Anomalous diffusion in the presence of external forces: Exact time-dependent solutions and their thermostatistical basis

Constantino Tsallis
*Department of Chemistry, Baker Laboratory, and Materials Science Center, Cornell University, Ithaca, New York 14853-1301 and Centro Brasileiro de Pesquisas Físicas, Rua Xavier Sigaud 150, 22290-180 Rio de Janeiro, Rio de Janeiro, Brazil**

Dirk Jan Bukman
Department of Chemistry, Baker Laboratory, Cornell University, Ithaca, New York 14853-1301
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Driven anomalous diffusions (such as those occurring in some surface growths) are currently described through the *nonlinear* Fokker-Planck-like equation $(\partial/\partial t)\,p^{\mu}=-(\partial/\partial x)\,[F(x)p^{\mu}]+D\,(\partial^2/\partial x^2)\,p^{\nu}$ $[(\mu,\nu)\in\mathcal{R}^2$; $F(x)=k_1-k_2x$ is the external force; $k_2\geqslant 0]$. We exhibit here the (physically relevant) *exact* solution for all (x,t). This solution was found through an ansatz based on the generalized entropic form $S_q[p]=\{1-\int du[p(u)]^q\}/(q-1)$ (with $q\in\mathcal{R}$), in a completely analogous manner through which the usual entropy $S_1[p]=-\int du p(u)\ln p(u)$ is known to provide the correct ansatz for exactly solving the standard Fokker-Planck equation $(\mu=\nu=1)$. This remarkably simple unification of normal diffusion $(q=1)$, superdiffusion $(q>1)$ and subdiffusion $(q<1)$ occurs with $q=1+\mu-\nu$. [S1063-651X(96)51209-2]

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Anomalous diffusion is intensively studied nowadays, both theoretically and experimentally. It is observed, for instance, in CTAB micelles dissolved in salted water, the analysis of heartbeat histograms in a healthy individual, financial transactions, chaotic transport in laminar fluid flow of a water-glycerol mixture in a rapidly rotating annulus, subrecoil laser cooling [1], particle chaotic dynamics along the stochastic web associated with a $d=3$ Hamiltonian flow with hexagonal symmetry in a plane [2], conservative motion in a $d=2$ periodic potential [3], transport of fluid in porous media (see [4] and references therein), surface growth [4], NMR relaxometry of liquids in porous glasses [5] and many other interesting physical systems. A paradigmatic class of (correlated) anomalous diffusions are currently described through the following generalized *nonlinear* Fokker-Planck equation:

$$\frac{\partial}{\partial t}[p(x,t)]^{\mu}=-\frac{\partial}{\partial x}\{F(x)[p(x,t)]^{\mu}\}+D\frac{\partial^2}{\partial x^2}[p(x,t)]^{\nu}, \tag{1}$$

where $(\mu,\nu)\in\mathcal{R}^2$, $D>0$ is a (dimensionless) diffusionlike constant, $F(x)\equiv -dV(x)/dx$ is a (dimensionless) external force (drift) associated with the potential $V(x)$, and (x,t) is a (dimensionless) $1+1$ space time. Since $D(\partial^2/\partial x^2)\,[p(x,t)]^{\nu}=(\partial/\partial x)\,\{D\nu[p(x,t)]^{\nu-1}\,(\partial/\partial x)\,p(x,t)\}$, the possible nonlinearity introduced in Eq. (1) has a simple physical interpretation for $\mu=1$. Indeed, there are various real situations in which the standard diffusion coefficient depends on $p(x,t)$. It is then clear that $\nu>1$ $(\nu<1)$ corresponds to the case where, for some reason, the p-dependent diffusion coefficient vanishes (diverges) for $p=0$. This is not a very rare case. Indeed, it occurs in percolation of gases through porous media ($\nu\geqslant 2$ [6]), thin saturated regions in porous media ($\nu=2$ [7]), thin liquid films spreading under gravity ($\nu=4$ [8]), radiative heat transfer by Marshak waves ($\nu=7$ [9]), solid-on-solid model for surface growth ($\nu=3$ [4]), among others (see also [10]). With no loss of generality we could assume that $\mu=1$ by renaming $\tilde{p}(x,t)\equiv[p(x,t)]^{\mu}$ and $\tilde{\nu}\equiv\nu/\mu$, but, for reasons that will become clear later on, we shall discuss Eq. (1) as it stands. We intend to consider here a specific (but very common) drift, namely characterized by $F(x)=k_1-k_2x$ $(k_2\geqslant 0$; $k_2=0$ corresponds to the important case of constant external force, and $k_1=0$ corresponds to the so called Uhlenbeck-Ornstein process). The particular case $\mu=\nu=1$ corresponds to the standard Fokker-Planck equation, i.e., to *normal* diffusion. The particular case $F(x)=0$ (no drift) has been considered by Spohn [4] for $\mu=1$ and arbitrary ν, and has been extended by Duxbury [11] for arbitrary μ and ν. The case $(\mu,k_1)=(1,0)$ has been considered by Plastino and Plastino [12]. Our present discussion recovers all of these as particular instances. On one hand, it will produce the *exact* solutions of Eq. (1) for all (x,t) and arbitrary (μ,ν,D,k_1,k_2), and on the other hand, it will enlighten their thermostatistical basis.

