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Phase transitions near the ''game of Life''

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Whether or not the cellular automaton (CA) ''game of Life'' is an example of self-organized criticality has been a controversial question. Here we view the problem from a different perspective by introducing probabilities into the rules. Thereby we extend the discrete space of deterministic CA to a hypercubic space of stochastic CA, where each corner represents a deterministic CA. We examine the scaling structure near the ''game of Life'' corner and identify a phase-transition line separating ''Life'' and ''Death.'' The transition line ends very close to but not in the ''game of Life.'' [S1063-651X(96)51408-X]

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The ''game of Life'' (GOL) is a two-dimensional cellular automaton which has been suggested to mimic aspects of complexity in nature [1,2]. It simulates by means of a simple algorithm the evolution of a society of cells (on a square lattice), where each cell is either ''dead'' (0) or ''alive'' (1). The evolution depends on the number of living cells among its 8 nearest and next-nearest neighbors (Table I): A dead cell will only come to life if it has exactly three living neighbors. A living cell will stay alive if it has two or three living neighbors, otherwise it will die. Starting from random initial conditions, ''Life'' will evolve through complex patterns eventually settling down in a stationary state with static or (time) periodic cell colonies.

Former reports on GOL have used a deterministic approach in investigating the critical properties of GOL [3–6]: After the system has relaxed to a stationary state the system was perturbed and the distribution of avalanches was observed. For a finite lattice the decay time to the stationary state is finite. Therefore the slow probing of a finite system may as well be simulated by changing the ''life'' conditions slightly. From this perspective we may study the critical properties near the GOL by introducing probabilities into the life conditions [7].

First, consider the general case, where all 0's and 1's in Table I are replaced by probabilities $p_{d,j}$ and $p_{l,j}$, where d and l refer to the cell being dead or alive, and $j=0,\ldots,8$ is the number of living neighbors. Assume further that $p_{d,0}=0$, such that ''death'' is an absorbing state. Then complete ''death'' may be viewed as the ordered state, where ''life'' provides the fluctuations. We are left with a 17-dimensional hypercubic cellular-automaton space, $(p_{d,j}[j=1,\ldots,8],p_{l,j}[j=0,\ldots,8])$ where the corners are deterministic cellular automata (CA). We are interested in separating the living states from the completely dead states, i.e., we wish to distinguish the (stochastic as well as deterministic) CA with a nonzero life density Φ from those with $\Phi=0$. At the border between $\Phi=0$ and $\Phi>0$, we may have a critical behavior with diverging length and time scales. If the border line between $\Phi=0$ and $\Phi>0$ lies close to a corner, the associated deterministic CA may also show a nearly critical behavior. We shall see that the border line lies very close to GOL.

Here, we illustrate the general idea by extending the rules in Table I with just two probabilities, p_d and p_l (Table II). We permit dead cells with two living neighbors to come to life with a nonzero probability p_d, generally raising the life density Φ. Moreover, a living cell with two living neighbors will only stay alive with a probability p_l, generally lowering the density Φ. Hence, we control Φ by raising p_d from the GOL value 0 and lowering p_l from the GOL value 1.

We have carried out simulations of the CA given by Table II at various values of p_d and p_l, with emphasis on the region near the GOL. The CA was initiated from a random configuration with 10% living cells, and we used a $L\times L=500\times 500$ lattice with periodic boundary conditions. The density of life, $\phi(t)$, was sampled over 10 000 time steps, which was sufficient to separate the asymptotic density profile from the transient behavior (see inset of Fig. 1). From the distribution of densities, $D(\phi)$, a peak is easily identified (Fig. 1). The peak is centered around a value Φ, which we determined as the average value over the last 1 000 time steps.

We observe a transition from $\Phi=0$ to $\Phi>0$ along a line ending very near the GOL corner (Fig. 2). At this line, length and time scales diverge as at a second-order phase transition (see below). The region near the GOL is magnified (insets of Fig. 2). We find that the transition line converges to the point $(p_d=0,p_l=p_l^*=0.9968\pm 0.0006)$, and not to the GOL point $(p_d=0,p_l=1)$. We discuss this further below.

We mention the suggestion by Langton [8] to characterize the CA according to their λ parameter, i.e., the number of

TABLE I. The state of a cell in the ''game of Life'' depends on the states of the cell and the 8 neighbors at the previous time step.

No. living neighbors	0	1	2	3	4	5	6	7	8
0 Dead cell	0	0	0	1	0	0	0	0	0
1 Living cell	0	0	1	1	0	0	0	0	0

TABLE II. Probability of life, depending on the states of the cell and the 8 neighbors at the previous time step. In the ''game of Life,'' $p_d=0$ and $p_l=1$.

