Dynamic scaling behavior of a growing self-affine fractal interface in a paper-towel-wetting experiment

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The dynamic scaling behavior of a growing self-affine fractal interface is examined in a simple paper-towel-wetting experiment. A sheet of plain white paper towel is wetted with red food dye solution, and the evolution of the interface is photographed with a 35-mm camera as a function of time. Each snapshot is scanned and digitized to obtain the interface height $h(x,t)$ as a function of time and position. From these the interface width $w(L,t)$ is determined as a function of time t and system size L. It is found that the interface width scales with system size L as $w(L,t) \sim L^{\alpha}$ with $\alpha = 0.67 \pm 0.04$ and scales with time as $w(L,t) \sim t^{\beta}$ with $\beta = 0.24 \pm 0.02$. It is also found that average height of the interface scales with time as $\langle h \rangle \sim t^{\delta}$ with $\delta = 0.33 \pm 0.02$. These results are assessed in comparison with the predictions of theoretical models and the results of other relevant experiments. [S1063-651X(96)11907-3]

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I. INTRODUCTION

Ever since the pioneering work of Mandelbrot [1] in 1982 on the fractal geometry of nature, much progress has been made in the field of fractal surfaces [2,3]. Much of the theoretical work has been directed toward an understanding of the dynamic scaling behavior of a growing interfacial surface. The dynamic scaling approach to the description of a growing interfacial surface is based on the anisotropic scaling invariance of self-affine fractals [1], and was initiated by Family [4,5]. Dynamic scaling behavior has been studied in a variety of theoretical models that include discrete particle, computer simulation models [4–10], analytical continuum models [11–14], and models that take into account the quenched disorder [15–18]. However our understanding of the dynamic aspect of a growing interfacial problem, has not been reinforced by extensive experimental studies. Only a handful of experimental investigations [15,16,19–24] have been reported on the subject at this point in time, and the experimental results that have been reported are not entirely consistent with the predictions of theoretical studies. For example, the Eden [9] model was originally proposed as a model of cell growth in biological systems, and yet in a bacterial growth experiment Vicsek, Cserzo, and Horvath [19] found the roughness exponent $\alpha = 0.78$, a value much higher than that of $\alpha = 0.5$ expected from the Eden model. In a paper-wetting experiment Barabasi and co-workers [15,16] found $\alpha = 0.63$, which is higher than the value of 0.5 expected from most of the conventional theoretical studies [7–14] (see Table I). To account for this anomalous roughening indicated by high value of α, investigators [15–17] took into account the quenched noise generated by the disorder in the medium, but they made no experimental study of the dynamic aspect of the growing interface to check the theoretical predictions of the model. In another paper-wetting experiment Family, Chan, and Amar [20] obtained $0.62 \leqslant \alpha \leqslant 0.78$, higher than the value of 0.5 expected from conventional theories [7–14]. In a paper-burning experiment by Zhang *et al.* [21], as well as in a paper-rupturing experiment by Kertesz, Horvath, and Weber [22] the roughness exponents were found to be much higher than the value predicted by conventional theoretical models [7–14]. As far as it is known to the authors, only a few experimental studies have been reported on the *dynamic* scaling behavior of growing self-affine fractal interfaces [20,23,24]. Furthermore, the agreement among experiments of similar nature is far from complete (see, for instance, p. 127, Ref. [2]). It is *not* difficult to see that additional experimental studies can enhance our understanding of dynamics of growing interfaces (see also pp. 3 and 449, Ref. [3]). This paper reports on an experimental study of dynamic scaling behavior observed in a simple paper-towel-wetting experiment. In particular, we report findings of scaling exponents, and incidentally a method of determining the growth exponent using an Excel spreadsheet.

In this paper we consider a $(d-1)$-dimensional surface that is flat at time $t=0$. The surface grows in time, and, as it grows, it roughens as a result of disorder in the medium generated by a random distribution of pore sizes as well as a random variation in the density of the medium, through or over which a fluid flows. We concentrate on a section of the surface having a size L perpendicular to the growth direction. We assume that the surface height can be described by a single valued function $h(x,t)$ in $d=2$ dimensions. We will neglect overhangs, and in case of overhangs, $h(x,t)$ will refer to the highest point at x. The average height of the interface over the length L at a given time t is defined by

$$\langle h(L,t) \rangle = L^{-1} \sum h(x,t), \tag{1}$$

where the summation extends over $x = 1,2,\ldots,L$. The average width of the interface at time t over the length L, $w(L,t)$, is defined by the root mean square value of the height fluctuations,

$$w(L,t) = [\langle h^2 \rangle - \langle h \rangle^2]^{1/2}, \tag{2}$$

TABLE I. Scaling exponents α, β, and δ from various theoretical models and experiments.