Indeed, the thermostatistical foundation of *anomalous* diffusion (as it is known for *normal* diffusion) is naturally highly desirable, and has long been looked for (see, for instance, [13] and references therein). This goal was recently achieved by Alemany and Zanette [14] and others [15,16] for Lévy-like anomalous diffusion, in the context of a generalized, not necessarily extensive (additive), thermostatistics that has been recently proposed [17]. In particular, within this framework, the ubiquity and robustness of Lévy distributions in nature has been thermostatistically founded on the Lévy-Gnedenko central limit theorem [16]. This thermostatistics has already been applied to a considerable variety of physical systems which include self-gravitating stellar objects, the hydrogen atom, the cosmic background radiation,

*Permanent address. Electronic address: tsallis@cat.cbpf.br

ferrofluidlike liquids, and apparent magnetism [18] (see also [19] for the linear response theory). It has also been applied with success to optimization techniques [20]. Finally, it has been experimentally checked for a specific pure-electron-plasma turbulence, the solar neutrino problem, and galaxy clusters [21]. This nonextensive formalism is based upon the entropic form

$$S_q[p] \equiv \frac{1-\int du[p(u)]^q}{q-1} \quad (q \in \mathcal{R}), \tag{2}$$

which reduces, in the $q \to 1$ limit, to the standard Boltzmann-Gibbs entropy

$$S_1[p] \equiv -\int du\, p(u) \ln p(u). \tag{3}$$

First, let us illustrate the procedure we intend to follow, by briefly reviewing normal diffusion ($\mu=\nu=1$). We wish to optimize S_1 [given by Eq. (3)] with the constraints

$$\int du\, p(u) = 1 , \tag{4}$$

$$\langle u-u_M \rangle_1 \equiv \int du(u-u_M)p(u) = 0 , \tag{5}$$

and

$$\langle (u-u_M)^2 \rangle_1 \equiv \int du(u-u_M)^2 p(u) = \sigma^2, \tag{6}$$

u_M and σ being fixed *finite* real quantities. The optimization straightforwardly yields the solution

$$p_1(u) = \frac{e^{-\beta(u-u_M)^2}}{Z_1} \tag{7}$$

with

$$Z_1 = \int du\, e^{-\beta(u-u_M)^2} = (\pi/\beta)^{1/2} \tag{8}$$

where $\beta \equiv 1/T$ is the Lagrange parameter associated with the constraint (6) and satisfies $\beta = 1/(2\sigma^2)$. On the basis of Eq. (7) we propose, for the $\mu=\nu=1$ particular case of Eq. (3) (i.e., the standard Fokker-Planck equation), the ansatz

$$p_1(x,t) = \frac{e^{-\beta(t)[x-x_M(t)]^2}}{Z_1(t)} \tag{9}$$

with

$$\frac{\beta(t)}{\beta(0)} = \left[\frac{Z_1(0)}{Z_1(t)}\right]^{\lambda}. \tag{10}$$

It follows straightforwardly that $\lambda = 2$,

$$\frac{\beta(t)}{\beta(0)} = \left[\frac{Z_1(0)}{Z_1(t)}\right]^2 = \frac{1}{\left[1-\dfrac{2D\beta(0)}{k_2}\right]e^{-2k_2 t} + \dfrac{2D\beta(0)}{k_2}} \tag{11}$$

and

$$\frac{dx_M(t)}{dt} = k_1 - k_2 x_M(t), \tag{12}$$

hence

$$x_M(t) = \frac{k_1}{k_2} + \left[x_M(0) - \frac{k_1}{k_2}\right] e^{-k_2 t}. \tag{13}$$

