Phase transitions in nonlinear oscillator chains

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It is shown numerically that a one-dimensional system of coupled disparate nonlinear oscillators undergoes a phase transition from a synchronized to a desynchronized state as the range of interactions is decreased. Using a coupling that decreases with distance as $r^{-\alpha}$, the functional dependence of the critical coupling exponent on the coupling constant $\alpha_c(K)$ is mapped out and the nature of the transition is discussed. Previously studied models and results are recovered in the appropriate limits of the coupling exponent. [S1063-651X(96)50909-8]

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Spontaneous synchronization among an ensemble of similar constituent elements is a phenomenon that allows for the proper functioning of many biological and physical systems. A dramatic example from biology occurs when the members of certain species of fireflies overcome their individual rhythms to begin flashing in unison [1]. This form of spontaneous collective behavior may also be important in physical systems. In applications of Josephson junctions, one is often interested in the behavior of a whole array. Within the array these quantum devices possess a distribution in intrinsic frequencies due to variations in resistances and critical currents. If the junctions can overcome this frozen disorder and attain the in-phase state, power output is expected to increase proportionally to the number of junctions [2]. Other examples of such collective behavior include cardiac rhythms, lasers, and neural activity [1,3].

These systems have been modeled as ensembles of coupled nonlinear oscillators with distributed frequencies [1,3–10]. Quantifying this approach, Wiesenfeld, Colet, and Strogatz recently mapped a series array of Josephson junctions onto such a model with mean-field interactions [4]. As a result, they were able to make quantitative predictions about the conditions necessary for such an array to synchronize. Although the junctions as well as the other given examples are nonequilibrium systems, usually considered at zero temperature, they show many similarities to ensembles from equilibrium statistical physics. For example, in spin systems the synchronized state might be thought of as a ferromagnetic arrangement. Thus it has become quite common to refer to ''phase transitions'' in such oscillator communities when some control parameter is varied [5–7].

In this spirit, Kuramoto [8] introduced a mean-field model from which he was able to solve for the critical coupling necessary for synchronization to occur. His results showed that a system possessing this type of ''all-to-all'' interaction would always be able to synchronize for some finite value of the coupling constant, provided the spread in the distribution of natural frequencies was not too large. Kuramoto and collaborators, as well as other investigators, extended this work by considering ensembles with nearest-neighbor interactions. For a one-dimensional chain, Strogatz and Mirollo [9] presented proof that synchronization was not possible in the thermodynamic limit. Similar conclusions were reached by Daido [10] using an analysis similar to renormalization group. One might expect that this would be due to rare or extreme fluctuations in the tails of the frequency distribution (a phenomenon known in other fields [11]), but it turns out that the inability to synchronize persists if the tails of the distribution are removed.

Thus it has previously been found that synchronization is an accessible state for a one-dimensional ensemble in the thermodynamic limit if it possesses mean-field interactions, while it is not attainable with nearest-neighbor coupling. It is the purpose of this article to investigate the requirements of interactions among the oscillators such that the synchronized state is spontaneously attainable in this limit. To this end, we introduce a one-dimensional model with interactions that decay with lattice separation according to a power law, $r^{-\alpha}$. For a fixed coupling constant, it is shown that this system of coupled nonlinear oscillators undergoes a transition from a state that will spontaneously synchronize to one that will remain desynchronized as the range of interactions is decreased. We find that there exists a critical coupling exponent dependent on the coupling constant $\alpha_c(K)$. The functional dependence is mapped out and the nature of the transition between the synchronized and desynchronized states is discussed. Since several of the systems mentioned as examples are effectively one-dimensional, these results show that the interactions must be sufficiently long ranged if the system is to reside in the synchronized state. The proper function of these systems may therefore be dependent on their ability to maintain a sufficiently long interaction range.

