Crossover from directed percolation to compact directed percolation

J. F. F. Mendes,[1,*] R. Dickman,[2,†] and H. Herrmann,[3,4,‡]
[1]*Center for Polymer Studies and Physics Department, Boston University, Boston, Massachusetts 02215 and Department of Physics, University of Porto, Rua do Campo Alegre 687, 4150 Porto, Portugal*
[2]*Department of Physics and Astonomy, Herbert H. Lehman College, City University of New York, Bronx, New York 10468*
[3]*P.M.M.H. (U.R.A. 857), E.S.P.C.I. Paris, 10 rue Vauquelin, 73231 Paris, France*
[4]*ICA 1, University of Stuttgart, Pfaffenwaldring 27, 70569 Stuttgart, Germany*
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We study critical spreading in a surface-modified directed percolation model in which the left- and rightmost sites have different occupation probabilities than in the bulk. As we vary the probability for growth at an edge, the critical exponents switch from the compact directed percolation class to ordinary directed percolation. We conclude that the nonuniversality observed in models with multiple absorbing configurations cannot be explained as a simple surface effect. [S1063-651X(96)51609-0]

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Recently, considerable effort has been devoted to understanding phase transitions in nonequilibrium systems. Many studies of models with a continuous transition to a *unique* absorbing state have established that such transitions belong generically to the class of directed percolation (DP) [1], as predicted by Janssen [2], Grassberger [3], and (for multicomponent models) by Grinstein *et al.* [4]. DP has been of great interest since its introduction by Broadbent and Hammersley [5], and is relevant to a vast range of models in biology, chemistry and physics [6,7]. Examples are catalytic reactions on surfaces [8,4], epidemics [9], transport in porous media [10], chemical reactions [11,12], self-organized criticality [13,14], electric current in a diluted diode network [15], Reggeon field theory [16–18], and, more recently, in damage spreading [19], and models of growing surfaces with a roughening transition [20].

Unlike systems with a unique absorbing configuration, understanding of models with multiple absorbing configurations [21–25] is far from complete. In such models the critical exponents that govern spreading from a seed vary continuously with the particle density in the initial configuration, and obey a generalized hyperscaling relation [24]. The exponents assume the usual DP values only for initial configurations having the ''natural'' particle density — that of absorbing configurations generated by the system, running at the critical point. Nonuniversality in these models remains a puzzle. In searching for an explanation, and remarking that the exponents depend on the environment into which the population spreads, one is led to investigate whether nonuniversality is also produced by modifying the process at the *surface* of the active region. In this work we consider surface-modified DP in one spatial dimension, and find it does not show continuously variable exponents, but rather a crossover between two different universality classes, compact and standard DP. Besides providing a simple example connecting the two classes, our study shows that the nonuniversality observed in systems with multiple absorbing states is not simply a surface effect.

In bond directed percolation on the square lattice, bonds connect each site (x,t) with $(x,t-1)$ and $(x-1,t-1)$. Each bond is ''wet'' with probability p; with probability $1-p$ a bond is ''dry.'' Suppose the origin is the only wet site in the layer $t=0$. One constructs the cluster $\mathcal{C}_0$ connected to the origin using the rule that $(x,t)\in\mathcal{C}_0$ if it is connected, by a wet bond, to $(x,t-1)$ or $(x-1,t-1)\in\mathcal{C}_0$. For $p<p_c$ such clusters are finite with probability 1; for $p>p_c$ there is a nonzero probability of a cluster growing indefinitely. The percolation threshold, p_c, marks a continuous phase transition. The model is readily generalized by introducing conditional probabilities $P(1|1,0)=P(1|0,1)=p_1$ and $P(1|1,1)=p_2$ for site (x,t) to be wet $(=1)$, given the states of $(x,t-1)$ and $(x-1,t-1)$. [$P(1|0,0)=0$, naturally.] The resulting Domany-Kinzel model exhibits a line of critical points in the DP class [26]. The end point of this line, $(p_1=1/2,p_2=1)$, describes a transition outside the DP class; it corresponds instead to *compact directed percolation* (CDP) [27]. The essential difference between DP and CDP is that in the latter, transitions from wet to dry cannot occur within a string of 1's; the evolution of a string of 1's is governed by a pair of random walks at its ends.

