PHYSICAL REVIEW E VOLUME 54, NUMBER 4 OCTOBER 1996

Critical dynamics of the contact process with quenched disorder

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(Received 24 April 1996; revised manuscript received 19 July 1996)

We study critical spreading in Monte Carlo simulations of the two-dimensional contact process (CP) with quenched disorder in the form of random dilution. In the pure model, spreading from a single particle at the critical point λ_c follows power laws with the critical exponents of directed percolation. With disorder, critical spreading is logarithmic not power law. Below λ_c there is a Griffiths phase in which the time dependence is governed by nonuniversal power laws. The effects of disorder are also apparent above λ_c, in the active phase, where the relaxation of the survival probability is algebraic, rather than exponential, as in the pure model. Our results support the conjecture by Bramson, Durrett, and Schonmann [Ann. Prob. **19**, 960 (1991)], that in two or more dimensions the disordered CP has only a single phase transition. [S1063-651X(96)51010-X]

PACS number(s): 05.50.+q, 02.50.−r, 05.70.Ln

Phase transitions between an absorbing state (one, that is, admitting no further evolution) and an active regime occur in models of autocatalytic chemical reactions, epidemics, and transport in disordered media. Critical phenomena attending absorbing-state transitions show a high degree of universality, characterized rather precisely in studies of the contact process (CP) and of directed percolation (DP) [1–3]. Since many-particle systems often incorporate frozen-in randomness, it is natural to investigate the effect of quenched disorder on an absorbing-state transition. Some time ago, Noest studied the critical behavior of disordered DP [4]. In this work we reexamine time-dependent critical phenomena at an absorbing-state transition in a disordered system. We focus on the two-dimensional CP, a simple lattice model of an epidemic [5]. Our primary interest is the effect of disorder on the spread of the critical process from a seed.

At a critical creation rate, λ_c, the pure CP exhibits a second-order phase transition with the same critical exponents as DP [1]. The well-known Harris criterion [6,7] states that disorder changes the critical exponents if $d\nu_\perp \leqslant 2$, where d is the dimensionality and $\nu_\perp$ is the correlation-length exponent of the pure model. Since $\nu_\perp \simeq 0.73$ for DP in $2+1$ dimensions, we expect quenched disorder to be relevant in the CP. Indeed, Noest's simulations of one- and two-dimensional stochastic cellular automata (belonging to the DP class) yielded critical exponents quite distinct from those of DP when the models were modified to incorporate quenched randomness [4]. A field-theoretic study by Obukhov [8] yielded qualitatively consistent results. Marques studied the effects of dilution on the phase diagram of the CP and related models in a mean-field renormalization group study [9]. Despite these efforts, the critical exponents for disordered DP are not known to good precision, due in part to the slow relaxation attending disorder [10].

In this paper we report extensive simulations of time-dependent critical behavior in the two-dimensional diluted contact process (DCP). In the contact process, each site of the square lattice $\mathbb{Z}^2$ is either vacant or occupied by a particle. Particles are created at vacant sites at a rate $\lambda n/4$, where n is the number of occupied nearest neighbors, and are annihilated at the unit rate, independent of the surrounding configuration. The order parameter is the particle density ρ; it vanishes in the vacuum state, which is absorbing. As λ is increased beyond $\lambda_c = 1.6488(1)$, there is a continuous phase transition from the vacuum to an active steady state; for $\lambda > \lambda_c$, $\rho \sim (\lambda - \lambda_c)^\beta$. We introduce disorder by randomly removing a fraction x of the sites. That is, for each $(i,j) \in \mathbb{Z}^2$ there is an independent random variable $\eta(i,j)$ taking values 0 and 1 with probability x and $1-x$, respectively. The DCP is simply the contact process restricted to sites with $\eta(i,j)=1$; those having $\eta(i,j)=0$ are never occupied. [Thus if exactly m neighbors of a given site have $\eta(i,j)=1$, the creation rate at that site is at most $m\lambda/4$.] Naturally, $1-x$ must exceed the square lattice site percolation threshold $p_c = 0.5927$ for there to be any possibility of an active state, since on finite sets the CP is doomed to extinction.