The oscillator model is governed by the equation

$$\dot{\theta}_j=\omega_j+\frac{K}{\eta}\sum_{r=1}^{N'} r^{-\alpha}[\sin(\theta_{j+r}-\theta_j)+\sin(\theta_{j-r}-\theta_j)], \tag{1}$$

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where $(j=1,\ldots,N)$, $N'=(N-1)/2$, and N is the number of oscillators taken to be odd. Periodic boundary conditions are assumed. The coefficient η is a normalization factor that allows interpolation between the nearest-neighbor and mean-field limits:

$$\eta=2\sum_{i=1}^{N'} i^{-\alpha}. \tag{2}$$

The natural frequencies ω_j are chosen at random from a unimodal distribution $g(\omega)$, taken to be a Gaussian. Since selecting the appropriate scaling transformation allows one to fix the variance, we chose $g(\omega)$ to have unit variance.

Equation (1) was integrated numerically using a fourth-order Runge-Kutta algorithm. The majority of these computations were performed on a MasPar single-instruction multiple-data (SIMD) massively parallel computer [12]. It turns out that finite-size effects can be quite important in this case. Thus, runs were performed for $N=51$, 401, 801, 901, 1501, and 2001 and finite-size scaling was used to extrapolate to the thermodynamic limit.

In the lower limit of the coupling exponent, $\alpha\to 0$ in Eq. (1), all of the oscillators become coupled to one another with equal strength, thereby resulting in the mean-field model studied by Kuramoto. Accordingly, our numerical simulations demonstrated his result for the critical coupling constant in this limit, $K_c=2/[g(0)\pi]$. To describe the collective behavior, Kuramoto used an order parameter defined by the expression

$$Re^{i\varphi}=\frac{1}{N}\sum_{j=1}^{N} e^{i\theta_j}. \tag{3}$$

In the synchronized state, the magnitude R goes to a fixed value in the thermodynamic limit, although in a finite system there may be fluctuations of the order $O(1/\sqrt{N})$, while in the desynchronized state, R will be very small and have fluctuations of the same type.

As the coupling exponent becomes large, $\alpha\to\infty$, the interaction terms in Eq. (1) reduce to local coupling, in the limit, resulting in the one-dimensional model with nearest-neighbor interactions. In this limit, the behavior of the oscillators may be well described in terms of the average frequency. The value at the jth site is defined by

$$\tilde{\omega}_j=\lim_{t\to\infty}\frac{\theta_j(t+T)-\theta_j(T)}{t}, \tag{4}$$

where T is chosen such that transients have decayed. In terms of this measure the synchronized state corresponds to all the oscillators having the same value. Although for $\alpha\to\infty$ in Eq. (1) synchronization is not possible in the thermodynamic limit, it will be attainable for a finite N provided K is sufficiently large [9]. The ability to reach the synchronized state is dependent on competition between the frozen disorder of the natural frequencies and the aligning interactions. If the coupling strength is large enough to constrain the disorder, then the ensemble will synchronize. The value of the coupling constant necessary for spontaneous synchronization should increase with system size as $\sqrt{N}$, a dependence

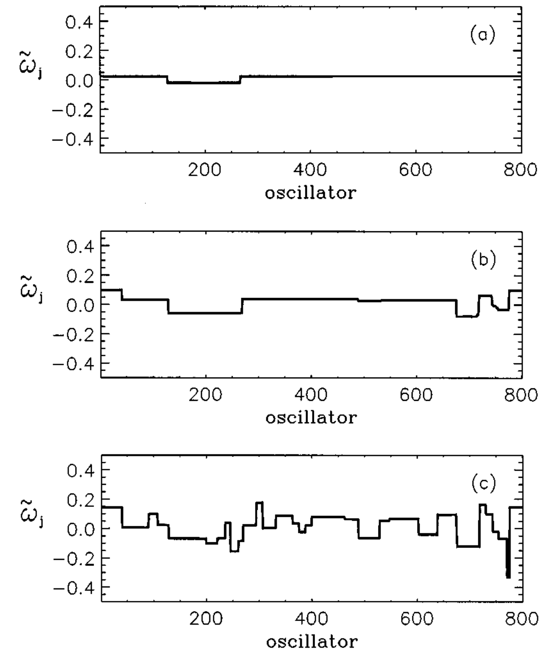

FIG. 1. The average frequency defined by Eq. (4) for representative values of the coupling exponent where $N=801, K=7.0$: (a) $\alpha=1.8$; (b) $\alpha=2.0$; (c) $\alpha=4.0$.

our simulations demonstrated. For a coupling constant just below the critical value, two plateaus of locally synchronized oscillators develop in the average frequency with a break between them: the system behaves effectively as two renormalized oscillators and is no longer globally synchronized. As the coupling is decreased, more and more plateaus in the average frequency quickly form.