To discuss the $k_2=0$ case we can use $e^{-2k_2 t} \sim 1-2k_2 t$, which implies $x_M(t) = x_M(0) + k_1 t$ and $1/\beta(t) = [1/\beta(0)] + 4Dt$, which, in the limit $t \to \infty$ (i.e., $t \gg 1/[4D\beta(0)]$), yields the familiar result $1/\beta(t) \sim 4Dt$. This result implies, in turn, the celebrated Einstein expression $\langle (x-x_M)^2 \rangle_1 \propto t$ for Brownian motion.

Let us now address the general (μ,ν) case. Following along the lines of Alemany and Zanette [14] (and the generic framework of the generalized thermostatistics [17]) we now wish to optimize S_q [given by Eq. (2)]. The constraints are Eq. (4),

$$\langle u-u_M \rangle_q \equiv \int du(u-u_M)[p(u)]^q = 0 \tag{14}$$

[which generalizes Eq. (5)] and

$$\langle (u-u_M)^2 \rangle_q \equiv \int (u-u_M)^2 [p(u)]^q = \sigma^2 \tag{15}$$

[which generalizes Eq. (6)]. This is an appropriate moment for commenting that the reason for using $[p(u)]^q$ [instead of the familiar $p(u)$] in the constraints (14) and (15) is the (very essential) fact that by doing so we preserve [17] the Legendre structure of thermodynamics and (through the non-negativity of C_q/q [22], where C_q denotes the specific heat) guarantee thermodynamic stability. Let us consistently stress that the constraint (14) is equivalent to $\langle u \rangle_q = \langle u_M \rangle_q$, but not to $\langle u \rangle_q = u_M$ (since, unless $q=1$, $\langle u_M \rangle_q \neq u_M$). All these peculiarities are, of course, originated by the *nonextensivity* that the index q introduces in the theory. For example, if we have two *independent* systems A and B [i.e., $p_{A*B}(u_A, u_B) = p_A(u_A) p_B(u_B)$], we immediately verify that $S_q(A*B) = S_q(A) + S_q(B) + (1-q) S_q(A) S_q(B)$.

It is straightforward to see that the above described optimization of S_q yields

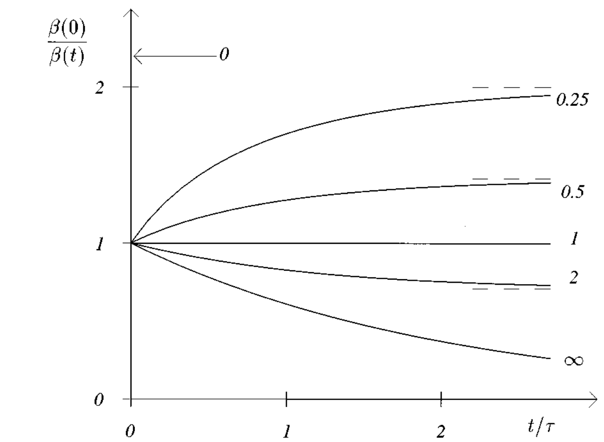
$$p_q(u) = \frac{[1-\beta(1-q)(u-u_M)^2]^{1/(1-q)}}{Z_q} \tag{16}$$

with

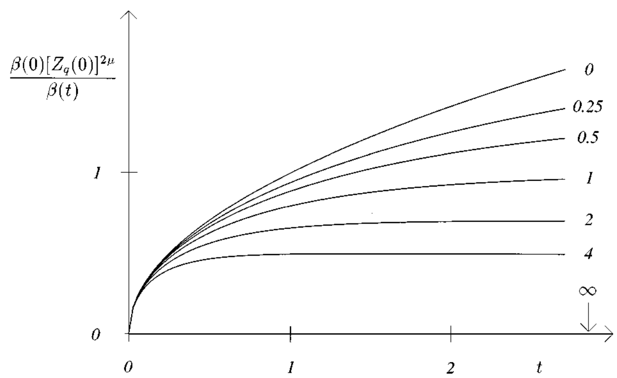
$$Z_q = \int du[1-\beta(1-q)(u-u_M)^2]^{1/(1-q)} \tag{17}$$