	α	β	δ
	THEORETICAL		
Random deposition [4–6]		$\frac{1}{2}$	
RD w/ surface diffusion [7]	$\frac{1}{2}$	$\frac{1}{4}$	
Ballistic deposition [4,8]	$\frac{1}{2}$	$\frac{1}{3}$	
Eden model [9]	$\frac{1}{2}$	$\frac{1}{3}$	
Restricted solid on solid [10]	$\frac{1}{2}$	$\frac{1}{3}$	
Langevin Eq. (EW) [11,12]	$\frac{1}{2}$	$\frac{1}{4}$	
Generalized LE (KPZ) [13,14]	$\frac{1}{2}$	$\frac{1}{3}$	
DPD simulation, pinned [15–17]	0.63 ± 0.02		
DPD simulation moving [15–17]	0.70 ± 0.05	0.70 ± 0.05	0.70 ± 0.05
	EXPERIMENTAL		
Paper wetting, pinned [15,16]	0.63 ± 0.04		
Bacterial colony [19]	0.78 ± 0.07		
Paper wetting [20]	0.62–0.78	0.29–0.40	0.72–0.74
Paper burning [21]	0.71 ± 0.05		
Paper rupturing [22]	0.63–0.72		
Forced fluid flow [23]	0.81	0.65	
Paper wetting [24]		0.56 ± 0.03	
Paper wetting (present)	0.67 ± 0.04	0.24 ± 0.02	0.33 ± 0.02

in which $\langle h^2\rangle$ is defined in a similar manner as $\langle h\rangle$ is defined in Eq. (1). Initially, the surface width grows with time because of the buildup of random fluctuations in the surface heights. The width of the surface is a measure of transverse correlation in the direction of growth. In the absence of any characteristic time or length scale, the function $w(L,t)$ is scale invariant. Therefore, $w(L,t)$ is a homogeneous function of two variables, which can be reduced to a function of a single variable [25],

$$w(L,t)\sim L^{\alpha}f(t/L^{z}), \tag{3}$$

where the dynamic exponent z is defined by $z=\alpha/\beta$. For $t\ll L^{z}$, Eq. (3) reduces to

$$w(L,t)\sim t^{\beta}, \tag{4}$$

and, for $t\gg L^{z}$, Eq. (3) simplifies to

$$w(L,t)\sim L^{\alpha}. \tag{5}$$

Thus the average width $w(L,t)$ scales with time, and the exponent β describes the growth of the width along the growth direction. As the interface evolves in time, the wavelength of the spatial fluctuations and the length over which the fluctuations are correlated also grow with time. The length L is the maximum spatial extent to which the correlations can grow in $d-1$ dimensions along the interface. The saturation value of the width in the limit of a long time, $w(L)$, is expected to have a power law dependence on the length L as given by Eq. (5). In the steady state the interface is a self-affine fractal, and its roughness is quantified by the exponent α. For this reason the exponent α is referred to as the roughness exponent. A great deal of studies related to the dynamics of rough interfaces has been directed toward obtaining the exponents α and β in various theoretical models. These models are either in discrete particle, computer simulation models [4–10,15–17] or in analytical continuum models [11–14,18], and their results are summarized in Table I along with available experimental results for quick reference and comparison.

II. EXPERIMENT

Figure 1 shows the experimental arrangement. A sheet of white paper towel, Scott brand, 240×280 mm^2, is placed over a piece of plate glass held at an angle θ to the horizontal, and at time $t=0$ the shorter edge of the paper is immersed about 1 cm into the aqueous solution of red food dye. The evolution of the interface is photographed with a 35-mm camera placed overhead. The shutter was operated manually, and the time was recorded on a digital clock placed nearby. Initially, the water front and the dye front emerged together, but after 10 s or so the water front moved ahead of the dye front. This may be due to the fact that organic dye molecules are larger than the water molecules. In about 30 s the water

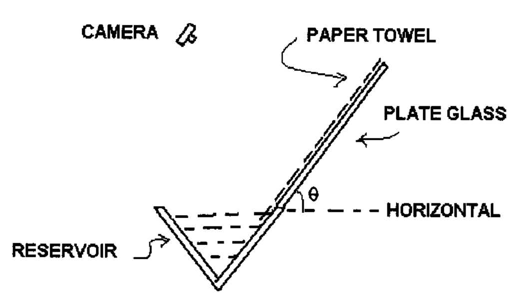

FIG. 1. Experimental arrangement. A sheet of white paper towel is placed over a piece of plate glass held at an angle θ to the horizontal. The reservoir holds an aqueous solution of red food dye.