By increasing the coupling exponent in Eq. (1) from its lower bound of zero, we observed a transition from a synchronized to a desynchronized state. To ensure that the ensemble would synchronize for $\alpha=0$, the coupling constant was chosen to be larger than the critical value in the mean-field limit. Figures 1 and 2 show the average frequency and order parameter defined by Eqs. (4) and (3), respectively, for $N=801$, $K=7.0$, and three selected coupling exponents. Figure 1(a) shows a state in which two plateaus in average frequency have just developed. This state may be thought of as consisting of two groups of oscillators moving with dissimilar average frequencies on the unit circle in the complex plane. As a consequence, the order parameter will undergo periodic behavior as shown in Fig. 2(a). Figure 1(b) shows a state in which several plateaus have developed, as can be seen in Fig. 2(b) by the large variations in amplitude in the order parameter. This state is basically a sum of several incommensurate periodic functions. Figure 1(c) shows a state where a great many plateaus are present and can be considered to be consisting of incoherent motion: small groups of neighboring oscillators are still locked, but the ensemble behaves in an essentially random fashion. Correspondingly, the order parameter R is seen in Fig. 2(c) to be fluctuating near zero.

To clarify this transition, consider the behavior of the average plateau size divided by the total number of oscillators, which we will denote $\overline{P}(\alpha)=[\overline{N}(\alpha)]/N$. This expression is equivalent to the inverse of the number of plateaus. Choos-

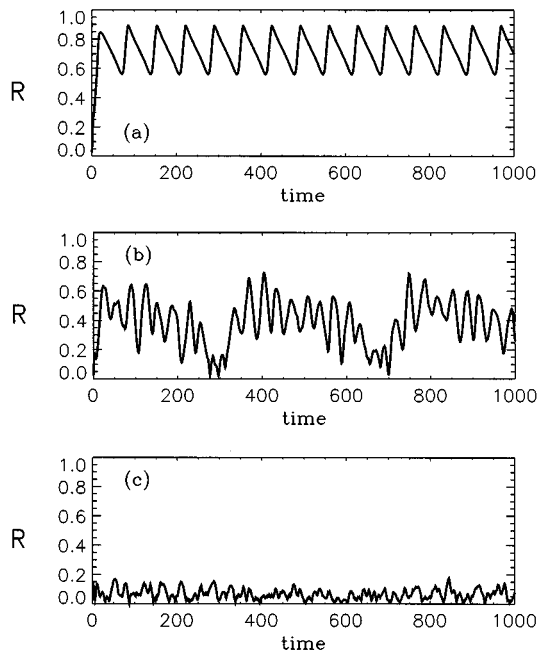

FIG. 2. The temporal behavior of the order-parameter magnitude defined by Eq. (3) for the same parameter values as shown in Fig. 1 ($N=801,K=7.0$): (a) $\alpha=1.8$; (b) $\alpha=2.0$; (c) $\alpha=4.0$.

ing the same K and N values as before, the behavior of this ratio is plotted in Fig. 3. The values of $\overline{P}$ shown in the figure resulted from averaging over six independent series of simulations, each with a particular realization of the natural frequencies. Perhaps the most striking aspect of this transition from a synchronized to a desynchronized global state is its abrupt nature. Since $\overline{P}=1$ corresponds to synchronization while $\overline{P}\approx 0$ corresponds to incoherent oscillations, the range of α values that result in more complex dynamics is remarkably small. The loss of synchronization is characterized by the emergence of plateaus of locally synchronized oscillators in the average frequency. The location and relative sizes of these plateaus is dependent on the particular realization of intrinsic frequencies. Despite this, the majority of our simulations demonstrated similar quantitative dependence of $\overline{P}(\alpha)$. The range of coupling exponent values over which