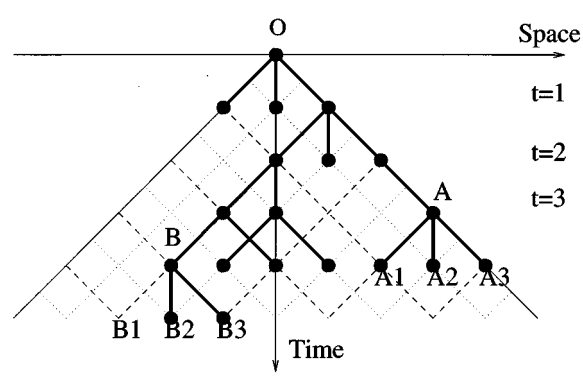

FIG. 1. The *centered square lattice* and a growing cluster.

*Electronic address: jfmendes@sid3.bu.edu
†Electronic address: dickman@lcvax.lehman.cuny.edu
‡Electronic address:hans@pmmh.espci.fr

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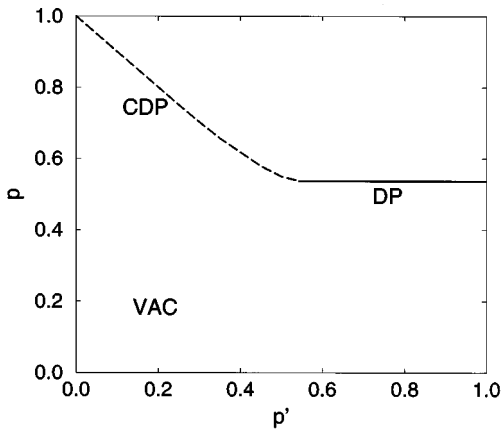

FIG. 2. Phase diagram in p-p' plane. The two regimes (CDP and DP) are shown.

In the present work we study directed percolation on the *centered square lattice*, in which (x,t) is connected to $(x-1,t-1)$, $(x,t-1)$, and $(x+1,t-1)$ by bonds that, as before, are wet with probability p (see Fig. 1). Earlier studies of this model [28,29] yielded a percolation threshold of $p_c=0.5387\pm0.0003$. We now modify the probabilities for introducing wet sites at the surface of the active region. Suppose (x_R,t) is the rightmost wet site in layer t. We set the probability for $(x_R+1,t+1)$ to be wet at p' rather than p, and similarly for $(x_L-1,t+1)$, if (x_L,t) is the leftmost wet site in layer t. In Fig. 1, for example, the probabilities to introduce wet sites at A_1, A_2, and A_3, are p, p, and p', respectively. (Similarly, we have probabilities p', p, and p for introducing wet sites at B_1, B_2, and B_3.)

We used *time-dependent* simulations to study the critical behavior. The method involves starting with a single wet site at $t=0$ and following the spread of wet sites in a large set of independent realizations. We measured the survival probability $P(t)$ (the probability that there is at least one wet site at time t), the average number of wet sites, $\overline{n(t)}$, and the average mean square distance of spreading from the origin, $\overline{R^2}(t)$. At criticality, these quantities follow power laws in the long-time limit, e.g., $P(t)\sim t^{-\delta}$, and similarly for the other quantities. In the subcritical phase $[p<p_c(p')]$ the survival probability decays exponentially, while in the supercritical phase $[p>p_c(p')]$, it approaches $P_\infty\equiv\lim_{t\to\infty}P(t)>0$, the *ultimate survival probability*. If we allow the exponents to depend on the surface probability, p', the simplest generalization of the usual scaling hypothesis [12] is

$$P(t)\sim t^{-\delta(p')}\Phi(\Delta t^{1/\nu_\parallel(p')}) \tag{1}$$

$$\overline{n(t)}\sim t^{\eta(p')}\Theta(\Delta t^{1/\nu_\parallel(p')}) \tag{2}$$