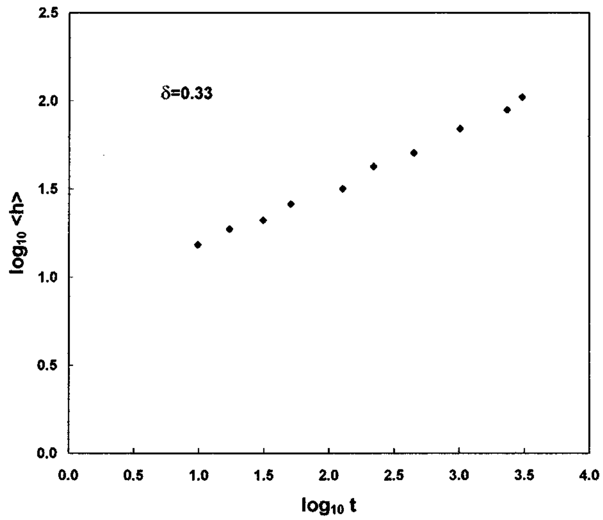

FIG. 2. $\text{Log}_{10}\langle h\rangle$ is plotted as a function of $\log_{10}t$ for the system size $L=2400$ digitized points representing length of 240 mm. The slope of the linear curve was found to be 0.33 ± 0.02.

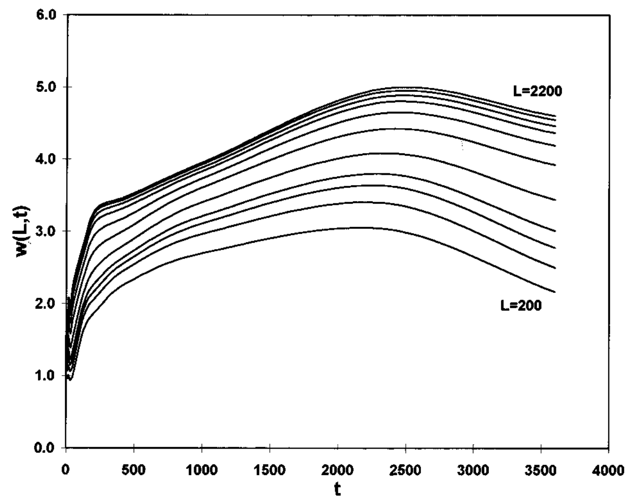

FIG. 3. The interface width $w(L,t)$ is shown as a function of time t for selected values of L. Starting from the bottom, $L=200$, 300, 400, 500, 700, 1000, 1300, 1600, 1800, 2000, and 2200. Each curve represents smooth joining of experimental points.

front led the dye front as much as 0.5 cm, and subsequently the separation gap grew to about 1.5 cm. Because the dye front produced a greater contrast and variation than the water front, we decided to follow the dye front.

To begin with the paper was held horizontally ($\theta=0°$), but the roughness of the interface appeared to grow without leveling off. When the paper was held vertically ($\theta=90°$) with the bottom edge immersed, the roughness appeared to grow for a while and then quickly decrease. When the paper was held at 60°, the roughness of the interface grew for a while and, after reaching a maximum, began to decrease slowly. The results presented in this paper are those for $\theta=60°$.

The evolution of interface was observed and photographed as a function of time at 10 different times in the interval 0 and 3602 s. Each snapshot, 4×6-in color print, was scanned with a Cannon Color Image Scanner IX-4015 with 400×800 dpi^2 resolution. Even though the picture was in color, it was scanned in black and white to increase the contrast and to facilitate subsequent digitizing. Each image, scanned in TIFF format, was converted to PCX format before it was digitized using Un-Scan-It software. Using this software, it was possible to digitize the horizontal length of the paper towel (240 mm) into 2400 points or ten points per mm. Henceforth, $L=2000$, for example, implies an actual physical length of 200 mm.