No. living neighbors	0	1	2	3	4	5	6	7	8
0 Dead cell	0	0	p_d	1	0	0	0	0	0
1 Living cell	0	0	p_l	1	0	0	0	0	0

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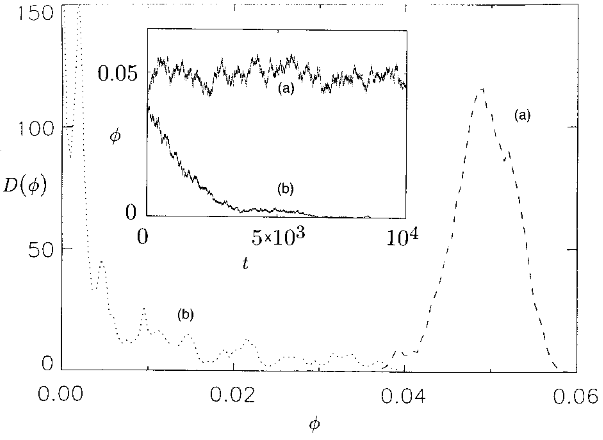

FIG. 1. The density distribution $D(\phi)$, (a) above the transition line ($p_d=0.02, p_l=0.985$), and (b) below the transition line ($p_d=0.02, p_l=0.978$). The distribution $D(\phi)$ was obtained over 3 runs of 10 000 time steps. $D(\phi)$ has a peak situated at $\Phi=4.9\%$ in case (a). This peak has moved to $\Phi=0$ below the transition line [case (b)]. Inset: Density of life ϕ as a function of time t.

transitions to a living state normalized by the total number of transitions. For the stochastic CA given by Table II, the fraction of transitions giving a living cell in the next time step (weighted by p_d and p_l) is $\lambda=7(4+p_d+p_l)/128$. At the GOL, $\lambda\approx=35/127\approx0.273$. However, λ is not constant along the transition line. Away from the GOL, the transition line bends downwards, and λ decreases. At $p_l=0$, we find $p_d=0.37$, and $\lambda\approx0.239$.

Let (p_{dc},p_{lc}) indicate a point on the transition line from $\Phi=0$ to $\Phi>0$. Right above the line we find a power-law behavior $\Phi(p_d=p_{dc},p_l)\propto(p_l-p_{lc})^{\beta_l}$, and $\Phi(p_d,p_l=p_{lc})\propto(p_d-p_{dc})^{\beta_d}$ (Fig. 3). Within error bars, we find β_l and β_d to be identical, $\beta_l=\beta_d=\beta$. Moreover, we obtain the same value for β at different transition points. The value of β is found to be $\beta=0.5\pm0.1$. In comparison, we note that the value for (2+1)-dimensional directed percolation is $\beta_{dp}=0.60$ [9,10].

The error bars given in Fig. 2 (inset) are determined using the condition that Φ follows a power law, $\Phi\propto(p_d-p_{lc})^{\beta}$. To obtain upper and lower bounds for the value of p_{lc}, we plot $\log(\Phi)$ vs $\log(p_l-p_{lc})$. At the lower and upper bounds for p_{lc} there is a clear curvature (Fig. 4). Our results for p_{lc} are where the points lie on a straight line. In the same way, we find upper and lower bounds for p_{dc}. We note that the error bars are distinctly smaller than $1-p_l^*=0.0032$.

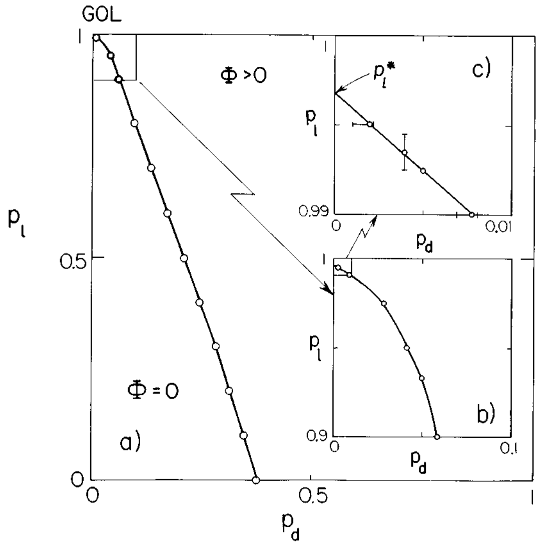

FIG. 2. (a) Phase diagram for the stochastic cellular automaton considered (Table II). Top left corner is the "game of Life" (GOL). The line is the phase transition line between asymptotic life density $\Phi=0$ and $\Phi>0$. (b) and (c): Magnifications of the region near the "game of Life." The transition line touches the p_l axis at $p_l^*=0.9968\pm0.0006$.

