Nonequilibrium corrections in the pressure tensor due to an energy flux

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The form $\mathbf{P}=a(u,v)\mathbf{U}+b(u,v)\vec{J}\vec{J}$ for the pressure tensor for a system submitted to an energy flux $\vec{J}$ ($\mathbf{U}$ being the identity matrix, u the specific energy, and v the specific volume) widely used for anisotropic radiation and proposed to be more general by Domínguez and Jou [Phys. Rev. E **51**, 158 (1995)] has been recently questioned by R. E. Nettleton [Phys. Rev. E **53**, 1241 (1996)]. We provide a physical basis, in a completely different way, for this expression for anisotropic radiation and ultrarelativistic gases and we criticize some previous physical interpretations. We recall the necessity of an understanding of this kind of expression in a thermodynamic framework. [S1063-651X(96)00312-1]

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Radiation hydrodynamics [3] is a subject of great interest in astrophysics, cosmology, and plasma physics. However, the numerical methods proposed to solve the transfer equation for the specific radiation intensity $\mathcal{I}$ are in many cases computationally too expensive. Therefore one usually considers the equations for the moments of $\mathcal{I}$ up to a given order m [4,5]. Due to the dependence of the equation for the moment m on the moment $m+1$ one needs to introduce a closure relation. If only the energy density u $(m=0)$ and the energy density flux $\vec{J}$ $(m=1)$ are considered, one must introduce a closure relation for the pressure tensor $\mathbf{P}$ $(m=2)$. The usual procedure is to introduce the so-called Eddington factor χ defined by

$$\mathbf{P}=u\left[\frac{1-\chi}{2}\mathbf{U}+\frac{3\chi-1}{2}\vec{n}\vec{n}\right], \tag{1}$$

where $\mathbf{U}$ is the identity matrix, $\vec{n}:=\vec{f}/f$ and $\vec{f}$ the normalized energy flux defined as $\vec{f}:=\vec{J}/cu$. In the limit of isotropic radiation (Eddington limit), $\chi(f=0)=1/3$, while in the free streaming case $\chi(f=1)=1$. A number of different expressions for the Eddington factor have been introduced in the literature [4] by interpolating between these limiting cases. Some of them have been obtained from maximum entropy principles. For instance, in [5,6] radiation under an energy flux is studied by exploiting the entropy inequality, i.e., by maximizing a generalized flux-dependent entropy under a set of constraints, and an Eddington factor given by

$$\chi=\frac{5}{3}-\frac{2}{3}\sqrt{4-3f^2} \tag{2}$$

is obtained. The same result is recovered in [1] from an information theoretical formalism, whereas different versions of this formalism have been used in [7,8] to obtain other variable Eddington factors.

Thus we observe that a flux-dependent pressure tensor with the form

$$\mathbf{P}=a(u,J^2)\mathbf{U}+b(u,J^2)\vec{J}\vec{J} \tag{3}$$

has been widely employed in radiative transfer. In order to obtain such a dependence, which apparently departs from local equilibrium, some authors [1,6,8] have considered a flux-dependent generalized entropy and Gibbs relation. In addition, (3) also appears in the study of an ultrarelativistic ideal gas under an energy flux by means of information theory [1].

However, in spite of what is claimed by the authors in [1,5,6], (2) can be obtained for the two simple cases of radiation and an ultrarelativistic gas without abandoning the local-equilibrium hypothesis. The equations of state and entropies appearing in [1,5,6] may be recovered as well.

First of all, let us notice that these systems are submitted to an *energy flux*, and not to a *heat flux*, because the condition of null global velocity has not been imposed. Therefore it is not difficult to show that the considered situation corresponds to equilibrium (i.e., a purely advective energy flux), in contradiction to what is assumed in [1,5,6]. In fact, due to the symmetry of the energy momentum tensor of a relativistic system (i.e., $T^{\mu\nu}=T^{\nu\mu}$), the energy flux $\vec{J}$ verifies

$$\vec{J}=c^2\vec{P}. \tag{4}$$

This property can also be obtained directly for a system of ideal relativistic particles, with energy $\epsilon_i=\sqrt{m^2c^4+p_i^2c^2}$ and velocity $\vec{v}_i=c\vec{p}_i/\sqrt{p_i^2+m^2c^2}$. The total energy flux is

$$\vec{J}=\sum \epsilon_i\vec{v}_i=:\sum \vec{j}_i, \tag{5}$$

and introducing the expressions for ϵ_i and $\vec{v}_i$, it can be readily verified that

$$\vec{j}_i=\epsilon_i\vec{v}_i=c^2\vec{p}_i. \tag{6}$$

Therefore Eq. (4) holds for this system.