Following Grassberger and de la Torre [1], we study a large ensemble of trials, all starting from a configuration very close to the absorbing state: a single particle at the origin. For $\lambda > \lambda_c(x)$ there is a nonzero probability that the process survives as $t \to \infty$; for $\lambda \leqslant \lambda_c(x)$ the process dies with probability 1. Of primary interest are $P(t)$, the survival probability at time t, $n(t)$, the mean number of particles (averaged over all trials, including those that die before time t), and $R^2(t)$, the mean-square distance of particles from the origin. At the critical point of the pure CP, these quantities follow asymptotic power laws,

$$P(t) \propto t^{-\delta}, \tag{1}$$

$$n(t) \propto t^{\eta}, \tag{2}$$

$$R^2(t) \propto t^{z}. \tag{3}$$

The exponents satisfy the hyperscaling relation $4\delta + 2\eta = dz$, in $d \leqslant 4$ dimensions [1]. For $\lambda < \lambda_c$, $P(t)$ and $n(t)$ decay exponentially, while for $\lambda > \lambda_c$,

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$$\lim_{t\to\infty} P(t) \equiv P_\infty \sim (\lambda-\lambda_c)^{\beta'}, \tag{4}$$

and $n(t) \sim t^d$.

While it is tempting to suppose that the critical point of the DCP is also distinguished by asymptotic power laws, one should, in fact, expect power-law relaxation of $P(t)$ over the range $\lambda_c(0) < \lambda \leq \lambda_c(x)$ [10,11]. In this Griffiths phase the long-time dynamics are governed by atypical regions in which the fraction of diluted sites is low, rendering the process locally supercritical [12]. Briefly, the argument for power-law relaxation may be given as follows. The probability of the seed landing in a favored region, of linear size L, in which the local density of diluted sites is such that $\lambda-\lambda_{c,eff}=\Delta$, is $\sim \exp(-AL^d)$. ($\lambda_{c,eff}$ is the critical creation rate for a system with the site density prevailing in this region.) The lifetime of the process in such a region $\sim \exp(BL^d)$. (Here the precise forms of A and B are unknown, but it is clear that they are positive, increasing functions of Δ for $\Delta>0$.) It follows that at long times

$$P(t) \sim \max_{\Delta,L} \exp[-(AL^d + te^{-BL^d})] \sim \max_{\Delta} t^{-A/B} \sim t^{-\phi}, \tag{5}$$

where the last step defines a (nonuniversal) decay exponent ϕ.

Thus the criterion of power-law evolution, so useful in locating absorbing-state transitions in nondisordered models, is not applicable to the DCP [13]. Our method for finding $\lambda_c(x)$ rests instead on an analysis of P_∞ and $n(t)$. We first determine the ultimate survival probability for a series of λ values. Plotting P_∞ versus λ yields a preliminary estimate for λ_c. To refine this estimate, we observe that for $\lambda>\lambda_c$, $n(t)$ must grow monotonically at long times; for $\lambda<\lambda_c$ it must decay. In the pure CP, for example, $n(t)$ grows for $\lambda \geq \lambda_c$, so λ_c is the smallest λ supporting asymptotic growth. In the present case we wish to stay clear of assumptions regarding the sign of $dn(t)/dt$ at critical; we simply note that growth (decay) rules out a particular λ as being subcritical (supercritical). Using these conditions to winnow the set of possible critical values, we eventually find a narrow range of λ for which $n(t)$ appears *steady* at long times.

We studied dilutions $x=$ 0.02, 0.05, 0.1, 0.2, 0.3, and 0.35, on square lattices of 2200 sites to a side, using samples of from 10^4 to 2×10^6 trials for each λ value of interest, each trial extending to a maximum time of $t_{max} \leq 2\times10^6$. (As is usual in this sort of simulation, the time increment associated with an elementary event — creation or annihilation — is $\Delta t = 1/N$, where N is the number of particles. The largest samples and longest runs were used at or near critical.) An independent realization of disorder [the variables $\eta(i,j)$], is generated for each trial. The procedure outlined above yields the estimates for $\lambda_c(x)$ given in Table I. For small x, $\lambda_c(x) \approx \lambda_c(0)/(1-x)$, as predicted by mean-field theory [9,14].