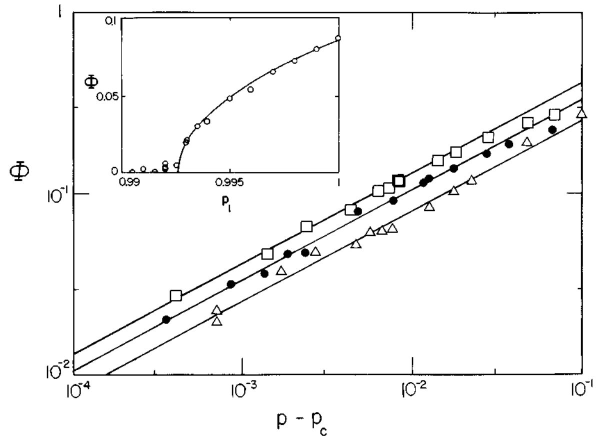

FIG. 3. Double-logarithmic plot of the life density Φ vs distance $p-p_c$ from the critical line. □: $p=p_d$ and $p_c=p_{dc}=0.0416$, for fixed $p_l=p_{lc}=0.95$. ●: $p=p_l$ and $p_c=p_{lc}=0.782$, for fixed $p_d=p_{dc}=0.111$. △: $p=p_d$ and $p_c=p_{dc}=0.0023$, for fixed $p_l=p_{lc}=0.995$. The straight lines have all slopes $\beta=0.5$. Inset: Life density Φ vs p_l for fixed $p_d=0.005$. Above $p_{lc}=0.9925$, Φ increases as $(p_l-p_{lc})^{\beta}$ with $\beta=0.5$.

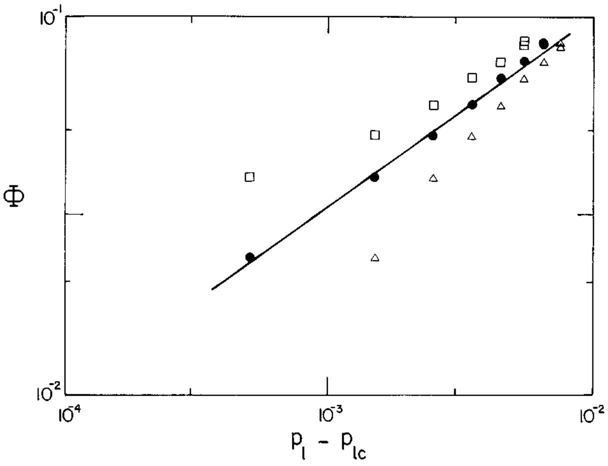

FIG. 4. Double-logarithmic plot of the life density Φ vs distance p_l-p_{lc} from the critical point ($p_{dc}=0.004, p_{lc}=0.9935$). If p_{lc} is chosen wrong a curvature is observed. ●: $p_{lc}=0.9935$. The best choice for p_{lc}. The straight-line fit has slope $\beta=0.5$. □: $p_{lc}=0.9945$. The choice for p_{lc} is too large, and a curvature is observed towards a non-zero value for Φ. △: $p_{lc}=0.9925$. The choice for p_{lc} is too small. Consequently Φ approaches zero at a finite value of p_l-p_{lc}.

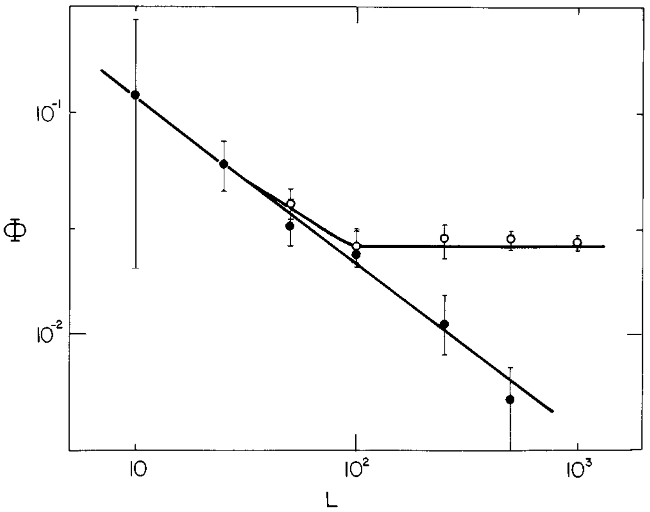

FIG. 5. Double-logarithmic plot of the life density Φ vs system size L at the critical point $(p_d=p_{dc}=0.005, p_l=p_{lc}=0.9925)$ (●). The straight-line fit has slope $-\beta/\nu_\perp=-0.75$. For $p_l=0.9935$ (○), slightly larger than p_{lc}, the density Φ saturates at large L.