Following (4), the equations of state for a system submitted to an energy flux $\vec{J}$ (without any additional restriction on the particle flow) must be the same as the equilibrium equations of state of a moving system with momentum $\vec{P}$, which can be obtained by simply performing a Lorentz boost to an

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equilibrium system at rest. Therefore the systems considered in [1,5,6] are nothing but moving equilibrium systems. The distribution functions of both cases can be obtained by the use of Lorentz transformations as follows. The rest frame (K_0) equilibrium distribution function can be written as

$$f=\frac{g}{e^{\alpha_0+\beta_0\epsilon_0}+a}, \tag{7}$$

where g is the degeneracy, $\alpha_0=-\beta_0\mu_0$ and $a=-1$ for bosons, $a=1$ for fermions, $a=0$ for particles obeying Boltzmann's statistics, and $a=-1,\mu_0=0$ for photons and phonons. We consider the cases of radiation and a classical ideal ultrarelativistic gas, so $\epsilon_0=p_0c$. An observer at rest in a frame K moving with momentum $-\vec{P}$ and velocity $-\vec{V}$ with respect to the K_0 frame measures an energy $\epsilon=pc$ for a particle with momentum p (and velocity $\vec{c}$) that verifies

$$\epsilon_0=\gamma(\epsilon-\vec{V}\cdot\vec{p}). \tag{8}$$

Substitution of (8) in (7) gives

$$f=\frac{g}{e^{\alpha+\beta\epsilon+\vec{I}\cdot\vec{p}c^2}+a}, \tag{9}$$

where we have defined $\beta:=\gamma\beta_0$ and $\vec{I}:=-\beta\vec{V}/c^2$. Note that $(\beta,\vec{I}c)$ is the so-called coldness 4-vector. If we now use (6) we obtain

$$f=\frac{g}{e^{\alpha+\beta\epsilon+\vec{I}\cdot\vec{j}}+a}. \tag{10}$$

Now, we can recover the distribution function used in [1] for radiation by simply setting $a=-1$, $\alpha=0$, $\epsilon=pc$, $\vec{j}=\epsilon\vec{c}$, $g=2$:

$$f=\frac{2}{e^{\beta pc+\vec{I}\cdot\vec{c}pc}-1}, \tag{11}$$

whereas for the classical ultrarelativistic gas we obtain the distribution function proposed in [1] setting $a=0$. Once this distribution function is fully justified, the whole procedure in [1] holds. Thus the results obtained in [1,5,6] are recovered and, in particular, defining the pressure tensor as the mean value of the operator

$$\hat{P}_{\alpha\beta}:=V^{-1}\sum_{i=1}^{N}p_i^{\alpha}v_i^{\beta}, \tag{12}$$

and using (1), (2) is obtained. However, the physical interpretation given by this derivation is completely different to that given in [1,5,6].

Clearly, the distribution of particles is anisotropic in the frame K due to the additional vectorial constraint (the global momentum $\vec{P}$). The distribution function (11) allows the study of anisotropic equilibrium radiation (being the anisotropy due to the relative motion) but not the study of nonequilibrium situations. We propose the following heuristic argument to understand the physical situation. Although in [1,5,6] different methods were used to arrive at the equations of state of nonequilibrium radiation submitted to an energy flux, the authors never imposed the constraint of no global motion of the system. Therefore, when they made use of the condition of maximum entropy, they found an equilibrium moving system because equilibrium situations have the maximum entropy and the moving system verifies the imposed constraint of nonzero energy flux.

