Tracking unstable steady states by large periodic modulation of a control parameter in a nonlinear system

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Suppression of chaos and stabilization of an unstable steady state in a dissipative nonlinear system are demonstrated numerically by means of large-amplitude slow (nonresonant) modulation of a control parameter. A wide domain of modulation amplitudes and frequencies is allowed. The steady state becomes influenced by the slow modulation, although in some cases this influence is very small. It can be said that suppression of chaos occurs, naturally, without any further action, in slowly modulated systems. [S1063-651X(96)03106-6]

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INTRODUCTION

Much attention has been devoted recently to the possibility of converting deterministic chaos appearing in nonlinear systems to regular behavior. The basic idea is to stabilize unstable periodic orbits embedded in chaos, which is performed by feedback [1] or nonfeedback [1(b),2] techniques. In the case of nonfeedback control methods a small periodic perturbation of the chaotic system is usually introduced [2,3]. On the other hand, it has been shown that control of chaotic behavior can also be achieved by stabilization of an unstable steady state. This has been performed, in particular, by means of the occasional proportional [4(a)] and continuous [4(b)] feedback techniques. In this work we show that suppression of chaos can also be achieved by large-amplitude slow periodic modulation of a control parameter of the system. In a wide domain of operating conditions, we show that by slowly modulating a chaotic system it can become anchored to an unstable fixed point in phase space, following quasistatically its slow forced periodic motion, instead of remaining on the chaotic attractor of the unperturbed system. A lower bound for the allowed domain of modulation amplitudes is found, which in some cases can be very low.

We show this phenomenon by considering three different examples taken from the field of laser dynamics, although the main conclusions that will be drawn can also apply to other nonlinear dissipative systems. These examples are of increasing complexity, and in each case modulation is applied to a different parameter of the system. In this way, each example will correspond, as will become evident below, to different conditions from the point of view of nonlinear dynamics.

I. RESONANT LORENZ-HAKEN LASER

The conceptually most fundamental class of laser is the so-called two-level Lorenz-Haken laser model, which is ruled by the following set of equations (see, for instance [5–9]):

$$\dot{E} = -\sigma E(1+i\delta) + \sigma A P,$$

$$\dot{P} = -P(1-i\delta) + ED,$$

$$\dot{D} = -b(D-1) - (b/2)(P^*E + E^*P), \qquad (1)$$

where the dimensionless complex variables E and P and the real variable D represent the laser field amplitude, medium polarization, and population inversion, respectively. A is the pump strength, $\delta = \delta_{CA}/(1+\sigma)$, and the dimensionless parameters σ, b, and δ_{CA} represent the cavity losses, atomic (or molecular) longitudinal relaxation rate, and cavity detuning—the difference between the cavity and atomic resonance frequencies—respectively, which are normalized with respect to the transverse relaxation rate $\gamma_\perp$ (time is expressed in units of $\gamma_\perp^{-1}$).

On resonance, i.e., for $\delta = 0$, Eqs. (1) reduce to the well-known real Lorenz-model equations [6] through the transformation $x = \sqrt{b}E$, $y = A\sqrt{b}P$, and $z = A(1-D)$, with $r \equiv A$. For $A > 1$, Eqs. (1) have a nontrivial stationary solution $\overline{E} = (A-1)^{1/2}$, which, in case $\sigma > b+1$ (''bad cavity'' condition), becomes unstable through a subcritical Hopf bifurcation when the pump reaches the value $A_{HB} = \sigma(\sigma+b+3)/(\sigma-b-1)$, which is known as the second laser threshold. Above this threshold the system falls into a chaotic attractor (the well-known Lorenz attractor), which exists in phase space above a certain pump threshold A_{CH}, smaller than A_{HB}. For pumping between A_{CH} and A_{HB}, the stable fixed point corresponding to the steady-state solution and the chaotic attractor coexist in phase space, the passage from the first to the second being possible only through hard-mode excitation. It is known, for example, that in case of an NH_3 far-infrared laser [10], typical values of the system parameters are $\sigma = 2.0$ and $b = 0.25$, for which $A_{CH} \approx 11$ and $A_{HB} = 14$.