$$\overline{R^2}(t)\sim t^{z(p')}\Omega(\Delta t^{1/\nu_\parallel(p')}) \tag{3}$$

where $\Delta=|p-p_c(p')|$ measures the distance to the critical point and $\nu_\parallel(p')$ is the critical exponent associated with the correlation length in the time direction ($\xi_\parallel\sim\Delta^{-\nu_\parallel(p')}$). Assuming that the scaling functions Φ, Θ, and Ω are nonsingular at the critical point, it follows that $P(t)$, $\overline{n(t)}$, and $\overline{R^2}(t)$ follow pure power laws as $t\to\infty$. A log-log plot of P, $\overline{n}$, or $\overline{R^2}$ as a function of time should yield a straight line at $p=p_c(p')$, permitting one to determine $p_c(p')$ rather precisely. The exponents δ, η, and z are given by the asymptotic slopes of the corresponding plots. In addition, the exponent β' controlling the approach to the critical point of the survival probability, $P_\infty(p)\sim[p-p_c(p')]^{\beta'(p')}$ [30], may also depend on p'. Our results (see Fig. 2) show that the percolation threshold p_c increases with decreasing p' for $p'<p_c^{DP}=0.538\,75$. This decrease in p' means that the percolating cone becomes narrow, and a larger value of p is needed for the cluster to survive.

Table I shows how the exponents vary with p'. The mean number of wet sites in surviving trials is $N(t)=\overline{n(t)}/P(t)$, and the fractal dimension d_f of aggregates surviving to time t is defined through $N(t)\sim R^{d_f}$, yielding $d_f=2(\delta+\eta)/z$. For DP, $d_f=0.752$. In the present study $d_f\approx1$ for $p'<p_c^{DP}$, which means that for this range of p' the clusters are no longer fractal but *compact*. From Fig. 3 we see that the exponent controlling the survival probability

TABLE I. Values of the critical exponents and threshold percolation for different values of p'. The values for DP were taken from Refs. [4], [31], [32], and [33]. The numbers in parentheses represent the uncertainty in the last digit(s).

p'	p_c	δ	η	$z/2$	β'
DP		0.1596(4)	0.312(2)	0.6321(4)	0.2767(4)
CDP		1/2	0	1	1
0.0	1.0				
0.10	0.9005(3)	0.495(7)	0.016(14)	0.502(8)	
0.25	0.7510(4)	0.496(5)	0.015(13)	0.505(7)	0.97(4)
0.30	0.7035(5)	0.498(8)	0.022(15)	0.509(13)	
0.35	0.6575(5)	0.493(4)	0.018(13)	0.505(11)	0.95(8)
0.45	0.5792(5)	0.497(7)	0.013(16)	0.504(6)	
0.50	0.55012(4)	0.499(5)	0.011(13)	0.506(8)	
p_c	0.53875(1)	0.159(1)	0.310(3)	0.632(2)	0.28(2)

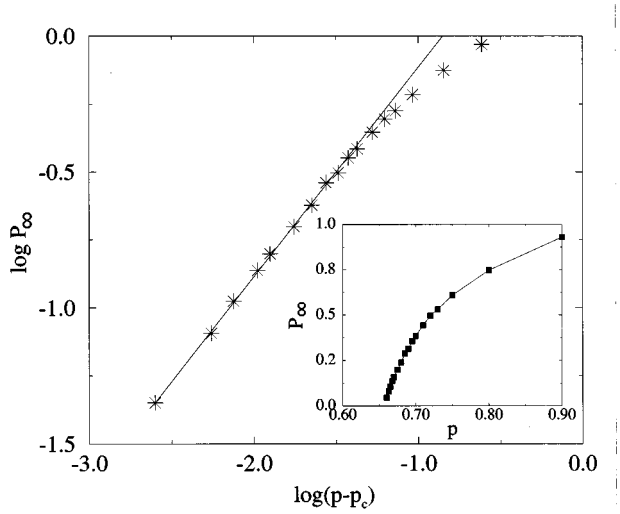

FIG. 3. The ultimate survival probability, P_∞ as a function of $p-p_c(p')$ in a log-log plot. The inset is a plot of P_∞ vs p. These results were obtained for $p'=0.25$ and the slope of the line is $\beta=1$.