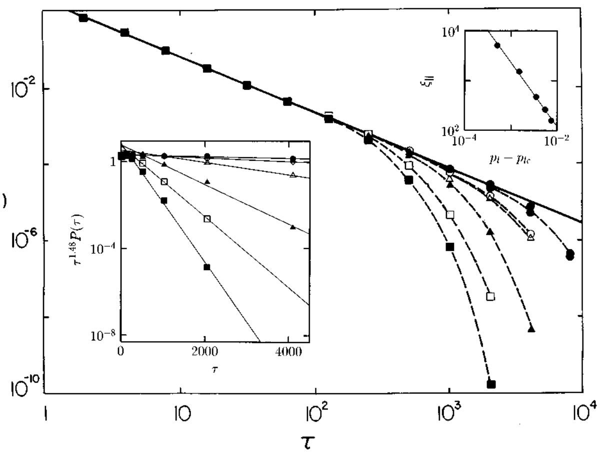

FIG. 6. Distribution $P(\tau)$ of "death times" τ for $p_d=0.005$ and at various values of p_l near $p_{lc}=0.9925$ (log-log plot). ○: $p_l=0.990$. ●: $p_l=0.992$. △: $p_l=0.994$. ▲: $p_l=0.996$. □: $p_l=0.998$. ■: $p_l=1$. The straight line has slope $-\alpha=-1.48$. For $p_l<p_{lc}$ and $p_l>p_{lc}$, the "death time" distributions are cut off exponentially at time scales lower than the total simulation time (10,000 time steps). Lower inset: Semilogarithmic plot of the rescaled "death-time" distribution $\tau^\alpha P(\tau)$, emphasizing the exponential tails. ◇ : $p_l=0.993$. The slopes have the value $-1/\xi_\parallel$, where $\xi_\parallel$ is the cutoff time. Upper inset: Double-logarithmic plot of time scale $\xi_\parallel$ vs p_l-p_{lc} $[p_d=0.005]$. The slope of the straight line is $-\nu_\parallel=-1.27$.

In order to further confirm the accuracy of our results for p_{lc}, we carried out a finite-size analysis. One finite-size effect is the rounding of Φ vs p_l-p_{lc} very close to p_{lc} (see, e.g., the inset of Fig. 3). Finite-size scaling predicts $\Phi(p_{dc},p_{lc})\propto L^{-\beta/\nu_\perp}$, where $\nu_\perp$ is the spatial correlation-length exponent. We determined the density Φ as the value where the density distribution has its peak. In Fig. 5, $\log\Phi$ is shown vs $\log L$ for $10\leqslant L\leqslant 1000$ at the transition point $(p_{dc}=0.005, p_{lc}=0.9925)$. On large lattices $(L>100)$, the density distribution was sampled over 1 to 10 runs (number decreasing with system size), each simulated 10 000 time steps. On smaller lattices $(L\leqslant 100)$, a large fluctuation in $\phi(t)$ may cause the system to go extinct, i.e., to enter the completely dead state within the simulation time. In this case the density distribution was sampled over 24 to 92 runs. We find $\beta/\nu_\perp=0.75\pm0.1$ (for directed percolation, $\beta/\nu_\perp=0.82$ [9]). For p_l chosen slightly above p_{lc} (for example, at the upper bound for p_c) $\log\Phi$ vs $\log L$ saturates at large L (Fig. 5). We conclude that the size of the error bars in Fig. 2 are reasonable.

When we move from the deterministic GOL system into the probabilistic regime, the stationary structures of the GOL (for example, the square-cell colonies) are perturbed—and they will therefore spread or shrink. Near the phase transition, the time evolution of the system depends crucially on p_d and p_l. Above the transition line the growth rate balances the death rate and the system keeps evolving. Below the line, the death rate exeeds the growth rate, and the system eventually reaches complete "death." As we go closer to the phase-transition line the characteristic time scale diverges. To examine the dynamics, we determined the distribution $P(\tau)$ of "death times," defined as the time from when a cell is vacated $(1\rightarrow 0)$ until it again becomes occupied $(0\rightarrow 1)$; see Fig. 6. At the transition line this distribution has a power-law behavior, $P(\tau)=\tau^{-\alpha}$, only cut off at time scales comparable to the total simulation length (10 000 time steps). We find α to be $\alpha=1.48$. Slightly above and below the line the distributions have an exponential cutoff. This can be seen by plotting $\tau^\alpha P(\tau)$ vs τ on a semilogarithmic scale (lower inset of Fig. 6). From standard percolation theory we expect to have a scaling form $P(\tau)=\tau^{-\alpha}f(\tau/\xi_\parallel)$ with $\xi_\parallel\propto|p-p_c|^{-\nu_\parallel}$, where $-\xi_\parallel^{-1}$ is the p_l-dependent slope in the lower inset of Fig. 6. In the upper inset of Fig. 6, $\xi_\parallel$ is plotted vs (p_l-p_{lc}) (double-logarithmic plot) near the transition point $(p_{dc},p_{lc})=(0.005,0.9925)$. From this plot we find $\nu_\parallel\simeq 1.27$ (this value is exactly the value obtained for directed percolation [9]).

