Synchronization of oscillators with random nonlocal connectivity

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In this paper we study the existing observation in literature about synchronization of a large number of coupled maps with random nonlocal connectivity [Chate and Manneville, Chaos **2**, 307 (1992)]. These connectivities which lack any spatial significance can be realized in neural nets and electrical circuits. It is quite interesting and of practical importance to note that a huge number of maps can be synchronized with this connectivity. We show that this synchronization stems from the fact that the connectivity matrix has a finite gap in the eigenvalue spectrum in the macroscopic limit. We give a quantitative explanation for the gap. We compare the analytic results with the ones quoted in the above reference. We also study the departures from this highly collective behavior in the low connectivity limit and show that the behavior is almost statistical for very low connectivity.

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I. INTRODUCTION

Of late, there has been considerable attention paid to the study of coupled map lattices (CML) in various contexts. They have been used as a computationally simple and analytically tractable model for spatiotemporal systems [1]. The studies on CML's have been either in one and two dimensional lattices or with global coupling, in which case there is no notion of lattice geometry [2]. The higher dimensional connectivities [3] or hierarchial connectivities [4,5] are studied very little.

One more system that has been studied is the CML with random nonlocal couplings [6]. The motivation is twofold. First, this is an effectively high dimensional system. The phenomenology in CML in higher dimensions has not been studied much, and needs further investigation. Second, there are systems like neural nets in which the local connections do not have any spatial significance. There also exist systems like electrical circuits [7] in which connectivity is at one's will and such a coupling can be easily realized. Thus the studies of different connectivities and their effects will be useful in designing well controlled systems. In this system of random nonlocal connectivity Chate and Manneville have presented preliminary results [6] which show that synchronization of a large number of oscillators is easily achieved with this connectivity.

Synchronization of oscillators in spatially extended systems such as coupled oscillators is important from various points of view. By achieving synchronization, one can try to build huge but more controllable and better behaved systems which are effectively low dimensional [7]. Sometimes, synchronization may serve other purposes, such as sending codes that are difficult to break [8]. In various contexts, this problem has been subjected to several studies in the past few years [9]. We would like to show that the phenomenon of synchronization can be generally understood by investigating the eigenvalue spectrum of the connectivity matrix and can help us to understand the existing observations. In this paper, we will explicitly illustrate how one can separate the mode leading to spatial homogeneity from the rest. We will show that there exists a finite gap between the growth rates, the spatially homogeneous mode, and the rest in the model studied in Ref. [6]. We will also study the departures from this behavior for lower connectivities.

For the linear stability analysis of the synchronized state, we will study the eigenvalue spectrum of the connectivity matrix with random nonlocal connectivity. We will also study the eigenspectrum of the product of such matrices. We would note that a similar model of random connectivity matrix of size $N\times N$ with k nonzero elements in each row has been investigated by Cook and Derrida in Ref. [10] in connection with the random energy model, the generalized random energy model, and directed polymers in random media. They have obtained exact analytic results for products of such matrices in the case where the matrices are sparse and the distribution function of nonzero elements is not a δ function. However, in the model studied by Chate and Manneville [6], all connections have the same weight, i.e., the distribution function of nonzero elements *is* a δ function.

II. RANDOMLY COUPLED CML AND THE LINEAR STABILITY ANALYSIS

The model is the following: there are N sites. Each site is coupled to k sites chosen randomly. The connection is not necessarily symmetric. A site can be connected to some other site more than once and can be connected to itself. The strength of coupling is proportional to the number of times the two sites are connected.

Let us define the ''neighborhood'' V_i for each site i in the above model. There are k sites (which site i is connected to) in V_i. These k sites are chosen randomly. Thus $V_i=\{c_1^i,c_2^i,\ldots,c_k^i\}$ where k randomly chosen sites $c_l^i, l=1,\ldots,k, i=1,\ldots,N$, to which the site i is connected, form the neighborhood of site i. As mentioned above, connections to all sites are equiprobable and the possibility that site i can be connected to some site j more than once is not ruled out. Now we define the interaction matrix I for the

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above connectivity. The matrix element $I_{i,j}$ is equal to the number of times site i is connected to site j. If site i is not connected to site j, obviously the matrix element $I_{i,j}$ has a zero value. Thus we define

$$I_{i,j}=\sum_{l=1}^{k} \delta_{j,c_l^i}. \tag{1}$$

The value of the element $I_{i,j}$ of the interaction matrix is the number of times site i is connected to site j, i.e., the number of times j occurs in V_i. Thus it could have values from 0 to k. [In the infinite lattice limit ($k<<N$), the possibility that two sites will be connected more than once is negligible. Thus the entries are 0 or 1. However, the summation in the above expression is introduced to take care of the possibility that the two sites may be connected more than once, i.e., V_i contains j more than once.] We also note that in the above model, the connectivity matrix I is not symmetric, i.e., $I_{i,j} \neq I_{j,i}$ in general.

For convenience, let us denote the ith row of matrix I by I_i. The only k elements with nonzero value in I_i will be at sites c_l^i's, $l=1,\ldots,k$.

