Stochastic Benjamin-Ono equation and its application to the dynamics of nonlinear random waves

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(Received 18 April 1996)

The stochastic Benjamin-Ono equation is introduced, which models the propagation of nonlinear random waves in a two-layer fluid system with and without uneven bottom topography. In the case of the flat bottom, the effect of the external random flow field on the evolution of both soliton and periodic wave is investigated. In particular, the mean value and the correlation function of these nonlinear wave fields are calculated exactly under the assumption that the flow field obeys the Gaussian stochastic process with a white noise. It is found that in the limit of large time, the mean value of an algebraic soliton approaches a Gaussian wave packet whereas that of a periodic wave is represented by Jacobi's theta function. In the case of the uneven bottom, a perturbation analysis is performed to evaluate the mean value of an algebraic soliton under the influence of random change of bottom topography. The large time asymptotic of the soliton is shown to exhibit a Gaussian wave packet with a small amount of the phase shift caused by the interaction between the soliton and the random bottom topography. [S1063-651X(96)05112-4]

PACS number(s): 47.55.Hd, 03.40.Kf, 05.40.+j, 02.50.−r

I. INTRODUCTION

The study of nonlinear wave propagation has made remarkable progress owing to the development of various exact methods of solutions for solving nonlinear evolution equations (NEE's). In particular, the inverse scattering method has enabled us to solve the initial value problems of a wide class of NEE's including the Korteweg–de Vries (KdV), nonlinear Schrödinger and sine-Gordon equations [1]. While the soliton solutions of these physically important NEE's are essentially nonlinear objects that cannot be obtained from the linear theory, they only give an idealized description of underlying physical phenomena. In most real physical situations, however, the system is affected by various perturbations. These may be occasionally random functions in space and time variables. It is probable that the medium parameters such as the temperature and the density of fluid have randomly fluctuating components. Furthermore, the randomness may originate from external forces as well as initial and boundary conditions subject to the system. Once these characteristics are introduced in the system, the resulting model equations would contain random variables and hence they become so-called stochastic evolution equations. For instance, in the context of water wave problems, the constant flow field imposed on the system may have randomly fluctuating components. Also, the irregular change of the bottom topography may act as a random force on the wave dynamics. The problem of the propagation of nonlinear random waves is of current interest in the world of physics and engineering. As for the present stage of its development, some review papers [2–4] and books [5,6] are now available that are mainly concerned with the dynamics of solitons under various random perturbations.

Recently, we have derived a model NEE describing the time evolution of interfacial waves in a two-layer fluid with uneven bottom topography [7]. The equation is written in a dimensionless form as follows:

$$\eta_t+(F-1)\,\eta_x-\frac{3\alpha}{2}\,\eta\eta_x-\frac{\Delta\delta}{2}\,H\eta_{xx}=\frac{\gamma\alpha F^2}{2}\,B_x\,. \tag{1.1a}$$

Here, $\eta=\eta(x,t)$ is the interfacial elevation, $B=B(x)$ represents the unevenness of the bottom topography, F is the Froude number, Δ is the density ratio, γ is a positive constant, H is the Hilbert transform operator given by

$$H\eta(x,t)=\frac{1}{\pi}\,P\int_{-\infty}^{\infty}\frac{\eta(y,t)}{y-x}\,dy, \tag{1.1b}$$

and the subscripts t and x appended to η and B denote partial differentiation. The parameters α and δ measure the magnitude of nonlinearity and dispersion, respectively, and it is assumed that $\delta=O(\alpha)$ in the derivation of Eq. (1.1). When the fluid bottom is flat, i.e., $B=0$, Eq. (1.1) reduces to the well-known Benjamin-Ono (BO) equation [8,9]. Its mathematical structure has been studied extensively [10,11]. Remarkably, the soliton solution has been found to be of the algebraic type unlike the usual soliton solution expressed in terms of exponential functions and hence it is sometimes called an algebraic soliton. In [7], the dynamics of an algebraic soliton have been investigated in detail on the basis of Eq. (1.1) while employing a direct soliton perturbation theory. We showed that when the Froude number is close to unity, the solutions to the dynamical equations for the soliton parameters exhibit a variety of phenomena, such as the capture and repulsion of the soliton by topography and the occurrence of solitonlike phase shift due to the interaction of the soliton with topography.