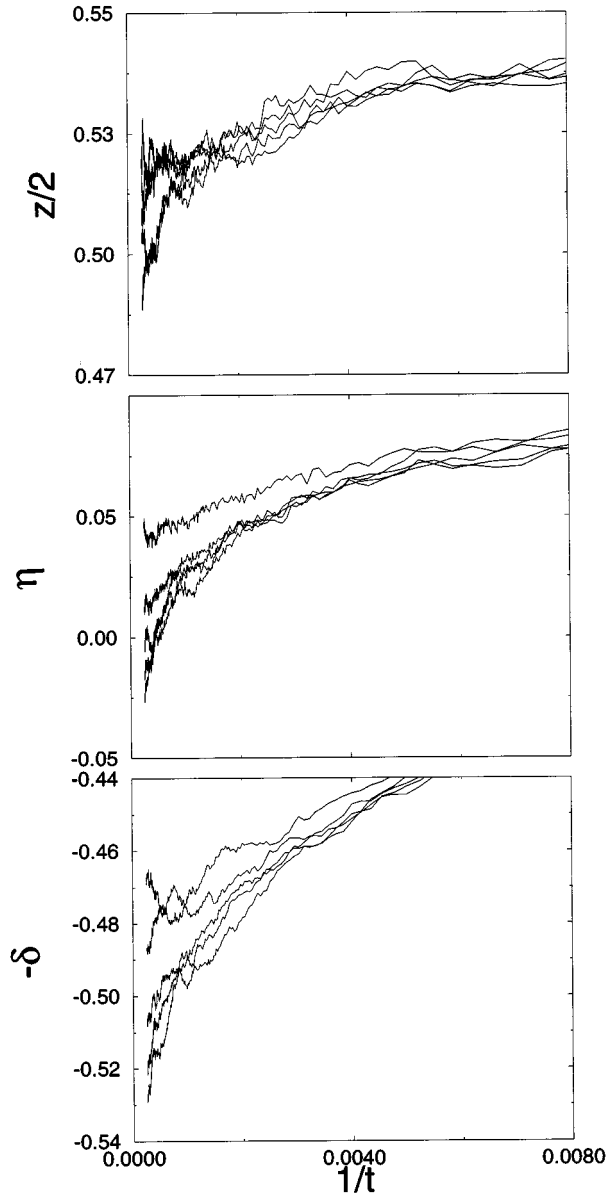

FIG. 4. The local slopes $-\delta(t)$ (upper), $\eta(t)$ (middle), and $z(t)/2$ (bottom), for the case $p'=0.25$. Each picture has five curves corresponding from bottom to top: $p=0.7506$, 0.7508, 0.7510, 0.7514, and 0.7516.

($\beta'=0.97\pm0.04$) is very different from that of DP ($\beta'=\beta=0.2767\pm0.0004$). For $p'<p_c^{DP}$, we find $\beta=0$, so that the generalized hyperscaling relation, $(1+\beta/\beta')\delta+\eta=dz/2$ [24], becomes $\delta+\eta=dz/2$, which describes compact growth [34]. The data are indeed consistent with compact hyperscaling. (For the point nearest p_c, P_∞ is obtained from data for $t>15\,000$, averaged over 10^5 independent runs.) Figure 4 shows the local slopes as a function of inverse time for $p'=0.25$. Similar curves were obtained for the other values of p'. (These results reflect averages over 10^5–5×10^5 independent trials, running to a maximum time of 2000 to 6000.) We also checked whether the scaling relations, Eqs. (1)–(3), are obeyed, by varying $\nu_{||}$ to optimize the data collapse. For $p'<p_c^{DP}$, we observe scaling for $\nu_{||}=2$, as expected for CDP. Our numerical results indicate that the