Now we define a spatiotemporal system as follows. Let us associate a real number $x_i(t)$ with the state of site i at time t. The evolution rule for the above dynamical system (coupled map system in the above work) is defined by

$$x_i(t+1)=k^{-1}\sum_j I_{i,j}f(x_j(t)), \tag{2}$$

where $i=1,\ldots,N$. The function $f:I\rightarrow I$ is some function from a real interval I onto itself.

Using Eq. (2) and the fact that $\Sigma_j I_{i,j}=k$ for all i, it is easy to verify that if one starts with the pattern in which all the points are in a coherent state, i.e., $x_1(t)=x_2(t)=\cdots=x_N(t)=x(t)$, they remain in the coherent state for all times $t'>t$. The connectivity is such that the time evolution does not destroy coherence and the evolution is like the evolution of a single map.

Thus a synchronized state is indeed an allowed pattern. In order that this "allowed" pattern is indeed realized in practice for at least some set of initial conditions which span a nonzero volume in the allowed phase space, this pattern should be stable against infinitesimal perturbation. In [4] the generic conditions for synchronized chaotic evolution in a macroscopic system are discussed. In this work we analyze the linear stability of a synchronized state on the lines of arguments in Ref. [4].

For the linear stability analysis the eigenvalues and eigenvectors of the matrix $J=\lim_{\tau\rightarrow\infty}J(\tau)$, where $J(\tau)=J_\tau\cdots J_2J_1$, are (asymptotically) relevant. The Jacobian matrix at time t, i.e., J_t is given by $J_t(i,j)=k^{-1}I(i,j)f'(x_j(t))$ and $x_j(t)=x(t)$ for all j. Thus the Jacobian matrix is $J=\lim_{\tau\rightarrow\infty}[I/k]^\tau f'(x_\tau)f'(x_{\tau-1})\cdots f'(x_1)$. The eigenvalues of J are $\lim_{\tau\rightarrow\infty}\lambda_i^\tau$, where $\lambda_i=v_i\lambda/k$ where $v_i, i=1,2,\ldots,N$ are the eigenvalues of the interaction matrix I and $\lambda=\lim_{\tau\rightarrow\infty}|f'(x(\tau))f'(x(\tau-1))\cdots f'(x(1))|^{1/\tau}$. The relevant eigenvectors are those of I, and the problem reduces to a study of the eigenvalues and eigenvectors of the interaction matrix I.

The fact that coherent patterns are allowed implies that a right eigenvector of the interaction matrix is $e_1=[1,1,\ldots,1]$. This is a characteristic of row stochastic matrices, and corresponds to the eigenvalue λ for the product of the J's. From Greshgorin's theorem [11] this is the largest eigenvalue. Consider a small deviation, $\Delta_0=[\delta_1,\delta_2,\ldots,\delta_N]$, from the homogeneous pattern $[x(0)$, x(0), ... ,x(0)]. We can reexpress Δ_0 in terms of its component along e_1 and the rest as Δ'.

$$\Delta_0=a_1e_1+\Delta'. \tag{3}$$

We will explicitly show that in this particular case it is possible to decompose the matrix I in a component along e_1 and along the $N-1$ dimensional matrix S in the subspace orthogonal to e_1. It is easy to check that using a similar technique it is possible to do the same for any row stochastic matrix. If the only eigenvalue with modulus greater than unity is $\lambda_1=\lambda$ and the matrix S which is a projection of matrix I in the $N-1$ dimensional subspace orthogonal to e_1 has all eigenvalues less than unity, then for large enough t we can write

$$\Delta_t \simeq a_1\lambda_1^t e_1. \tag{4}$$

The perturbation grows along the direction $e_1=[1,1,\ldots,1]$ and any random deviation will eventually be homogenized. Thus the necessary condition for the synchronized pattern to exist (and evolve chaotically in time) is that λ_1 is the only eigenvalue greater than unity and all others in a subspace orthogonal to e_1 are less than unity in magnitude. Since we achieve this decomposition by a simple similarity transformation, one can put the above statement as a linearly stable coherent pattern—in the infinite lattice limit—which therefore requires a finite *gap* in the eigenvalue spectrum of the interaction matrix.

Now the question is whether the interaction matrix I mentioned above has a finite gap in the spectrum.

One more interesting observation in Ref. [6] is the following. They select a different interaction matrix each time. The number of connections that each site i has is still k. However, the sites to which it is connected changes every time. (The neighborhood of site i, V_i, still has k elements but elements keep on changing in time.) Thus the interaction matrix I depends on time. It is easy to see that for this case the condition is that the product of the interaction matrices should have a gap. In other words the effective interaction matrix I', where $I'^\tau=I_\tau I_{\tau-1}\cdots I_1$, has a gap in its spectrum in the asymptotic limit.

We will show that the interaction matrix or the effective interaction matrix mentioned above indeed has a gap. The largest eigenvalue is k. The magnitude of the second largest eigenvalue is of the order $\sqrt{k}$ in the infinite lattice limit.