The purpose of the present paper is to study the dynamics of nonlinear waves when the randomness comes into the system under consideration. To be more specific, we consider

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the situation where the flow field U imposed on the system is close to the phase velocity c_0 of the linear wave. In addition, we assume that U is a slowly varying random function of time. In terms of the Froude number defined by $F=U/c_0$, these statements may be represented as

$$F=1+\alpha\Gamma(\alpha t), \tag{1.2}$$

where Γ is a random function whose characteristics will be specified later. At this stage, it is convenient to rescale the variables according to $t\to(\Delta\delta/\alpha^2)t$, $x\to(\delta/\alpha)x$, $\eta\to(8/3)u$, and $B\to(16/3F^2)B$ in which t is transformed into a slow time due to the ordering $\delta=O(\alpha)$. Then, Eq. (1.1) takes the following form:

$$u_t+\Gamma(t)u_x-4uu_x-Hu_{xx}=\gamma B_x, \quad u=u(x,t). \tag{1.3}$$

While the above equation has been guessed simply from Eq. (1.1), its exact derivation can be made along the same line as that used in the derivation of Eq. (1.1) [7]. In fact, it is only necessary for this purpose to change the upstream boundary condition for the external flow field U in the basic system of hydrodynamic equations. It turns out that under the ordering $\delta=O(\alpha)$, the resulting evolution equation for η takes the same form as Eq. (1.1), the only difference being that the Froude number F $(=U/c_0)$ is not a constant but depends on time.

In Eq. (1.3), the randomness may come from the fluctuation $\Gamma(t)$ of the flow field and/or the irregular bottom topography $B(x)$. In this paper, we shall particularly investigate the following three cases separately: (i) Γ a random function, $B=0$; (ii) Γ=const, B a random function; (iii) Γ=const, B_x a random function. In Sec. II, we consider case (i). We then easily observe that Eq. (1.3) becomes a completely integrable NEE. In view of this fact, various statistical quantities such as the mean value and the autocorrelation function for both soliton and periodic waves are calculated *exactly* under the assumption that the random field Γ obeys the Gaussian process with a white noise. Subsequently, the large time asymptotics of these averages are examined in some detail. In Sec. III, we consider cases (ii) and (iii) where an exact treatment similar to that for case (i) is impossible because of the presence of external random forces. Here, we shall perform an analysis based on a direct soliton perturbation theory developed recently [12,13] to calculate the mean value of an algebraic soliton under the assumption of small external forces, i.e., $\gamma\ll 1$ with B (B_x) being a Gaussian white noise in case (ii) [case (iii)]. The asymptotic behavior of the averaged soliton field is also elucidated. In Sec. IV, we summarize the results achieved in this paper and refer to some problems left for a future work. In the Appendix, the formulas for certain integrals are presented that are useful in evaluating various mean values.

II. RANDOM FLOW FIELD

In this section, we shall focus our attention on the dynamics of nonlinear random waves governed by Eq. (1.3) in the special case of the flat bottom topography. Equation (1.3) is now simplified as

$$u_t+\Gamma(t)u_x-4uu_x-Hu_{xx}=0, \quad u=u(x,t). \tag{2.1}$$

This equation may be called the stochastic BO equation since the coefficient $\Gamma(t)$ of u_x is a random function of time. We note that Eq. (2.1) can be reduced to a standard form of the BO equation by a Galilean transformation $T=t$ and $X=x-\int_0^t\Gamma(s)ds$, implying that Eq. (2.1) is a completely integrable NEE.