III. RESULTS AND DISCUSSION

For each given value of time t (or each snapshot) the interface height $h(x,t)$ is obtained for $x=1,2,\ldots,2400$, the height being measured in units of mm. The average height $\langle h\rangle$ as defined by Eq. (1) is determined as a function of time for the maximum system size $L=2400$. In Fig. 2, $\log_{10}\langle h\rangle$ is plotted as function of $\log_{10}t$, and the slope of the linear curve was determined to be 0.33 ± 0.02; therefore the average height scales with time as $\langle h\rangle\sim t^{\delta}$, with $\delta=0.33\pm0.02$. Our value of δ is about the half the value obtained by Family, Chan, and Amar [20], $0.72\leqslant\delta\leqslant0.74$, in a similar experiment using different papers. Barabasi *et al.* [15] obtained $\delta=0.70\pm0.05$ in directed percolation depinning (DPD) simulation. These values are much higher than the theoretically expected value of 0.5, while our value is lower than the theoretically expected value [26]. The difference between ours and that of Ref. [20] may be due to the different nature of the papers used.

For each given time t the interface width $w(L,t)$ as defined by Eq. (2) is obtained as the average of 2400 L values of $w(L,t)$ evaluated over all possible correlated intervals: $[x,x+L]$ for $x=1,2,\ldots,2400$ and for selected values of $L=10$, 20, 30, 40, 60, 100, 200, 300, 400, 500, 700, 1000, 1300, 1600, 1800, 2000, and 2200. For large L values the error margin was large because of the small number of samples over which the average was obtained. Calculation of $w(L,t)$ was facilitated by the use of built-in function STDEV (standard deviation) in Excel.

In Fig. 3 we show $w(L,t)$ as s function of time t for selected values of L with $L=200$, 300, 400, 500, 700, 1000, 1300, 1600, 1800, 2000, and 2200. Each curve represents a smooth joining of experimental points. The figure shows two unexpected features of fluid flow through a porous medium at the beginning ($t<30$ s) and at the end ($t>2500$ s). For $t<30$ s, $w(L,t)$ shows an irregular behavior, indicating the transient nature of the fluid flow. This was unexpected but understandable. The behavior is similar to the behavior of an underdamped oscillator. Since the flow velocity is proportional to the porosity [26], and because the porosity of the paper towel is large, and also because of the closeness of the interface to the fluid reservoir, the fluid initially rushed forward, but because of a lack of damping due to the thinness of the paper towel, the relaxation of the fluid flow is not critically damped but rather underdamped. A similar behavior was observed in the relaxation of spin-spin correlations [27]. For the time interval 30 and 2500 s $w(L,t)$ shows a linear behavior in time, reflecting a steady laminar fluid flow. As the interface nears the maximum height, the fluid flow slows down because of the increased distance from the reservoir or a decrease in the pressure gradient due to capillary action. After reaching a saturation value $w(L)$ near 2500 s, the in-

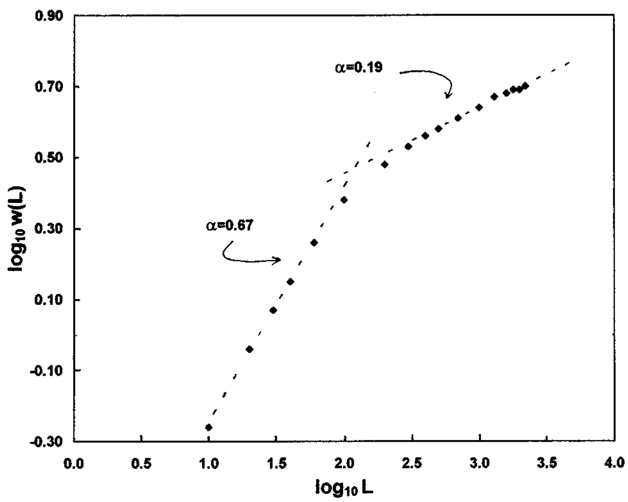

FIG. 4. $\mathrm{Log}_{10}w(L)$ is plotted as a function of $\log_{10}L$. For the short length region the slope indicates $\alpha=0.67$, and for the large length region the slope is 0.19.