First let us give a qualitative explanation. Let us take an example of interconnectivity matrix with $N=6$ and $k=2$. Let the matrix I be

$$I=\begin{pmatrix} 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}. \tag{5}$$

Thus $V_1=\{c_1^1,c_2^1\}=\{2,5\}, V_2=\{1,6\}, \ldots$ and $V_6=\{1,4\}$.

It is clear that $e_1=[1,1,\ldots,1]$ is a right eigenvector with eigenvalue $k=2$. Let the next eigenvector in the space orthogonal to e_1 be $e_2=[u_1,u_2\ldots,u_N]$. Orthogonality with e_1 would imply that $\Sigma_{i=1}^{N}u_i=0$. Since the connectivity is nonlocal, the ''correlation length'' does not have a meaning. Correlations, if any, are expected to span the entire lattice. Thus the correlations can only be expected in the zeroth Fourier component which is the vector $[1,1,1\cdots 1]$. However, in the space perpendicular to this vector we expect the components to be δ correlated. Thus a mean value of u_i is zero and e_2 is a vector which has components which are random elements with zero mean. If this is an eigenvector with eigenvalue λ_2, i.e., $Ie_2=\lambda_2 e_2$, the equation implies that the sum of k of the random elements (on the left hand side) scales with the element itself as λ_2. However, by the law of large numbers, one would expect the largest scaling factor of the sum of k random numbers with itself to be of the order of $\sqrt{k}$. Thus one could guess that λ_2 should be at most of the order $\sqrt{k}$. (This is something like displacement from the origin after t time steps in symmetric random walk in 1D, which is like sum of t random numbers with zero mean scales as $\sqrt{t}$.) Thus the spectrum has a gap for $k>1$.

Now let us try to give a formal proof for the above statement. We will carry out a similarity transformation of the above matrix to separate its component along the eigenvector e_1 and a matrix in an $N-1$ dimensional space orthogonal to it.

The Fourier matrix of order N is given by $F_N(m,n)=\omega^{(m-1)(n-1)}$ while its inverse F_N^{-1} is given by $F_N^{-1}(m,n)=\omega^{-(m-1)(n-1)}$, where $\omega=e^{2\pi i/N}, i=\sqrt{-1}$. For example, the Fourier matrix of order 6 will be

$$F_6=\frac{1}{\sqrt{N}}\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \omega & \omega^2 & \omega^3 & \omega^4 & \omega^5 \\ 1 & \omega^2 & \omega^4 & \omega^6 & \omega^8 & \omega^{10} \\ 1 & \omega^3 & \omega^6 & \omega^9 & \omega^{12} & \omega^{15} \\ 1 & \omega^4 & \omega^8 & \omega^{12} & \omega^{16} & \omega^{20} \\ 1 & \omega^5 & \omega^{10} & \omega^{15} & \omega^{20} & \omega^{25} \end{pmatrix} \tag{6}$$

and $F_6^{-1}=F_6^*$.

For convenience let us denote the ith row (column) of F as $f_i=1/\sqrt{N}(1,\omega^{(i-1)},\omega^{2(i-1)},\ldots,\omega^{(N-1)(i-1)})$ and the ith row (or column) of F^{-1} as $f_i^{-1}=1/\sqrt{N}(1,\omega^{-(i-1)},\omega^{-2(i-1)},\ldots,\omega^{-(N-1)(i-1)})$.

The transformed matrix is $K=F_N^{-1}IF_N$ given by

$$K=\frac{1}{N}\begin{pmatrix} kN & N(I_1\cdot f_2,I_2\cdot f_2,\ldots,I_N\cdot f_2)\cdot f_1^{-1} & \cdots & N(I_1\cdot f_N,I_2\cdot f_N,\ldots,I_N\cdot f_N)\cdot f_1^{-1} \\ 0 & N(I_1\cdot f_2,I_2\cdot f_2,\ldots,I_N\cdot f_2)\cdot f_2^{-2} & \cdots & N(I_1\cdot f_N,I_2\cdot f_N,\ldots,I_N\cdot f_N)\cdot f_2^{-1} \\ 0 & N(I_1\cdot f_2,I_2\cdot f_2,\ldots,I_N\cdot f_2)\cdot f_3^{-1} & \cdots & N(I_1\cdot f_N,I_2\cdot f_N,\ldots,I_N\cdot f_N)\cdot f_3^{-1} \\ 0 & \cdots & \cdots & \\ 0 & \cdots & \cdots & \\ 0 & N(I_1\cdot f_2,I_2\cdot f_2,\ldots,I_N\cdot f_2)\cdot f_N^{-1} & \cdots & N(I_1\cdot f_N,I_2\cdot f_N,\ldots,I_N\cdot f_N)\cdot f_N^{-1} \end{pmatrix} \tag{7}$$