Let us now modulate, in the case $\delta = 0$, the pump parameter A in the form $A(t) = A_0 + m\cos(\Omega t + \varphi)$, where A_0, m, Ω, and φ denote the average pump value and the modulation amplitude, frequency, and phase, respectively. We choose

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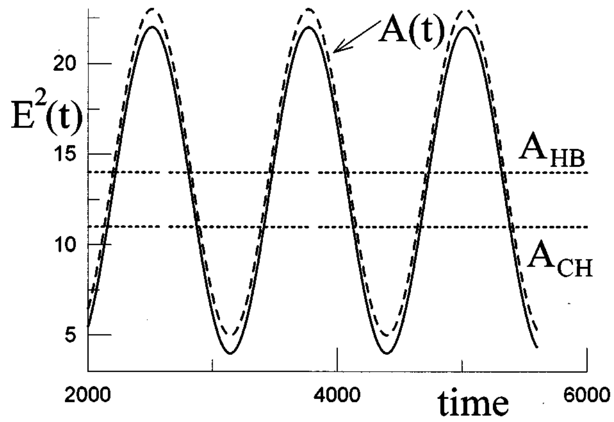

FIG. 1. Time evolution of the field intensity $E^2(t)$ in the Lorenz-Haken model (continuous line) for $\sigma=2$, $b=0.25$, $A_0=14$, $m=9$, $\Omega=0.005$, and $\varphi=0$. Also shown are the modulated pump $A(t)=A_0+m\cos(\Omega t+\varphi)$ (dashed line), the threshold for chaos A_{CH}, and the Hopf bifurcation threshold A_{HB}.

Ω to be well below the natural relaxation rates of the system σ, b, and $\gamma_\perp$ (i.e., $\Omega \ll 1$), and the values of A_0 and m are taken in such a way that $A(t)$ crosses up and down the thresholds for chaos, A_{CH} and A_{HB} (dotted straight lines in Fig. 1), at each modulation period. For instance, we take $A_0=14$ and $m=9$. Figure 1 shows the time evolution of the modulated pump $A(t)$ (dashed line) and of the laser field intensity $E^2(t)$ corresponding to this case (continuous line), when $\Omega=0.005$ (in units of $\gamma_\perp$) and $\varphi=0$. It can be observed that, at each time t, $E^2(t)$ coincides almost exactly with the intensity of the stationary solution corresponding to the instantaneous value of the pump parameter $A(t)$; i.e., $E^2(t)=\overline{E^2}(t)=A(t)-1$. This equality is verified with an accuracy of at least five significant digits, or even more for smaller modulation frequencies. This means that the system prefers to remain at any time very close to the steady-state solution, instead of falling into chaos when the pump $A(t)$ surpasses the threshold values A_{CH} and A_{HB}. This result is independent of the value of φ and, more interesting, of the initial conditions of the system. This means that, even in the case when the system has already fallen on the chaotic attractor, application of pump modulation drives the system out of chaos and stabilizes it, tracking the steady-state solution (this is shown below, with the next example, Fig. 4). There is a wide domain of values of A_0, m, and Ω for which stabilization occurs. As a rule, Ω must be of the order of 10^{-3}–10^{-2}, although the modulation frequencies and amplitudes strongly depend on the value of A_0: the closer A_0 to A_{HB}, the larger m and the smaller Ω must be. For instance, for $A_0=12$ and $\Omega=0.01$ the modulation amplitude m has to be equal to or larger than 3 $(=0.25A_0)$, whereas for $A_0=14$ and the same value of Ω, m must be about 6 $(=0.43A_0)$ or above (more detailed analysis will be given elsewhere).

