$\prod_{j=1}^N |x_j|^{2b} e^{-x_j^2} \prod_{1 \le j < k \le N} |x_k - x_j|^2, \quad b > -\tfrac{1}{2}. \tag{8}$$

They prove that in the thermodynamic limit, with each x_j scaled $x_j \mapsto X_j/\sqrt{2N}$ so that the bulk density is $1/\pi$, the corresponding n-level distribution is given by

$$\rho_n(X_1,\dots,X_n)=\det[K_1(X_j,X_k)]_{j,k=1,\dots,n}, \tag{9}$$

where $K_1(x,y)$ is given by Eq. (4) [for $b\notin Z_{\geqslant 0}$, $x(y)<0$, $x(y)$ in the denominator needs to be replaced by $|x|(|y|)$; however, below we will only consider the case $b\in Z_{\geqslant 0}$].

It follows from Eq. (9) (see, e.g., Ref. [1]) that the probability $E(0;(-t,t))$ of an interval $(-t,t)$being free of eigenvalues in the ensemble (8) is given by $\det(1-K_1)$. Since in the case $b=1$, Eq. (8) is precisely the eigenvalue PDF of the GUE with an eigenvalue fixed at the origin, the result (3) follows. In fact this interpretation of Eq. (8) suggests another derivation of Eq. (9) in the case $b=1$. Thus with $b=1$, Eq. (9) must be equal to the $(n+1)$-level distribution of the GUE (see, e.g., Ref. [1]),

$$\rho_{n+1}^{\rm GUE}(X_1,\dots,X_{n+1})=\det\left[\frac{\sin(X_j-X_k)}{\pi(X_j-X_k)}\right]_{j,k=1,\dots,n+1}, \tag{10}$$

with one of the levels, X_{n+1}say, fixed at the origin. Setting $X_{n+1}=0$ in Eq. (10) and performing Gaussian elimination so that all entries below the first in the final column are zero gives Eq. (9) in the case $b=1$.

To derive Eq. (6) we introduce the quantities

$$(1-K_1)^{-1}\doteq\rho(x,y),\quad K_1(1-K_1)^{-1}\doteq R(x,y),$$
$$R(t,t):=R \tag{11}$$

(the symbol $\doteq$ denotes ''has kernel'') and

$$Q(x):=(1-K_1)^{-1}\phi,\quad q:=Q(t^-)$$
$$P(x):=(1-K_1)^{-1}\psi,\quad p:=P(t^-) \tag{12a}$$
$$u:=\int_{-t}^{t}Q(y)\phi(y)dy,\quad w:=\int_{-t}^{t}P(y)\psi(y)dy,$$

where

$$\phi(x)=\sqrt{x/2}J_{b+1/2}(x),\quad \psi(x)=\sqrt{x/2}J_{b-1/2}(x) \tag{12b}$$

(note that $K_1(x,y)=[\phi(x)\psi(y)-\phi(y)\psi(x)]/(x-y)$).

Using the facts that ϕ and ψ satisfy a pair of coupled first order differential equations, and that for b odd (even), $\phi(x)$ is even (odd) and $\psi(x)$is odd (even), from the theory of [9] we can deduce that the following equations hold:

$$tR=2(-b+u-w)pq+t(p^2+q^2)+2(pq)^2, \tag{13}$$
$$tq'=(-b+u-w)q+tp, \tag{14}$$
$$tp'=-tq-(-b+u-w)p, \tag{15}$$
$$(tR)'=p^2+q^2, \tag{16}$$
$$u'=2q^2,\quad w'=2p^2. \tag{17}$$

Also, it is easy to check from the definitions that

$$\frac{d}{dt}\ln(1-K_1)=-2R. \tag{18}$$

To derive Eq. (5a) we set

$$\sigma_1(2t):=-2tR \tag{19}$$

and integrate Eq. (18) [the factor $\pi\rho$ in the upper terminal of Eq. (5) results from changing the mean eigenvalue spacing from π to $1/\rho$]. To derive Eq. (6) we multiply Eq. (14) by p, multiply Eq. (15) by q, add and use Eq. (17) to obtain

$$(pq)'=p^2-q^2=\tfrac{1}{2}(w'-u') \tag{20}$$

and consequently

$$pq=\tfrac{1}{2}(w-u). \tag{21}$$

Substituting Eqs. (21) and (16) into Eq. (13) gives

$$tR=-2b(pq)-2(pq)^2+t(tR)', \tag{22}$$

which relates tR to pq. On the other hand, another equation relating these two quantities is obtained by squaring Eq. (16) and the first equality in Eq. (20) and subtracting:

$$((pq)')^2-((tR)')^2=-4(pq)^2. \tag{23}$$

Solving Eq. (22) for pq (the negative square root is to be taken) and $(pq)'$, substituting in Eq. (23) and introducing the notation (19) gives Eq. (6). The boundary condition (7) follows from the fact that $R(s,s)\sim K_1(s,s)$ as $s\to 0$ and the corresponding behavior of $K_1(s,s)$ deduced from Eq. (4).

To summarize, the exact evaluation of the PDF $S_{nn}(t)$ for the spacing between nearest neighbor levels in the infinite GUE has been given in terms of a certain solution of the nonlinear equation (6). This PDF can be readily calculated from empirical eigenvalue data, so our exact evaluation provides a statistical test for the hypothesis that the data have the distribution of the eigenvalues of a random Hermitian matrix. Applying this test to the zeros of the Riemann ζ function on the critical line, we have found further evidence supporting the validity of the GUE hypothesis.

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