(Though this separation between largest eigenvalue and its minor has been explicitly carried out in this case, it is easy to check that it is possible for any row stochastic matrix.) In the particular case above for which $k=2$, $N=6$, the above matrix can be written as

$$K=\frac{1}{N}\begin{pmatrix} 2N & (\omega^{c'(1,1)}+\omega^{c'(1,2)})+(\omega^{c'(2,1)}+\omega^{c'(2,2)})\cdots+(\omega^{c'(6,1)}+\omega^{c'(6,2)}) & \cdots \\ 0 & (\omega^{c'(1,1)}+\omega^{c'(1,2)})+\omega^{-1}(\omega^{c'(2,1)}+\omega^{c'(2,2)})\cdots+\omega^{-5}(\omega^{c'(6,1)}+\omega^{c'(6,2)}) & \cdots \\ 0 & (\omega^{c'(1,1)}+\omega^{c'(1,2)})+\omega^{-2}(\omega^{c'(2,1)}+\omega^{c'(2,2)})\cdots+\omega^{-10}(\omega^{c'(6,1)}+\omega^{c'(6,2)}) & \cdots \\ 0 & \cdots & \cdots \\ 0 & \cdots & \cdots \\ 0 & (\omega^{c'(1,1)}+\omega^{c'(1,2)})+\omega^{-5}(\omega^{c'(2,1)}+\omega^{c'(2,2)})\cdots+\omega^{-25}(\omega^{5c'(6,1)}+\omega^{c'(6,2)}) & \cdots \end{pmatrix} \tag{8}$$

where $c'(i,j)=c_j^i-1, i=1,2,\ldots,N, j=1,\ldots,k$. A typical matrix element $K(i,j)$ in this matrix is $K(i,j)=1/N\Sigma_{m=1}^{N}(\Sigma_{l=1}^{k}\omega^{(j-1)c'(m,l)})\omega^{-(m-1)(i-1)}$. Thus $K(i,1)=1/N\Sigma_{m=1}^{N}k\omega^{-(m-1)(i-1)}$, and is 0 for $i\neq 1$ since the sum of roots of unity is zero except for unity itself.

The above matrix can also be written as

$$K=\left(\begin{array}{c|ccccc} k & - & - & - & - & - \\ - & - & - & - & - & - \\ 0 & & & & & \\ 0 & & & & & \\ \vdots & & S & & & \\ 0 & & & & & \end{array}\right), \tag{9}$$

where S is an $N-1$ dimensional matrix in the subspace orthogonal to e_1.

For convenience, we define matrices S', S'', and S''' by

$$S=\frac{1}{N}S'=\sqrt{k}S''=\frac{\sqrt{k}}{\sqrt{N}}S'''. \tag{10}$$

The eigenvalues of the transformed matrix are k and the eigenvalues of the minor of the $K_{1,1}$, i.e., S. In S' each element is the sum of Nk randomly chosen roots of unity. Now let us rewrite the modulus of one individual element of S as $|S_{i,j}|=1/N|S'_{i,j}|=\sqrt{k}|S''_{i,j}|$ (by the above definition). One can write $S_{i,j}$ as $S_{i,j}=(\sqrt{k}/\sqrt{N})(1/\sqrt{Nk})(S'_{i,j})=(\sqrt{k}/\sqrt{N})S'''(i,j)$. A typical element in S''' is $S'''(i,j)=(Nk)^{-1/2}\Sigma_{l=1}^{Nk}\exp(\theta_l)$, where θ_l are randomly chosen. The variance of the modulus of this typical element

$$\langle|S'''(i,j)|^2\rangle=(\langle\{[\Sigma_{l=1}^{Nk}\cos(\theta_l)]/\sqrt{Nk}\}^2\rangle + \langle\{[\Sigma_{l=1}^{Nk}\sin(\theta_l)]/\sqrt{Nk}\}^2\rangle.$$

The law of large numbers implies that both the first and second terms have an expectation value 1/2. Since the second moment is defined for the terms, the central limit theorem asserts that the distribution will be Gaussian. (Rigorously speaking, distribution is approximately Gaussian and the approximation becomes more and more exact for large values of Nk.) Thus the modulus of the sum is a quantity which has a Gaussian distribution with variance 1. Exploiting the fact that different Fourier components of the same random vector are independent of each other and that components of different random vectors form a random vector [see Eq. (7)], one can conclude that the matrix elements are independent of each other. In brief, S''' is an $N-1$ dimensional matrix with independent identically distributed variables, modulus of whose elements has a Gaussian distribution with variance unity. Thus the matrix $S'''/\sqrt{N}=S''$ has elements such that for large N, $N\langle|S''_{i,j}|^2\rangle=1$, $i=1,\ldots,N-1$, $j=1,\ldots,N-1$. Now S'' is an asymmetric complex random matrix whose elements are i.i.d. and such that their variance goes as $N\langle|S''_{i,j}|^2\rangle=1$. We know that [12,13] eigenvalues of such matrices lie in the unit circle in the complex plane. Thus the eigenvalues of S should lie in a circle of radius $\sqrt{k}$ in the infinite lattice limit.