In the limit $q \to 1$, these equations reduce to Eqs. (7) and (8), respectively. The corresponding ansatz for solving Eq. (1) now is

$$p_q(x,t) = \frac{\{1-\beta(t)(1-q)[x-x_M(t)]^2\}^{1/(1-q)}}{Z_q(t)} \tag{18}$$



(a)



(b)

FIG. 1. The $\mu/\nu=1/3$ example: (a) Time dependence of $\beta(0)/\beta(t)=[Z_q(t)/Z_q(0)]^{2\mu}$ for $Z_q(0)\neq 0$ and typical values of K_2 (indicated at the right of each curve). The curve for $K_2=0$ lies on the vertical axis. For $K_2=0.25$, 0.5, and 2 the asymptotic values for $t/\tau\to\infty$ are shown by the dashed lines [τ is defined in Eq. (23)]. (b) Time dependence of $\{\beta(0)[Z_q(0)]^{2\mu}\}/\beta(t)=[Z_q(t)]^{2\mu}$ for $Z_q(0)=0$, $\beta(0)[Z_q(0)]^{2\mu}\neq 0$, and typical values of $K_2'\equiv k_2/\{2\nu D\beta(0)[Z_q(0)]^{2\mu}\}$ (indicated at the right of each curve). The curve for $K_2=\infty$ coincides with the horizontal axis. All curves saturate at a finite value as $t\to\infty$, except that for $K_2'=0$, which is proportional to $t^{2\mu/(\mu+\nu)}$ for all t.

with

$$\frac{\beta(t)}{\beta(0)}=\left[\frac{Z_q(0)}{Z_q(t)}\right]^{\lambda} \tag{19}$$

[as before, $\beta(t)$ and $Z_q(t)$ are nothing but the scaling of space with time]. A tedious (but straightforward) calculation yields $\lambda=2\mu$, and $q=1+\mu-\nu$. An equation for $Z_q(t)$ is also found; namely,

$$2\nu D\beta(0)[Z_q(0)]^{2\mu}-k_2[Z_q(t)]^{\mu+\nu} - \frac{\mu}{\mu+\nu}\frac{d[Z_q(t)]^{\mu+\nu}}{dt}=0, \tag{20}$$

which can be solved by substituting $\tilde{Z}(t)=Z_q(t)^{\mu+\nu}$. The resulting solution is (for all values of k_1)

$$Z_q(t)=Z_q(0)\left[\left(1-\frac{1}{K_2}\right)e^{-t/\tau}+\frac{1}{K_2}\right]^{1/(\mu+\nu)}, \tag{21}$$

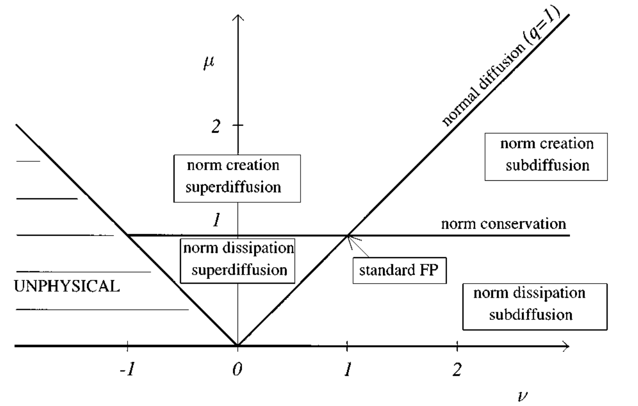


FIG. 2. ‘‘Norm conservation’’ means that $N\equiv\int dx p_q(x,t)$ is time invariant; ‘‘norm creation’’ means that N monotonically increases (decreases) with time if $K_2<1$ ($K_2>1$); ‘‘norm dissipation’’ means that N monotonically decreases (increases) with time if $K_2<1$ ($K_2>1$). ‘‘Normal diffusion,’’ ‘‘superdiffusion,’’ and ‘‘subdiffusion’’ refer to the fact that, for $k_2=0$, $(x-x_M)^2$ scales like t, faster than t, and slower than t, respectively. The standard Fokker-Planck equation corresponds to $\mu=\nu=q=1$. For the precise meaning of ‘‘unphysical,’’ see the text. On the $\mu=1$ line we have $q=2-\nu$; consequently, when ν varies from ∞ to -1, q varies from $-\infty$ to 3, which is precisely the interval within which Eq. (4) [and, consistently, $\int dx p_q(x,0)=1$] can be satisfied.