Examples of $P(t)$, $n(t)$, and $R^2(t)$ are shown in Fig. 1. Of note is the slow approach of $P(t)$ to its limiting value, P_∞, in the supercritical regime, where we find that at long times $P(t) \approx P_\infty + \text{const}\times t^{-y}$, with y ranging from 1/2 (quite near critical) to 1 (at larger λ). [In cases for which $P(t)$ has yet to attain its limit at t_{max}, we use expressions of this form to estimate P_∞.] Also evident in Fig. 1 is the power-law behavior in the subcritical, Griffiths phase. Figure 2 (inset) shows P_∞ versus λ for $x=0.1$; the data for other dilutions look similar. Least-squares linear fits to plots of $\ln P_\infty$ vs $\ln(\lambda-\lambda_c)$, as in Fig. 2, yield the estimates for β' listed in Table I. For $x \geq 0.05$ the β' estimates cluster near unity; the mean is 0.99(3), not far from Noest's result, $\beta=1.10(5)$ [4]. (Our preliminary results on the stationary density yield $\beta \simeq 1$ for $x=0.35$.)

TABLE I. Critical parameters from simulations of the DCP. Numbers in parentheses indicate uncertainties.

x	λ_c	β'	a	c
0	1.6488(1)	0.586(14)		
0.02	1.6850(3)	0.566(7)		
0.05	1.7409(1)	0.97(10)		8.6(3)
0.1	1.84640(5)	0.89(4)	4.6(1)	8.1(1)
0.2	2.1080(5)	0.99(4)	3.64(14)	6.3(2)
0.3	2.470(3)	1.07(3)	3.05(15)	5.30(6)
0.35	2.719(2)	1.01(5)	2.72(5)	4.78(5)

Having located the critical point $\lambda_c(x)$, we turn to the spreading behavior. Log-log plots of $P(t)$, and $R^2(t)$ at λ_c, as shown in Fig. 3, present substantial curvature at late times, prompting us to ask whether spreading is power law or slower. (The local slopes of these graphs, commonly employed to extract estimates for spreading exponents [15], here show all three exponents decreasing sharply at long times.) By contrast, the same data approach linear asymptotes when plotted, as in Fig. 4, versus $\ln(\ln t)$. (Logarithmic decay of the survival probability has been observed in previous, less extensive simulations of the DCP [11].) For $x \geq 0.1$ expressions of the form $P(t) \sim (\ln t)^{-a}$ and $R^2(t) \sim (\ln t)^c$ fit the data over a larger range of times than do power laws. For $t \geq \tau_P(x)$, $P(t)$ is well described by a logarithmic time dependence; τ_P decreases from about 3500, for $x=0.1$, to about 60 for $x=0.35$. The approach of R^2 to a logarithmic growth law typically occurs earlier, at around

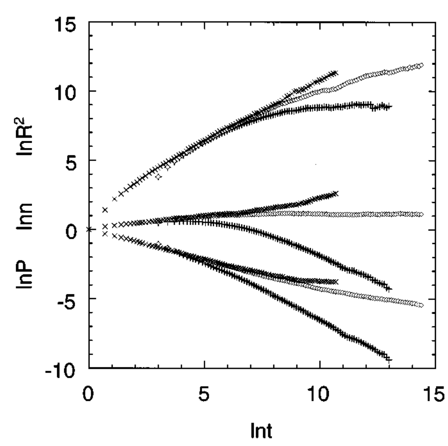

FIG. 1. Survival probability P, mean population n, and mean-square distance of particles from the origin R^2, in the diluted contact process (dilution $x=0.3$). $\times$: $\lambda=2.50$; $\diamond$: $\lambda=2.47$; $+$: $\lambda=2.40$.