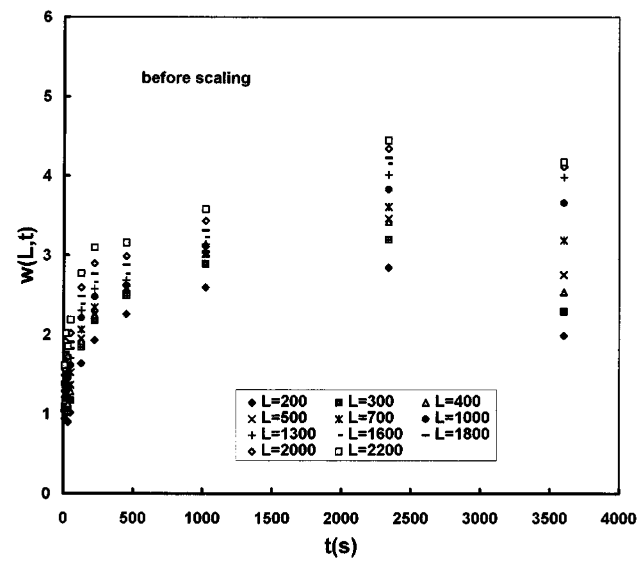

FIG. 5. The interface width $w(L,t)$ is shown as a function of time t for selected values of L using discrete experimental points without joining them with smooth lines.

terface width begins to decrease, instead of remaining constant as expected from theoretical models (see, for instance, Fig. 2.3, p. 22, Ref. [2], or Fig. 3.2, p.77, Ref. [3]). This behavior may be peculiar to the paper-towel-wetting experiment. In paper towel wetting the roughness reaches a maximum value while the paper is still wet. After that, details of the roughness begin to erode slowly, smoothing out the roughness in the interface. In other growth models we have no reason to expect that the width will peak at a certain time τ and then decrease, so that $w(L,\tau)>w(L,t\rightarrow\infty)$. In our experiment $w(L)=w(L,\tau)$, with $\tau\approx2500$ s.

We took these peak values of $w(L,\tau)$ shown in Fig. 3 as the saturation value $w(L)$, and plotted $\log_{10}w(L)$ as a function of $\log_{10}L$, as shown in Fig. 4. The slope of the initial portion indicates that $\alpha=0.67\pm0.04$. This value agrees well with the value obtained by Family, Chan, and Amar [20], (see Table I). It is also in accord with the prediction of the DPD simulation in the moving phase [15–17]. It is interesting to note that, in our experiment, after the interface has reached the maximum height, the saturated width $w(L)$ did not remain constant as expected from theoretical model studies. Instead, the width $w(L)$ began to decrease, though slowly. This, we believe, is due to the fact that details of the roughness erode as the red food dye smears out in the neighborhood region. Therefore, we believe that the peak values of the width shown in Fig. 3 should be taken as saturated values of the width $w(L)$. If we waited until the paper was completely dry and interface reached a pinned phase before we digitized it, many of the details of structures in the interface would have been lost. In a way our result may be viewed as consistent with the result of Barabasi *et al.* [15], in that they obtained $\alpha=0.63$ for the pinned phase in their experiment as well as in their DPD model simulation, but $\alpha=0.70$ for the moving phase in their DPD model simulation (see Table I for details).

To determine the growth exponent β, we first plotted $w(L,t)$ as shown in Fig. 5 using discrete experimental points, without joining them with smooth lines as was done in Fig. 3, for the purpose of showing some of the details in