II. DETUNED LORENZ-HAKEN LASER

We consider again Eqs. (1) but now pumping will be kept constant and modulation will be applied to the detuning parameter δ. Let us first recall that a detuned laser behaves ‘‘less chaotically’’ than a resonant laser [7,8,11]. For $A \gtrsim A_{CH}$, the laser emission is stable for large $|\delta|$, and when $|\delta|$ is decreased the steady-state solution undergoes a Hopf bifurcation which is subcritical at small pump and supercritical at large pump [7] and leads to the appearance of a limit cycle in phase space. By further decreasing $|\delta|$ this limit cycle undergoes a sequence of period-doubling bifurcations defining a complete Feigenbaum scenario toward chaos. Chaos occurs within a domain of values of δ centered around $\delta=0$ and symmetric with respect to the sign of δ. Thus the main difference between this case and the previous one is that whereas in case 1 the fixed-point solution to be tracked and the chaotic attractor to be avoided are well separated in phase space, in the present case they are connected by a sequence of local bifurcations.

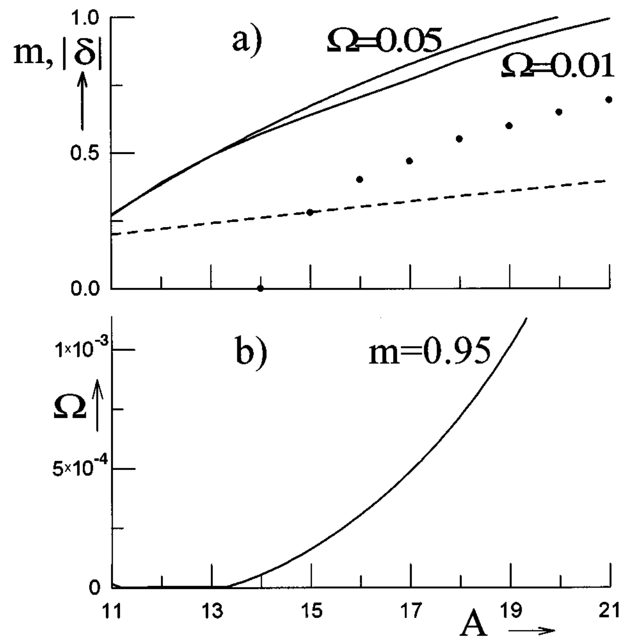

FIG. 2. (a) Continuous line: minimum allowed value for the detuning modulation amplitude m, in the Lorenz-Haken laser model, for $\sigma=2$, $b=0.25$, $\Omega=0.05$, and 0.01, as function of pumping rate A. Dots: values of detuning $|\delta|$ at which the steady-state solution undergoes a Hopf bifurcation when $|\delta|$ is decreased. Dashed line: values of $|\delta|$ at which the first period-doubling bifurcation occurs (see text). (b) Minimum values of Ω leading to steady-state stabilization, for $m=0.95$, as a function of A.

Modulating detuning in the form $\delta(t)=\delta_0+m\cos(\Omega t+\varphi)$, we find again that deep modulation at slow frequencies $\Omega \ll 1$ leads to suppression of chaos and stabilization of the steady state. The condition for this is that detuning, along its sinusoidal time evolution, must cover all the domain of instability of the steady-state solution. For instance, Fig. 2(a) shows the minimum values that m must take in order to stabilize the steady state, as a function of pumping, in a specific case with $\delta_0=0$ and for two different values of Ω (continuous line). For the sake of comparison Fig. 2(a) also shows the values of $|\delta|$ at which the steady-state solution undergoes a Hopf bifurcation when $|\delta|$ is decreased (dots), as well as the values of $|\delta|$ at which the limit cycle originating at the Hopf bifurcation point undergoes the first

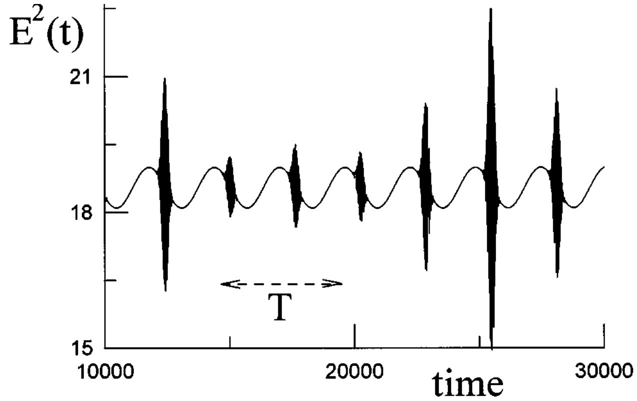

FIG. 3. Laser field intensity $E^2(t)$ as a function of time for $\sigma=2$, $b=0.25$, $A=20$, detuning modulation amplitude $m=0.95$, $\Omega=0.0012$, $\varphi=0$, and $\delta_0=0$. The detuning modulation period $T=2\pi/\Omega$ is twice the laser intensity modulation period.