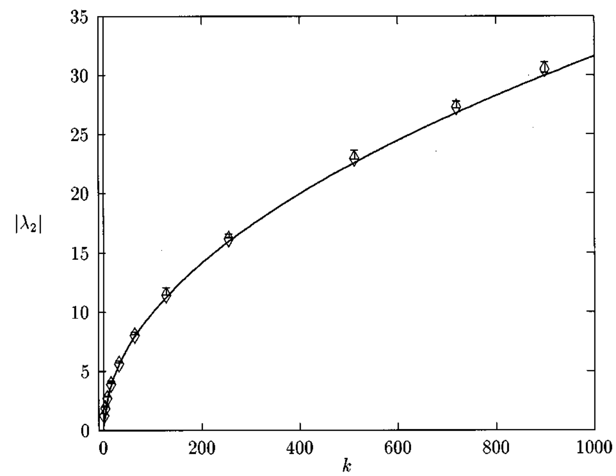

FIG. 1. This figure shows an eigenvalue of the interaction matrix which is the second largest in magnitude as a function of k. The values closely follow the curve $\sqrt{k}$. The points are obtained by averaging over 10–15 configurations and fluctuations are of the size of error bars.

We have numerically confirmed the above statement by determining the second largest eigenvalue in large matrices. We operate a vector which is orthogonal to the largest eigenvector (and orthogonalize it repeatedly) and determine the magnitude of the second largest eigenvalue. Figure 1 shows the eigenvalue which is the second largest in magnitude for different values of k. The points are obtained by averaging over 10–15 different configurations for $N=1000$ and the fluctuations are of the size of the error bars. It is very clear that within numerical accuracy the second largest eigenvalue is indeed $\sqrt{k}$. Figure 2 shows the eigenvalues of 10 configurations of matrices with $N=100$ superposed over each other for $k=16$. It is very clear that while one of the eigenvalues is

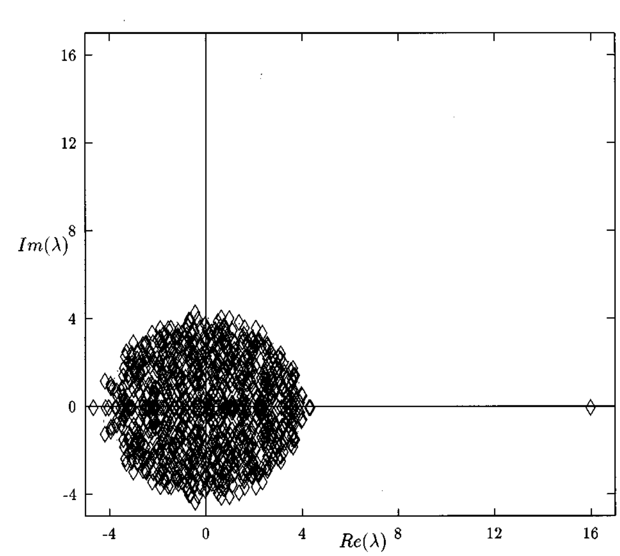

FIG. 2. This figure shows the eigenvalues of 10 configurations of matrices with $N=100$ superposed over each other for $k=16$. While one of the eigenvalues is 16, the rest are almost in the circle of radius 4.

16, the rest are in a circle of radius 4.

Now let us turn to the product of such matrices. Here we do not have the exact proof. We argue as follows. Each of the matrices can be Fourier transformed. Since the Fourier matrix depends only on dimensionality of the matrix and not the elements [see eq. (6)], the product of these matrices has the same eigenvalues as the product of transformed matrices. Let us consider the matrix

$$\begin{aligned} I'^{\tau} &= I_{\tau} I_{\tau-1} \cdots I_1 \equiv F_N^{-1} I_{\tau} I_{\tau-1} \cdots I_1 F_N \\ &= F_N^{-1} I_{\tau} F_N F_N^{-1} I_{\tau-1} F_N \cdots F_N^{-1} I_1 F_N \\ &= K_{\tau} K_{\tau-1} \cdots K_1, \end{aligned} \tag{11}$$

where $K_t = F_N^{-1} I_t F_N$ and has the same form as Eq. (9). This product will have the same structure and one of the eigenvalues of the product of τ. Such matrices will clearly be k^{τ}. Let us denote the minor of the element $K_t(1,1)$ by S_t. The minor of the first element in the first row of this matrix will be $\Pi_{t=1}^{\tau} S_t$. For clarity, let us write the above matrix as

$$I^{\tau} \equiv \left(\begin{array}{c|cccc} k^{\tau} & ----- & ---- & - & - \; - \\ --- & ----- & ---- & - & - \; - \\ 0 & & & & \\ 0 & & & & \\ \vdots & & S_{\tau} \cdots S_1 & & \\ 0 & & & & \end{array} \right). \tag{12}$$