with

$$K_2\equiv\frac{k_2}{2\nu D\beta(0)[Z_q(0)]^{\mu-\nu}} \tag{22}$$

and

$$\tau\equiv\frac{\mu}{k_2(\mu+\nu)} \tag{23}$$

(see Fig. 1). The function $x_M(t)$ is the same as in the case of normal diffusion, Eq. (13), since it only describes the motion of the average of the distribution $p_q(x,t)$, and does not depend on the way in which it spreads. $\beta(0)$ and $Z_q(0)$ are determined by the initial condition [i.e., by $p_q(x,0)$]. For $k_2=0$, Eq. (21) becomes

$$Z_q(t)=\left\{[Z_q(0)]^{\mu+\nu} + \frac{2\nu(\nu+\mu)D\beta(0)[Z_q(0)]^{2\mu}}{\mu}t\right\}^{1/(\mu+\nu)} \tag{24}$$

which, for $t\to\infty$, asymptotically recovers Duxbury's solution [11], namely, $1/\beta(t)\propto[Z_q(t)]^{2\mu}\propto t^{2\mu/(\mu+\nu)}$. As we see, $\mu/\nu=1$, >1 and <1 respectively imply that $[x(t)-x_M(t)]^2$ scales like t (normal diffusion), faster than t (superdiffusion), and slower than t (subdiffusion). The limits $\mu/\nu=0$ and $\mu/\nu=\infty$ correspond to ‘‘no diffusion’’ and ballistic motion, respectively. For $(\mu,k_1)=(1,0)$, the present set of equations reduces to that of Plastino and Plastino [12]. Finally, by using Eq. (19) with $\lambda=2\mu$, we can verify that

$$\int dx p_q(x,t)=[Z_q(t)/Z_q(0)]^{\mu-1}\int dx p_q(x,0). \tag{25}$$

Consequently, the norm (''total mass'') is generically conserved for all times only if $\mu=1$ ($\forall K_2$) or if $K_2=1$ ($\forall \mu$). For $0 \leqslant K_2 < 1$ (a common case), the norm monotonically increases (decreases) with time if $\mu>1$ ($\mu<1$). If $K_2>1$, it is the other way around.

Before ending let us mention that when t grows to infinity, the solutions we have found must be physically meaningful. This imposes $\mu/\nu>-1$. Indeed, if $k_2 \neq 0$, τ in Eq. (21) must be positive, which implies $\mu/\nu>-1$. Also, if $k_2=0$, x must scale with increasing function of t; hence, $\beta(t)$ must decrease with t, which implies [through Eqs. (19) and (24)] $2\mu/(\mu+\nu)>0$, hence, the already mentioned restriction applies once again. The entire picture that emerges is indicated in Fig. 2 (we have not focused on the $\mu<0$ region because that would force us to discuss the stability of the solutions with respect to small departures, and this lies outside of the scope of the present work).

To summarize, on general grounds, we have shown that thermostatistics allowing for nonextensivity constitute a theoretical framework within which a rather nice unification of normal and correlated anomalous diffusions can be achieved. Both types of diffusions have been founded, on equal footing, on primary concepts of (appropriately generalized) thermodynamics and information theory. Moreover, the Lévy-like anomalous diffusion (see [16,23,24] and references therein) is possibly describable (suggestion by Zaslavsky [2]) by a *linear* Fokker-Planck-like equation with *fractional* (time and space) derivatives, and the present correlated anomalous diffusion is described by a *nonlinear* Fokker-Planck-like equation with *integer* derivatives (in contrast with normal diffusion, which corresponds to the *linear* Fokker-Planck equation with *integer* derivatives). Since both types of anomalous diffusions can be handled within the present generalized thermostatistics, the conjecture is allowed that further unification can possibly be achieved by considering the generic case of a *nonlinear* Fokker-Planck-like equation with *fractional* derivatives. On specific grounds, we have obtained, for a generic linear force $F(x)$, the physically relevant exact (space, time)-dependent solutions of a considerably generalized Fokker-Planck equation, namely, Eq. (1).

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