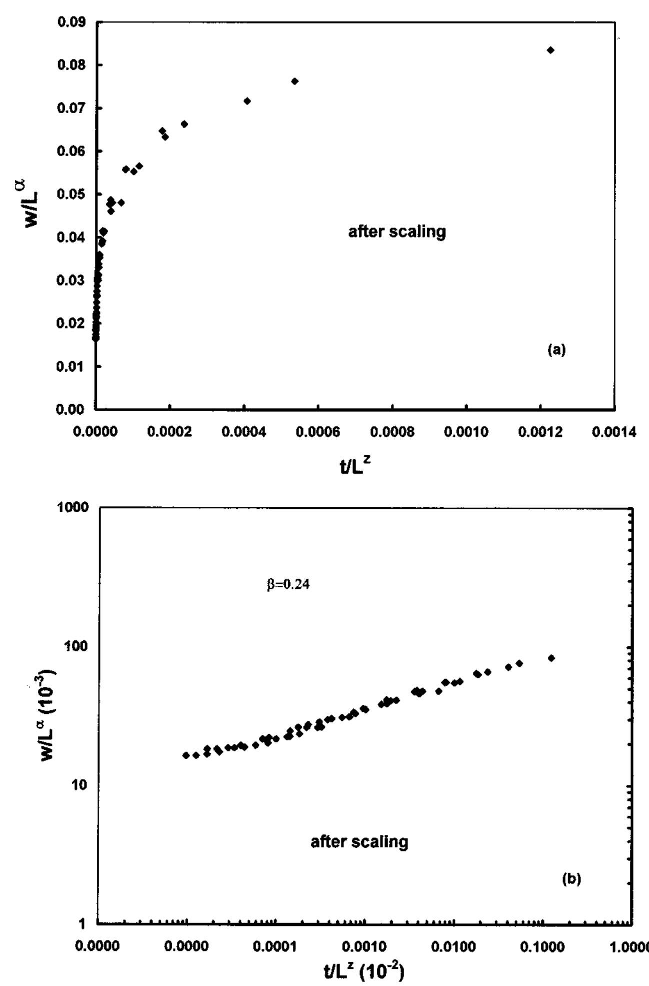

FIG. 6. Scaled width is shown as a function of scaled time using a regular scale in (a), and using a log-log scale in (b). It is seen in both of these graphs that those scattered points in Fig. 5 collapse into a single line indicating scaling behavior.

the early time region ($t<30$ s). Next we plotted the scaled width $w(L,t)/L^{\alpha}$ as a function of the scaled time t/L^{z} for each selected value of L, using the regular scale in Fig. 6(a), and using the log-log scale in Fig. 6(b). To do this we had to assume a value for β; to begin with we set $\beta=0.333$. We displayed both of these graphs in the window screen, and gradually varied β until those scattered points, as shown in Fig. 5, collapsed into a single line as shown in Fig. 6(a), and into a single straight line shown in Fig. 6(b). Inasmuch as many points fell on top of one another in Fig. 6, no effort was made to distinguish those points belonging to different L values as was done in Fig. 5. The optimum value thus arrived at is $\beta=0.24\pm0.02$. This value was confirmed by calculating the average of slopes obtained in 11 different L values. That is, we calculated the slope in each $\log_{10}[w(L,t)/L^{\alpha}]$ vs $\log_{10}(t/L^{z})$ curve for 11 different L values, and an average of the slopes was obtained. We believe that this method of determining β is new and original. Our value of $\beta=0.24$ is close to the value of 0.25 predicted by the Lengevin equation (EW) [11,12]. Our value is slightly lower than the values, 0.29–0.40, obtained by Family, Chan, and Amar [20]. Neither our value, $\beta=0.24\pm0.02$, nor the value of Family, Chan, and Amar [20], $0.29\leqslant\beta\leqslant0.40$, agree with the simulation value predicted by the DPD model of Barabasi and coworkers [15–17], $\beta=0.70\pm0.05$. It is interesting to note that a recent value of $\beta=0.56\pm0.03$ obtained by Horvath and Stanley [24] in a paper-wetting experiment is also lower than the DPD simulation value.

The scaling behavior we observed in our experiment is valid for a limited time interval and spatial range. The time interval in which the scaling behavior is observed in our experiment is 30 and 2500 s. This time interval is wider than the time interval in which Family, Chan, and Amar [20] observed their scaling behavior (615 and 1985 s) (see Fig. 5 of Ref. [20]). The spatial extent over which scaling is observed is 2 and 22 cm or 200 and 2200 in digitized points. This appears to be quite reasonable, in that for a length less than 2 cm or 200 digitized points a consistent statistical behavior may not be expected.

Family, Chan, and Amar [20] briefly discussed the possibility of crossover in the scaling behavior of the interfacial width. At a larger length scale a different scaling behavior may be expected, possibly due to the existence of a characteristic length scale at a later time (see Fig. 5 of Ref. [20]). Buldyrev *et al.* [16] went a step further to determine $\alpha=0.73$ in region I, where $\xi_{\parallel}<L$, and $\alpha=0.5$ in region II, where $\xi_{\parallel}>L$ (see Fig. 5 of Ref. [16]). We have also found two distinct regions, as shown in Fig. 4. In the short-L region we found $\alpha=0.67$, and in the longer L region we found $\alpha=0.19$. We went a step further. We searched for but failed to find another value of β which, when paired with $\alpha=0.19$, will scale the interface width.

Through a simple experiment that does not require sophisticated equipment, it has been possible to study the dynamic scaling behavior of a growing self-affine fractal interface. We find that the interface width scales with time t and with system size L. We also find that the average interface height scales with time. However, the scaling behavior is valid only over a limited time and space. The dynamic scaling approach, nevertheless, is a useful means of describing fluctuating far-from-equilibrium phenomena, which cannot be described in terms of a standard formalism in equilibrium statistical mechanics.

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