period-doubling bifurcation when $|\delta|$ is decreased (dashed line)—the crossing between the two curves is due to the fact that the Hopf bifurcation is subcritical for small A and supercritical for large A. Clearly, along its time evolution $\delta(t)$ crosses back and forth across all the instability domains. Note, however, that in spite of this fact the detuning modulation amplitude m can be quite small (below unity, i.e., below the natural width $\gamma_\perp$ of the lasing transition). Note also that for smaller modulation frequencies, smaller modulation amplitudes m can be used.

Figure 2(b) shows, for a fixed value of the detuning modulation amplitude ($m=0.95$), the minimum values of Ω that can be used to stabilize the steady-state solution, as a function of pumping. A remarkable fact is that Ω can be as small as 10^{-3}–10^{-4}. One would have expected that at these slow modulation frequencies the steady-state solution would become perturbed by the sequence of period-doubling bifurcations affecting it when detuning is adiabatically varied, but clearly this is not the case. For pumping $A\leqslant A_{HB}=14$ the system always tracks the steady state, even for extremely low modulation frequencies (we observed stabilization down to $\Omega=2\times10^{-5}$ which corresponds to ca. 20 Hz for $\gamma_\perp=6\times10^{-6}\ \mathrm{s}^{-1}$ [10,12]). For $A\geqslant A_{HB}$ and for Ω below the curve of Fig. 2(b), the system is not able to track the steady state completely, as shown in Fig. 3. Within each modulation period two bursts of irregular behavior appear. Note that these bursts are affected by a dynamic delay, because instead of appearing just at the maxima of the modulated signal [i.e., when $\delta(t)=0$, the value of δ for which the unperturbed system behaves chaotically], they appear at later times. This dynamic delay decreases when Ω decreases. The behavior shown in Fig. 3 is a manifestation of a type of intermittency characterized by periodic alternations of regular and chaotic motions, similar to the one found by Qu *et al.* [2] in a nonautonomous system with an additional harmonic perturbation, called the "breathing effect" by these authors.

There is also an upper bound for the modulation frequency Ω. Tracking of the steady-state solution fails when Ω approaches unity, because in these conditions resonance effects between the modulation frequency and the natural response frequencies of the system change the dynamics.

As in case 1, the stabilization effect occurs for any value of φ and for any initial conditions of the laser system. For instance, Fig. 4 shows what occurs when the system is initially in a chaotic state (corresponding to $\delta=0$) and modulation is applied at the time indicated by a vertical dashed line. Clearly, chaos is suppressed and the steady state is accurately stabilized. The time needed by the system to stabilize depends on the initial value of the modulation signal, i.e., on the phase φ [compare Fig. 4(a) with Fig. 4(b)], as well as on the instantaneous value of the system's variables when the modulation is switched on.