The eigenvalues of the transformed matrix are $(kN)^{\tau}$ and the eigenvalues of $\Pi_{t=1}^{\tau} S_t$. Each of these matrices S_t has entries which are independent identically distributed random variables. In the case of real matrices it has been proven that under certain conditions the noncommutative nature of the matrices can be ignored [12,14]. Thus, using the law of large numbers the largest eigenvalue of the product converges to λ^{τ}, where λ is the expectation value of the largest eigenvalue of an individual matrix. The necessary condition for this theorem holds for complex matrices also [15]. The condition is that the distribution of $||S_t(\mathbf{z}/||\mathbf{z}||)||$ is independent of $\mathbf{z} \neq 0$ in $\mathscr{C}^N$. Cohen and Newman [14,12] have also given a sufficient condition for the above. In particular, it holds for the product of random matrices where the elements are jointly Gaussian variables and the columns are independent and identically distributed. If we assume the sufficient condition to be true for Gaussian *complex* random matrices [16], one can claim that the largest Lyapunov exponent of the product will asymptotically converge to $k^{\tau/2}$. Thus the effective interaction matrix I' has the second largest Lyapunov exponent with magnitude $\sqrt{k}$. The departures of the finite time Lyapunov exponent from this value will be less and less in the infinite time limit. Now let us address the question of whether the above analysis explains the results obtained by Chate and Manneville [6].

The answer is in the affirmative. The largest eigenvalue in the above evolution is λ, which is the same as the one in the single map evolution. The second largest is $\lambda\sqrt{k}/k = \lambda/\sqrt{k}$.

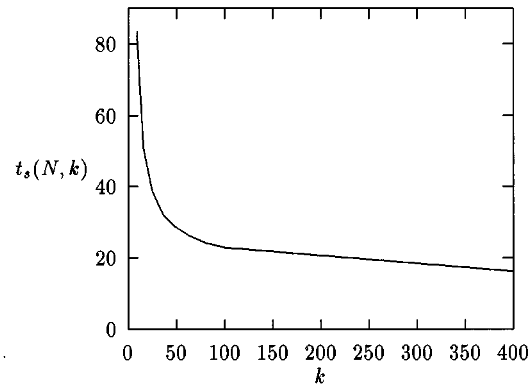

FIG. 3. Dependence of average time needed for synchronization $t_s(N,k)$ as a function of connectivity k for $N=1000$.

The maps explored in the above paper are logistic maps for which the largest value of λ is 2.

Thus for $k>4$ the maps must get synchronized (Fig. 3 of Ref. [6]) if the second largest eigenvalue is less than or equal to the expected value. However, for a given realization there could be fluctuations and thus one may not observe synchronization and might have to go to slightly higher values of k. Thus if one wants the convergence for all realizations, the value could be a little higher in this case.

However, in the case in which the interaction matrix is newly selected each time, one observes that $k=4$ is indeed the transition point. This clearly reflects the fact that in the case of the product of matrices, the fluctuations away from the expected value are quickly averaged out.

Now let us turn to the question of how the time needed from synchronization depends on N and k. We denote the average time needed to synchronize a system of N maps with each site connected to k sites as $t_s(N,k)$. Figure 3 shows the dependence of the averge time needed for synchronization $t_s(N,k)$ as a function of k for $N=1000$. The average time needed for synchronizing N maps decreases with k and it saturates for large values of k. If one investigates variation in $t_s(N,k)$ as a function of N for given k, one can see that for a large enough value of N the time required is virtually unchanged. Figure 4 shows the dependence of this average time as a function of N for $k=9$. It is interesting that the time required for synchronization does not increase for large N. This may have to do with the fact that the connectivity is ''locally treelike'' and thus the information at any point spreads quickly all over the lattice. The higher average time at smaller N may have to do with higher fluctuations in the second largest eigenvalue for lower values of N.

III. TRANSITION FROM COLLECTIVE BEHAVIOR TO STATISTICAL BEHAVIOR

Now we consider departures from the synchronized state. If the function is not highly chaotic, i.e., λ is small, the synchronization is achieved fairly quickly. Thus in order to understand the departures, we should consider a highly chaotic function, i.e. one with a large Lyapunov exponent. Let the function $f(x)$ in Eq. (2) be a thrice iterated logistic map,

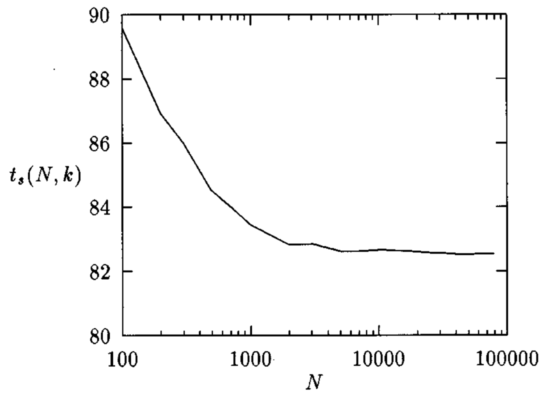

FIG. 4. Dependence of average time needed for synchronization $t_s(N,k)$ as a function of number of sites N for $k=9$.