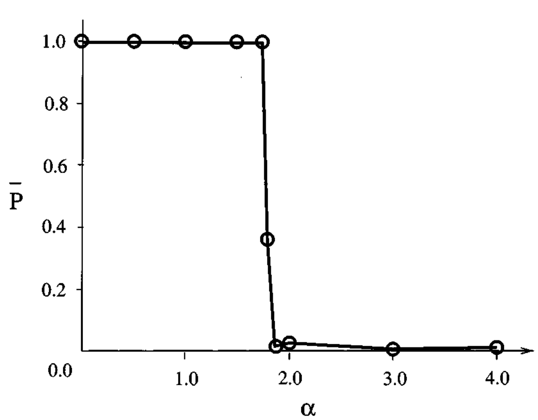

FIG. 3. Dependence of the average plateau size on the coupling exponent. The circles represent the average over six series of simulations with independent natural frequencies.

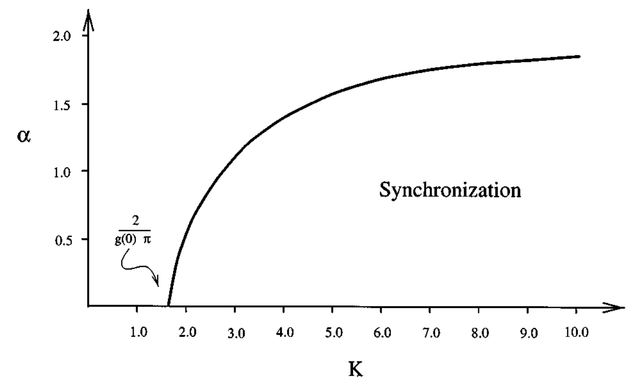

FIG. 4. Critical coupling exponent as a function of the coupling constant $\alpha_c(K)$ in the thermodynamic limit. An infinite size system whose coupling parameters lie within the region labeled synchronization will spontaneously evolve to a synchronized state.

synchronization was lost was approximately 0.2 for the various realizations, while qualitatively all of the individual curves showed the same behavior as the average. It should be noted that the range of α values over which the transition takes place could be broadened by some atypical realizations of the natural frequencies. In numerics other than those averaged we observed such behavior for a single realization of the natural frequencies.

In view of the sharpness of the transition, it is possible to identify a ''critical'' interaction exponent α_c (for a given K and N) such that the oscillator system cannot synchronize if $\alpha>\alpha_c$. Since our central interest is the behavior in the thermodynamic limit, finite-size scaling was used to extrapolate the $\alpha_c(K)$ results at the various N values leading to the curve shown in Fig. 4. This was performed by plotting α_c versus $1/N$ and extrapolating for $N\to\infty$ [12]. As should be expected, Kuramoto's analytical result for the critical coupling in the mean-field limit ($\alpha=0$) was recovered. In the limit of very large coupling constants, α_c appears to be approaching an asymptotic value of 2.

The similarity of the present results to those for spin systems is interesting. For the one-dimensional Ising and XY models with an interaction of the form $r^{-\alpha}$, it is known (see Ref. [13], and references therein) that ferromagnetism is not possible at finite temperatures for $\alpha>2$, while ferromagnetic order becomes an accessible state at low temperatures if $\alpha\leq 2$. Likewise, for the one-dimensional spin-glass model a phase transition is found for $\frac{1}{2}<\alpha<1$ [14]. We note that in those cases the critical α value does not depend on the coupling constant. The analogy between spin systems and oscillator ensembles is intriguing and may be a useful guide for intuition, but it remains to be seen if it is more than just a useful similarity.

Summarizing, for a one-dimensional chain of interacting disparate nonlinear oscillators in the mean-field limit, global order is possible, while it is not in the limit of nearest-neighbor coupling. Using a decaying power-law interaction, we examined the loss of synchronization. We found that if $\alpha\leq\alpha_c$ a synchronized state exists for some finite K, while if $\alpha>\alpha_c$ no finite coupling will synchronize the ensemble. We mapped out the functional dependence of the critical exponent on the coupling constant $\alpha_c(K)$. The nature of the transition between the two extremes was found to be abrupt and might help explain why some systems suddenly lose their ability to synchronize.

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