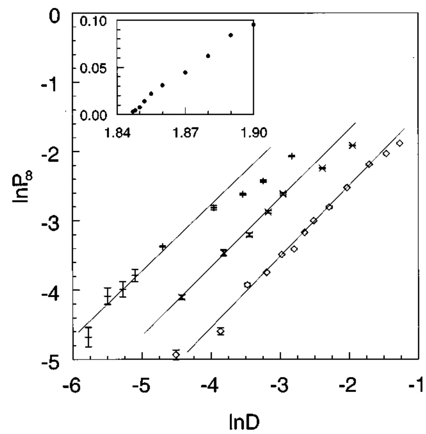

FIG. 2. Logarithmic plot of the ultimate survival probability vs $D=\lambda-\lambda_c(x)$ for dilutions $x=0.05$ (+), 0.2 (×), and 0.35 (◇). The straight lines are least-squares fits to the 6 points, 5 points, and 11 points nearest λ_c, for $x=0.05$, 0.2, and 0.35, respectively. The inset shows P_∞ vs. λ for $x=0.1$.

$\tau_P/3$. For the weakest disorder studied ($x=0.02$), we observe only (pure) DP-like spreading on the time scale of our simulations. The somewhat larger dilution of $x=0.05$ presents an intermediate case, in which the mean-square spread follows $R^2\sim t^{1.18}$ for $t<400$, and $R^2\sim(\ln t)^c$ for $t>1600$, but the survival probability is better described by a power law, $P\sim t^{-0.53}$, for $t<t_{max}=4\times10^5$. (Note that the exponent estimates are fairly close to those of the pure CP.) For these small dilutions we expect a crossover to the logarithmic forms at larger t, but have been unable to verify this, due to computational limitations. The rapid decrease in τ_P with increasing dilution can be understood by noting that for weak disorder, the process must spread over a rather large area before randomness becomes manifest; for small x, sizable regions of the lattice look nearly regular.

Since our results for critical spreading are best characterized by logarithmic time dependences, they are formally consistent with δ, η, and z all being zero. The powers a and c in the logarithmic fits for P and R^2 vary systematically, and over a substantial range, as the dilution is varied (see

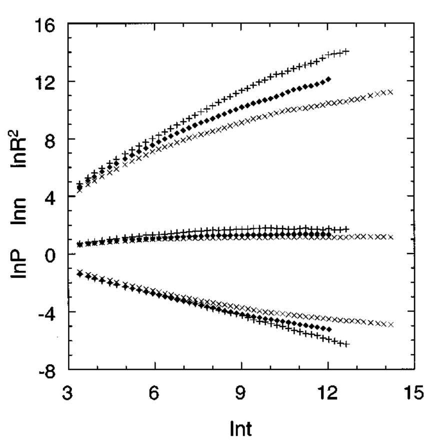

FIG. 3. Survival probability P, mean population n, and mean-square distance of particles from the origin R^2, in the critical DCP. ×: $x=0.35$, $\lambda=2.72$; ◇: $x=0.2$, $\lambda=2.108$; +: $x=0.05$, $\lambda=1.7408$.

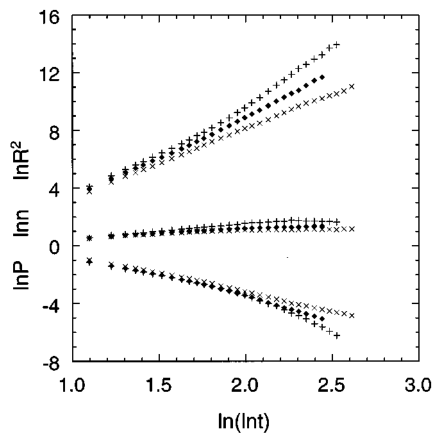

FIG. 4. The same data as in Fig. 3, but plotted versus $\ln(\ln t)$.

Table I). While we are confident that the critical exponent $\eta\simeq0$, it is possible that $n(t)\sim(\ln t)^b$ with some small $|b|$. More precise determinations of λ_c and/or of $n(t)$ at long times are required to resolve this question. The only previous determination of a spreading exponent we are aware of (for a model in this class), is Noest's result for the spreading dimension, $\hat{d}=1.61(5)$ for disordered DP in 2+1 dimensions [4]. In our notation, $\hat{d}=1+\eta+\delta$, so our simulations yield $\hat{d}=1$ (logarithmic spreading). [We obtain the same value if we extract the exponent directly from the data for $n(t)/P(t)$.] It is worth noting that our studies extend about 10 to 100 times longer in time (to at least 5×10^4, compared with 4×10^3 in Ref. [4]), and employ samples two to three orders of magnitude larger. The latter is of particular significance, since rare events appear to dominate the critical behavior in disordered systems.