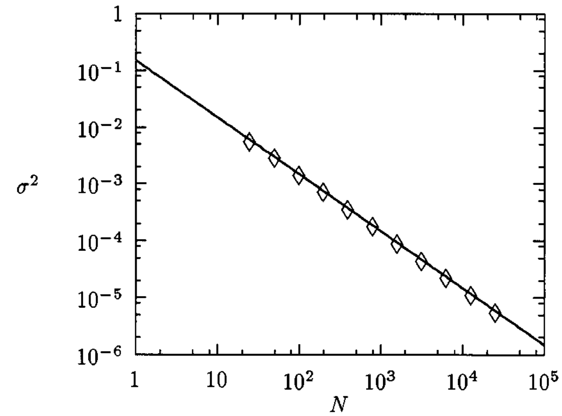

FIG. 6. The mean square deviation of the mean field as a function of total number of sites for CML with $k=4$. The map $f(x)=g^3(x)$, where $g(x)=4x(1-x)$.

i.e., $f(x)=g^3(x)$ where $g(x)=4x(1-x)$. The exponent λ for a single map $g(x)$ is 2 and that for $f(x)$ is 8. This function f being the third iterate of the logistic map is an eighth order polynomial with four maxima and four minima. For $\lambda=8$, one would expect complete synchronization of a large number of oscillators for $k>64$, though the onset will differ a little for a given configuration. As in [6], let us define the collective variable $h(t)=\Sigma_{i=1}^{N}x_i(t)$. In Fig. 5 we plot the return map of $h(t)$ for various connectivities. In the case of an exact synchronization, all the variables $x_i(t)$ are identical and the whole array behaves like a single map. Thus one expects the return map to be the function f itself in this case. This expectation is fulfilled even if the maps are not exactly synchronized and the connectivity k is below the threshold required for synchronization. As the connectivity is reduced further, there are further departures of this return map from the function f, and for very low connectivities one does not see any coherence. In Fig. 5, we plot the return map of

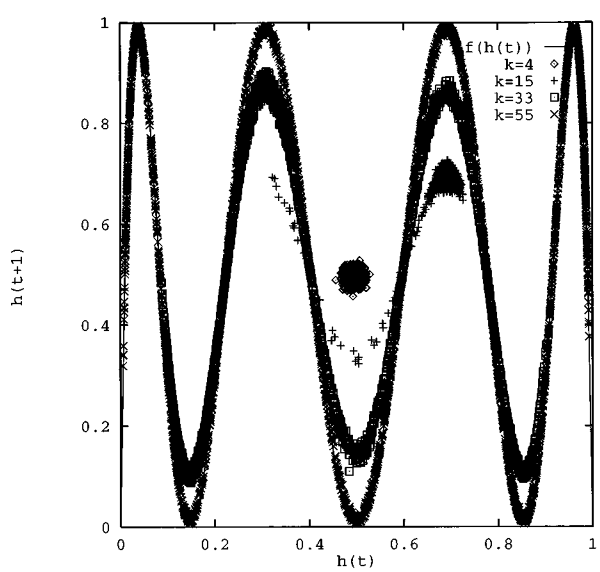

FIG. 5. The return plot of the collective variable $h(t)$ for $k=4$,15,33, and 55 for a typical configuration. It is clear that while for $k=4$ this variable behaves like a sum of i.i.d. random variables, it exhibits highly collective motion for larger values of k.

$h(t)$ for various values of connectivity for a typical realization, namely for $k=4$, $k=15$, $k=33$, and $k=55$. While for $k=55$ the return map closely follows the map $f(x)$ (though a synchronized state is not reached), it departs from this behavior for lower connectivities. For $k=4$ we do not see any coherence emerging between the variables and the return map is a filled ellipse whose size decreases rapidly with N. This is the behavior that one would expect if $x_i(t)$'s were i.i.d. random variables. This is further confirmed by checking the variance of $h(t)$ for different values of N for $k=4$. Figure 6 shows the plot of variance of $h(t)$ for various values of N and it is clear that the variance decays as $1/N$. It is not unexpected that for small connectivities the variables will indeed be uncorrelated. For example, for $k=1$, the evolution is as if N independent maps are evolving but are labeled differently each time. For $k>1$, the same explanation will not be exactly applicable. However, one seems to have a statistical behavior for small connectivities and a highly collective synchronized evolution for larger values of connectivity.

IV. DISCUSSION

Our result on synchronization is indeed significant from an experimental point of view. There have been attempts to have macroscopic synchronized chaos in one dimension with asymmetric coupling and open boundary conditions. The problem with this type of connectivity is that due to convective instabilities one cannot have synchronization in systems with really large sizes [17]. However, since in this model there is no preferred direction in which instabilities can ''flow and grow,'' it does not have the above problem. Auerbach [18] suggested system size dependent controls to overcome this problem while Gade, Cerdeira, and Ramaswamy suggested [4] the tree type connectivity so that the problem is less pronounced. However, it is interesting to note that random connectivities can achieve the same thing. Since one can indeed achieve asymmetric connectivities in systems like electrical circuits, it could be useful for those who want to build huge but better controlled and effectively low dimensional systems.