As shown in the phase diagram (Fig. 2), we have $\Phi>0$ $(\Phi=0)$ for $p_l>p_l^*=0.9968$ $(p_l<p_l^*)$, where p_l^* is the

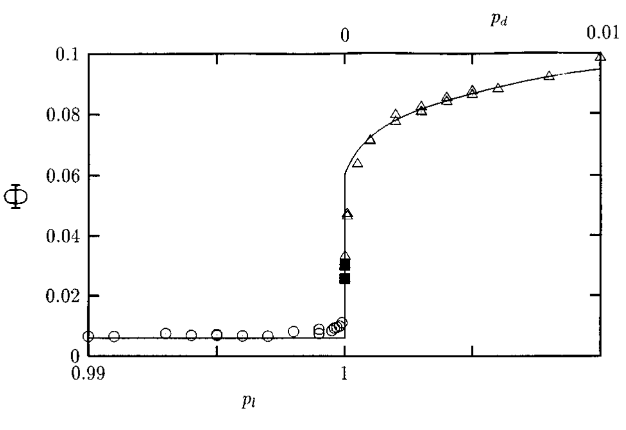

FIG. 7. Life density Φ along the two edges $(p_d=0,p_l)$ [left side] and $(p_d,p_l=1)$ [right side] near the "game of Life," situated in the mid-point $(p_d=0,p_l=1)$. A jump in the value of Φ is seen on both sides of the "game of Life". The jumps are slightly smoothened, due to the finite simulation time.

asymptotic value for p_l for $p_d \to 0$. However, we may encounter ''first-order'' jumps in density Φ as we go from $p_d > 0$ to $p_d = 0$, simply because some of the static or periodic structures known from the GOL may become stable, and because certain perturbations produced at $p_d > 0$ are absent at $p_d = 0$. Similar considerations apply to the case where we go from $p_l < 1$ to $p_l = 1$. We therefore examined these situations carefully. In the latter case ($p_l < 1$ to $p_l = 1$), we found that $\Phi(p_l \to 1) = \Phi(p_l = 1)$ [$p_d > 0$]. However, in the former case ($p_d > 0$ to $p_d = 0$), we do find a jump in the value for Φ. This is a consequence of the fact that along the edge ($p_d = 0, p_l$) the square-cell colony remains stable. It consists solely of cells with 3 living neighbors and is thus unaffected by the value of p_l. Accordingly, we find that $\Phi(p_d = 0, p_l)$ has a nonzero p_l-independent value [$p_l < 1$], given by the asymptotic density of square-cell colonies. We find that this value is $\Phi(p_d = 0, p_l) = 0.6\%$. For $p_d > 0$, the neighbor cells to a square-cell colony will eventually come to life and perturb the colony. The perturbation leads to extinction or growth, dependent on the value of p_l.

Finally, we consider the GOL point, $(p_d, p_l) = (0,1)$. At this point, ''blinkers'' and other cell colonies become stable. Along the edge ($p_d = 0, p_l$) we therefore observe a jump in life density from $\Phi = 0.6\%$ to $\Phi = \Phi_{\mathrm{GOL}} = 2.6\%$ (Fig. 7). Along the other edge ($p_d, p_l = 1$), important perturbations are produced as soon as $p_d > 0$. The perturbations cause the life density to increase abruptly from $\Phi_{\mathrm{GOL}} = 2.6\%$ to $\Phi(p_d \to 0, p_l = 1) \approx 6.0\%$ (Fig. 7).

In summary, we have identified a phase-transition line in a stochastic environment of the deterministic ''game of Life'' cellular automaton. The transition line is of the ''second-order'' type with power-law diverging length and time scales (exponents consistent with directed percolation). Our simulations show that the ''game of Life'' is very near, but not on the critical transition line.

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