As noted above, the decay of $P(t)$ should be governed by a power law in the range $\lambda_c(0)<\lambda<\lambda_c(x)$; examples of P, n, and R^2 in this regime are shown in Fig. 5. This plot confirms power-law decay, and shows that the exponents ϕ and ζ governing P and n ($\sim t^\zeta$) are nonuniversal in this regime, as expected [10,11]. When $x=0.35$, for example, we

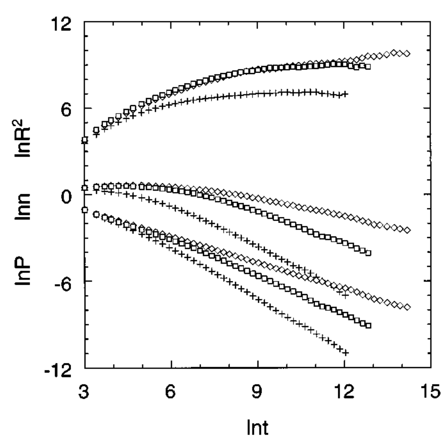

FIG. 5. Survival probability P, mean population n, and mean-square distance of particles from the origin R^2, in the subcritical DCP (Griffiths phase). +: $x=0.45$, $\lambda=3.0$; ◇: $x=0.35$, $\lambda=2.65$; □: $x=0.3$, $\lambda=2.40$.

find $\phi \approx 2.2$ for $\lambda = 2.4$, and $\phi \approx 0.6$ for $\lambda = 2.65$; the corresponding values of ζ are -2.0 and -0.4. In all cases studied, however, the asymptotic growth (if any) of R^2 seems slower than power law. (Prior to reaching a plateau, R^2 exhibits logarithmic growth.) Figure 5 includes data for $x = 0.45$, i.e., a site concentration below the percolation threshold. In this regime power-law relaxation of $P(t)$ is expected for *any* $\lambda > \lambda_c(0)$.

Bramson, Durrett, and Schonmann studied a one-dimensional CP with disorder in the form of a death rate randomly taking one of two values (independently) at each site [16]. They demonstrated that this model possesses an intermediate phase in which survival (starting, e.g., from a single particle) is possible, but the active region grows more slowly than linearly; sublinear growth has also been observed in simulations [17]. (In the pure CP the radius of the active region grows $\propto t$ for any $\lambda > \lambda_c$.) In two or more dimensions, Bramson *et al.* conjectured, there is no intermediate phase. Our results for various dilutions support this conjecture. For example, simulations at $x = 0.1$, with λ close to, but slightly above λ_c [to be precise, $\lambda = 1.86$ and 1.87, corresponding to $(\lambda - \lambda_c)/\lambda_c = 0.007$ and 0.013, respectively], showed $n(t) \sim t^2$[(and similarly for $R^2(t)$], consistent with the radius of the active region growing $\sim t$. Thus a sublinear-growth phase, if it exists at all, is confined to a very narrow range of creation rates. While our model incorporates dilution rather than a random death rate, one would expect such an intermediate phase to be a rather general feature of disordered contact processes, so that its apparent absence here argues for the validity of the conjecture.

In summary, we find that quenched disorder induces a radical change in the critical spreading of the contact process. In contrast to the well-known power laws in the pure CP, we observe logarithmic time dependence. Although our results are restricted to dilutions $0.05 \leq x \leq 0.35$, we expect a crossover to logarithmic behavior for all $0 < x < 1 - p_c$, albeit at very long times for small x. While we are inclined to suppose that the DCP is but one member of a universality class encompassing all disordered models with a continuous transition to a unique absorbing configuration, studies of absorbing-state transitions in other disordered models are needed to verify the universality hypothesis.

We are very grateful to Peter Grassberger for comments that helped us to identify a misconception in an earlier version of this paper. We thank Dani Ben-Avraham for extensive discussions and Geoff Grinstein for helpful suggestions. A.G.M. was supported by CNPq (Brazil). She thanks the staff of Lehman College, City University of New York, for their hospitality during her visit.

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