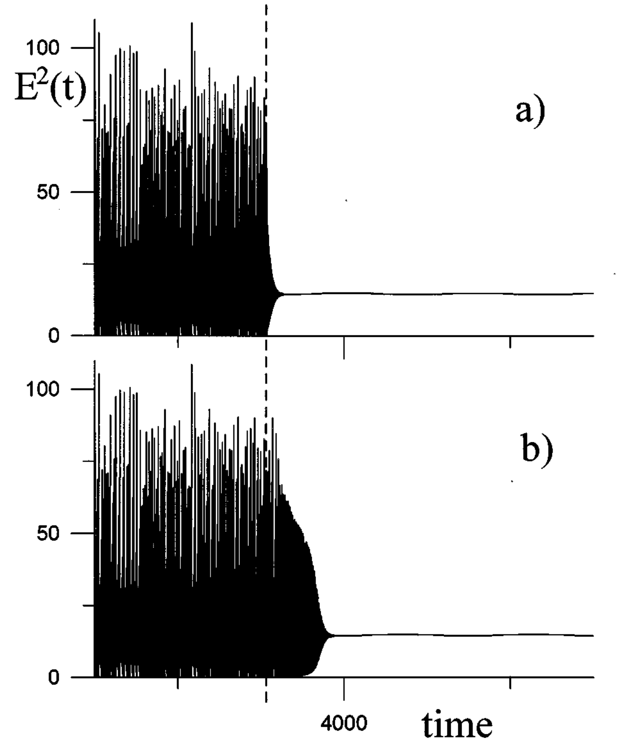

FIG. 4. Laser field intensity $E^2(t)$ as a function of time for $\sigma=2$, $b=0.25$, $A=16$, detuning modulation amplitude $m=0.7$, $\Omega=0.002$, and $\delta_0=0$. (a) $\varphi=2.9$; (b) $\varphi=1.57$.

Figure 4 shows another interesting feature. Since the detuning modulation amplitude is small ($m=0.7$) its influence on the stabilized laser emission intensity is very weak, resulting in an intensity modulation of only $\pm2\%$, barely perceptible in Fig. 4. Thus this technique provides a method to stabilize up to a good degree the chaotic output of a resonant laser: it suffices to slightly modulate the cavity length around its resonance value to get quasi-steady-state behavior. This allows one to greatly extend the domain of steady-state emission of a bad-cavity laser toward larger values of the pumping parameter A, far beyond the instability threshold A_{HB}. For instance, for a far-infrared ($\lambda=81\ \mu$m) gas laser 1 m long the cavity-length modulation necessary to get detuning modulation with $m\sim1$ and $\Omega\sim0.001$ [Fig. 2(b)] should be of amplitude $\sim0.3\ \mu$m and frequency ~1 kHz, which is very easy to implement.

III. OPTICALLY PUMPED LASER WITH PUMP-FIELD POLARIZATION MODULATION

We have tested also the present method of controlling chaos in a model for an optically pumped $J=0\rightarrow J=1$

$\rightarrow J=0$ three-level laser sensitive to field polarization [13,14]. For linearly polarized pump and laser fields the autonomous system is ruled by a system of 14 real first-order differential equations, and is known [14] to exhibit very different dynamics for different (fixed) values of the angle θ between the polarization directions of the pump and laser fields. By modulating θ in the form $\theta(t)=\theta_0+\Omega t$ (rotation of the pump polarization plane at constant angular velocity Ω) we have again observed inhibition of chaos and stabilization and tracking of the steady state for any value of θ_0 and frequencies Ω in the range from $10^{-5}\gamma_\perp$ to $0.2\gamma_\perp$ (a detailed account of these results will be reported elsewhere).

In conclusion, we have shown in this paper that control of chaos resulting in accurate stabilization of an unstable steady state can be accomplished by large-amplitude slow modulation of a control parameter for a wide domain of modulation amplitudes and frequencies. Although our analysis has been concentrated on laser systems, the fact that the method works for different laser models and different parameters to which modulation is applied, allows us to conjecture that it could be applied to many different nonlinear dissipative systems. This method is easy to apply and, unlike feedback techniques, it does not require any feedback loop. Its disadvantage is that the slow modulation applied to the system influences the steady state, so that the system gives a modulated output. Thus it can be concluded that the present method allows one to extend control of chaos to modulated (nonautonomous) systems, with the remarkable fact that, in this case, *nothing else* has to be done: by the simple fact of applying the external modulation chaos disappears and the steady state becomes stabilized.

Another remarkable feature is that in some cases—as in our example 2—the lower bound for the allowed domain of modulation amplitudes is very low; in these cases, as well as in cases where the modulation affects only weakly the main variables of the system, the system's behavior is approximately constant in time. In these conditions, the present method can be considered as able to stabilize the autonomous system.

We think this method could be easily tested experimentally on, for instance, a far-infrared ammonia laser or an electronic circuit, among other possible systems. Further characterization of the method, and mathematical interpretation, as well as study of its possible extension to control of chaos in conservative systems, are interesting issues that could be addressed in the immediate future.

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