Let us note another interesting feature of (11) related to the physical meaning of temperature in this moving system. The distribution function (11) can also be viewed as a Planck distribution with an effective $\beta_{\rm eff}$ given by

$$\beta_{\rm eff}:=\beta+Ic\cos\theta=\beta\left(1-\frac{V}{c}\cos\theta\right). \tag{13}$$

This expression is used, for instance, in cosmology in the study of the cosmic microwave background radiation (CMBR) in order to take into account the relative movement between the Earth and the reference frame defined by the CMBR. By averaging over the angular dependence with the distribution function (11), it is obtained that

$$\langle\beta_{\rm eff}\rangle=\beta\left(1-\frac{I^2}{\beta^2}\right)=\frac{\beta}{\gamma^2}=\frac{\beta_0}{\gamma}, \tag{14}$$

so it is possible to define an effective mean temperature given by

$$T_{\rm eff}:=\frac{1}{k_B\beta_{\rm eff}}=\gamma T_0, \tag{15}$$

where $T_0:=1/k_B\beta_0$. Therefore $T_{\rm eff}$ is found to be simply the Lorentz transformation of T_0, according to Ott's transformation law [9]. This gives a simple interpretation of Ott's temperature, whose physical bases were controverted during the 1960s [10].

In [11], it has been argued that in some situations it is not possible to apply the methods of nonequilibrium thermodynamics to radiation. This is the case, for example, for two planar surfaces fixed at different temperatures T_1 and T_2 which exchange energy through a radiation field. Photons traveling in one direction are characterized by T_1, and the ones traveling in the opposite direction by T_2, so it is not possible to assign a single temperature to radiation. Based on these arguments, recently Nettleton [2] has criticized the thermodynamical methods used in [5,6] in their analysis of anisotropic radiation. However, we have proved that the system considered in [5,6] is, in fact, an equilibrium moving system and therefore the criticism does not hold in this case. Let us remark that in the system considered by Essex [11] the distribution function is characterized by a double peak, whose relative heights depend on direction, while ours has a single peak whose position varies with direction. Thus it is possible to define an angle-dependent effective temperature.

In addition, in [2] the validity of expressions like (3) for the pressure tensor has been questioned, both for gases and for radiation. We have seen that in the cases of equilibrium moving radiation or an ultrarelativistic gas, the pressure tensor adopts an anisotropic form due to the presence of an additional vectorial constraint (i.e., $\vec{J}$). We think that these simple problems can serve as a guide to more complicated nonequilibrium situations. Therefore it seems a plausible possibility that for a nonequilibrium system submitted to an energy flux and zero mass flow, the pressure tensor also depends on the energy flux, as in equilibrium. If that is the case, the dependence must have the form in (1) because, from a purely algebraic point of view, the most general tensor that may be built up in the presence of a vector $\vec{J}$ must have the form $a(J^2)\mathbf{U}+b(J^2)\vec{J}\vec{J}$ and according to the definition for the pressure tensor tr$\mathbf{P}=u$. However, this question remains open, and such a form for the pressure tensor is not free of difficulties, as pointed out in [2]. Taking into account these criticisms, and the fact that expressions of the form (3) are widely used in radiation hydrodynamics, the convenience of finding a consistent thermodynamic scenario for these systems arises. We should also note that some variable Eddington factors χ have been proposed [7] using maximum entropy principles without a careful interpretation of the generalized flux-dependent entropies that naturally appear in the formalism.

A plausible framework in which to understand these nonequilibrium flux-dependent entropies appearing in radiation transfer may be extended irreversible thermodynamics (EIT) [12]. According to EIT, both temperature and thermodynamic pressure should be modified in nonequilibrium situations, if a generalized flux-dependent entropy function is considered. Up to second order in the fluxes, one has

$$s(u,v,\vec{J})=s_{eq}(u,v)+\alpha(u,v)\vec{J}\cdot\vec{J} \tag{16}$$

and if pressure and temperature are defined, as usual, by the derivatives of the entropy function, one can easily obtain flux-dependent equations of state:

$$\frac{1}{\theta}=\frac{1}{T}+\frac{\partial\alpha}{\partial u}\vec{J}\cdot\vec{J}, \tag{17}$$

$$\frac{\pi}{\theta}=\frac{p}{T}+\frac{\partial\alpha}{\partial v}\vec{J}\cdot\vec{J}, \tag{18}$$

where T is the kinetic or local-equilibrium temperature and p the local-equilibrium pressure and θ and π their generalized flux-dependent counterparts. In addition, the resulting pressure tensor was supposed in [1] to adopt the form

$$\mathbf{P}=\pi\mathbf{U}+\psi\vec{J}\vec{J}, \tag{19}$$

where ψ is determined by the requirement that tr$\mathbf{P}=u$.

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