One more advantage with this kind of connectivity over globally coupled lattices or any other type of regular connectivity is the lack of symmetries in the model. In the presence of symmetries, one can have many equivalent attractors present simultaneously. All these equivalent attractors will be equally attracting and have their own basin of attraction. Thus the basin of attraction of a synchronized state, even in the case where it exists and is stable, could be very small. For example, in Josephson junction arrays (JJA's), a similar phenomenon of attractor crowding is reported where a small change in initial conditions makes one jump from one attractor to other [19]. The number of competing attractors explodes even for a small number of oscillators. This is in contrast with the present model where almost all initial conditions lead to a synchronized state [20].

We have also shown that for lower connectivities, the collective motion slowly disappears and for very low connectivities statistical behavior appears in this model. We note that for this type of coupling, the array looks like a tree with k branches locally. We note that in the case of coupled maps on trees [4], the model indeed resembles a similar model in statistical mechanics [21] and displays a mean-field type behavior. We would like to mention that such behavior (or rather its absence) has been widely investigated in globally coupled maps (which is an analog of the mean field model in statistical mechanics) and it is claimed that the reason for an apparent lack of nonstatistical behavior emerging in this model is due to nonstationary evolution of the model [22]. It is interesting that this behavior is observed with finite nonlocal couplings.

In brief, we have tried to explain the existing observations in the literature about the models with random nonlocal connectivity. We have shown that the interaction matrix has a gap in the eigenvalue spectrum which is of the order of the number of connections. We have pointed out the practical importance of this observation. We have also studied the departures from this collective behavior. We observe that the maps become highly uncorrelated and their sum shows a statistical behavior in the low connectivity limit.

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[1] J. P. Crutchfield and K. Kaneko, in *Directions in Chaos*, edited by Hao Bai-Lin (World Scientific, Singapore, 1987), Vol. II, and references therein.

[2] K. Kaneko, Phys. Rev. Lett. **63**, 219 (1989).

[3] H. Chaté and P. Manneville, Prog. Theor. Phys. **87**, 1 (1992).

[4] P. M. Gade, H. A. Cerdeira, and R. Ramaswamy, Phys. Rev. E **52**, 2478 (1995).

[5] M.G. Cosenza and R. Kapral, Phys. Rev. A **46**, 1850 (1992).

[6] H. Chate and P. Manneville, Chaos **2**, 307 (1992).

[7] J. F. Heagy, T. L. Caroll, and L. M. Pecora, Phys. Rev. E **50**, 1874 (1994).

[8] K. Cuomo and A. V. Oppenhein, Phys. Rev. Lett. **71**, 65 (1993); G. Perez and H. A. Cerdeira, *ibid.* **74**, 1970 (1995).

[9] A few of the references are L. Pecora and T. M. Caroll, Phys. Rev. Lett. **64**, 821 (1990); Phys. Rev. A **44**, 2374 (1991); N. Gupte and R. E. Amritkar, Phys. Rev. E **48**, 1620 (1993); K. Pyragas, Phys. Lett. A **181**, 203 (1993).

[10] J. Cook and B. Derrida, J. Stat. Phys. **61**, 961 (1990).

[11] See, for example, S. Barnett, *Matrices, Methods and Applications* (Oxford University Press, Oxford, 1990).

[12] A. Crisanti, G. Paladin, and A. Vulpiani, *Products of Random Matrices* (Springer-Verlag, Berlin, 1993), Chap. 1.

[13] For the extension of the theorem in [12] to complex matrices see H. J. Sommers *et al.*, Phys. Rev. Lett. **60**, 1895 (1988).

[14] J. E. Cohen and C. M. Newman, Ann. Prob. **12**, 283 (1984).

[15] P. Bougerol and J. Lacroix, *Products of Random Matrices with Applications to Schrodinger Operators* (Birkhauser, CITY, 1985).

[16] One can easily confirm numerically that the largest eigenvalue of the product of a few matrices converges to the expected value for a single matrix. However, we are not aware of a rigorous mathematical extension of the sufficient condition of this theorem [14] for the complex case.

[17] I. Aranson, D. Golomb, and H. Sompolinsky, Phys. Rev. Lett. **68**, 3595 (1992), see also I. Aranson *et al.*, Nonlinearity **3**, 639 (1990).

[18] D. Auerbach, Phys. Rev. Lett. **72**, 1184 (1994).

[19] K. Y. Tsang and K. Wiesenfield, Appl. Phys. Lett. **56**, 495 (1990).

[20] We would like to point out that the recent work (after submission of this paper) on coupled Josephson junction arrays with disorder [Y. Braiman, W. L. Ditto, K. Wiesenfeld, and M. L. Spano, Phys. Lett. A **206**, 54 (1995)] gives a similar explanation for the observation that introducing disorder in JJA's improves the synchronization considerably.

[21] See, e.g., R. J. Baxter, *Exactly Solvable Models in Statistical Mechanics* (Academic, New York, 1982).

[22] K. Kaneko, Phys. Rev. Lett. **65**, 1391 (1990); G. Perez *et al.*, Physica D **63**, 341 (1993); M. Ding and L. T. Wille, Phys. Rev. E **48**, 1605 (1993); A. S. Pikovsky and J. Kurths, Phys. Rev. Lett. **72**, 1644 (1994).