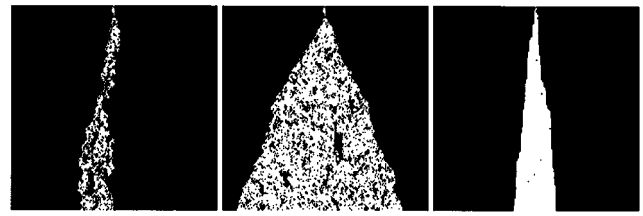

FIG. 5. Typical evolutions from a single wet site for $p=p'=0.45$, $p=p'=0.5387$, and $p=0.9$, $p'=0.1$ respectively. Time increases downward.

model exhibits a crossover from directed percolation to compact directed percolation as the surface growth rate, p', is varied (See Fig. 5). We can understand this by examining a surface-modified contact process (CP) [9]. The CP (in d space dimensions) is a sequentially updated version of DP (in $d+1$ dimensions). In the CP, each lattice site is either vacant or occupied by a particle. Particles die at unit rate, and give birth at rate λ. (In a birth event a new particle appears at one of the sites neighboring the parent, if that site is vacant.) The CP exhibits a phase transition in the DP class, as λ is increased through λ_c.

Now consider the one-dimensional CP, with x_R and x_L denoting the positions of the rightmost and leftmost particles at any instant, and modify the rules so that the rate for a particle to appear at x_R+1, or at x_L-1, is $\lambda'/2$. If we let $L=\langle x_R-x_L\rangle$, the mean being taken over all surviving trials at time t, starting from a single particle, then $dL/dt=\lambda'-2g$, where g is the mean distance from x_L to the next occupied site (and similarly at the right edge). If we set $\lambda'=\lambda$, there is a critical point at λ_c, in which case the interior density approaches zero as $t\to\infty$. Since $L\sim t^{z/2}$ at the critical point, $dL/dt\to0$ as $t\to\infty$ as well. (For $\lambda<\lambda_c$, $dL/dt<0$, and vice versa.) Hence, for $\lambda'=\lambda=\lambda_c$, g must tend to $\lambda_c/2$ at long times. The ultimate survival probability P_∞ can only be nonzero if $dL/dt>0$ at any finite time. On the other hand, g is a decreasing function of λ and λ'. It follows that if $\lambda'<\lambda=\lambda_c$, the process must die. We can compensate the reduced surface birth rate by increasing λ, and should again have survival for $g(\lambda,\lambda')\leqslant\lambda'/2$. Having forced $\lambda>\lambda_c$, surviving trials are compact, i.e., have a nonzero particle density as $t\to\infty$. The region between x_L and x_R is not simply a string of occupied sites, as it would be in CDP, but the scaling behavior should nonetheless be characteristic of compact growth. When the surface growth rate *exceeds* the bulk critical value, we expect a continuous (DP-like) transition at $\lambda=\lambda_c$. For $\lambda'>\lambda_c$, the advance of the edges may be enhanced at short times, but the process cannot survive for $\lambda<\lambda_c$ because the interior approaches the vacuum. The density in surviving clusters is localized at the edges, so the number of particles remains $\mathcal{O}(1)$, and the process dies with probability 1. To summarize, our argument shows why surface-modified DP exhibits compact growth for $p\geqslant p_c(p')$ when $p'<p_c^{DP}$, and why there is a DP-like transition at p_c^{DP} for $p'\geqslant p_c^{DP}$, As in CDP, the transition is *first order* for $p'<p_c^{DP}$.

Clearly, the surface probability p' does not play the same

role as the initial density in models with multiple absorbing configurations. Conversely, the continuously variable exponents observed in the latter class of models cannot be explained as a simple surface effect. Finally, it is interesting to note that while the surface modification explored in the present work represents a nonlocal *spatial* interaction (x_R and x_L are defined globally), the dynamics of models with multiple absorbing configurations involves long *memory* effects [35].

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