Let us now specify the nature of the random function Γ. While various types of randomness appear in real physical problems, we shall here confine ourselves to the Gaussian white noise defined by the averages

$$\langle\Gamma(t_1)\Gamma(t_2)\cdots\Gamma(t_n)\rangle=\begin{cases}\sum\Pi\langle\Gamma(t_i)\Gamma(t_j)\rangle, & n \text{ even}\\ 0, & n \text{ odd},\end{cases} \tag{2.2a}$$

$$\langle\Gamma(t)\rangle=0, \quad \langle\Gamma(t_i)\Gamma(t_j)\rangle=2D\delta(t_i-t_j). \tag{2.2b}$$

Here, the ensemble average is denoted by $\langle\cdots\rangle$, the symbol $\Sigma\Pi$ means that we multiply $n/2$ products $\langle\Gamma(t_i)\Gamma(t_j)\rangle$ and sum over the $(n-1)!!$ different combinations, D is a positive constant characterizing the strength of the correlation, and $\delta(t_i-t_j)$ is Dirac's delta function. Relation (2.2b) shows that the correlation time is negligibly short relative to other time scales. This is not essential in the following analysis and introduced only for the purpose of simplifying the calculation. Indeed, one can replace the delta function by an arbitrary function $f(|t_i-t_j|/t_c)$ with t_c being the correlation time. In calculating statistical quantities, the following formulas are quite useful that follow with the use of (2.2) [14]:

$$\left\langle\exp\left(ik\int_0^t\Gamma(s)ds\right)\right\rangle=e^{-k^2Dt}, \tag{2.3a}$$

$$\left\langle\exp\left(ik\int_0^t\Gamma(s)ds+ik'\int_0^{t'}\Gamma(s)ds\right)\right\rangle$$
$$=\exp(-[k^2Dt+kk'D(t+t'-|t-t'|)+k'^2Dt']), \tag{2.3b}$$

$$\langle e^f\rangle=e^{(1/2)\langle f^2\rangle}, \tag{2.4}$$

$$\langle fe^g\rangle=\langle fg\rangle e^{(1/2)\langle g^2\rangle}. \tag{2.5}$$

Here, f and g are linear functions of Gaussian random functions with zero mean.

One advantage of introducing completely integrable stochastic NEE's is that various statistical quantities can be calculated exactly without recourse to any approximation. We shall now perform the calculation for both soliton and periodic waves. In the case of the soliton, we shall calculate the mean values of u^n $(n=1,2,\ldots)$, uHu_x, and Γu^2 as well as the equal-time autocorrelation function. For the periodic wave, we shall restrict our consideration only to the mean value and the equal-time autocorrelation function. We show that in the long-wave limit, these quantities reduce to the corresponding ones for the soliton. The analysis developed below is easily generalized to the case of multisoliton and multiperiodic wave solutions.

A. Soliton

1. Mean value

Equation (2.1) exhibits on a soliton solution of the form [8,9]

$$u(x,t)=\frac{a}{a^2(x-\xi)^2+1}, \tag{2.6a}$$

with

$$\xi=\int_0^t \Gamma(s)ds-at+\xi_0, \tag{2.6b}$$

where a and ξ are the amplitude and position of the soliton, respectively, and ξ_0 is an initial value of ξ. Note that ξ becomes a stochastic variable through the dependence of the random function Γ.

In order to calculate the mean value of $u(x,t)$, it is convenient to introduce the Fourier transform of u and its inverse transform by the relations

$$\hat{u}(k,t)=\int_{-\infty}^{\infty} u(x,t)e^{-ikx}dx, \tag{2.7a}$$

$$u(x,t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{u}(k,t)e^{ikx}dk. \tag{2.7b}$$

It immediately follows from (2.6a) and (2.7a) that

$$\hat{u}(k,t)=\pi e^{-ik\xi-|k|/a}. \tag{2.8}$$

By substituting (2.8) into (2.7b) and taking the ensemble average with the aid of the formula (2.3a), we arrive at the following integral representation for the mean value of u:

$$\langle u(x,t)\rangle=\int_0^{\infty} e^{-Dtk^2-k/a}\cos k(x+at-\xi_0)dk\equiv I(\sqrt{p},z,b), \tag{2.9a}$$

with

$$p\equiv 1/a^2,\quad z\equiv x+at-\xi_0,\quad b\equiv Dt, \tag{2.9b}$$

where we have introduced the integral I defined by (A1) in the Appendix. It is worthwhile to remark here that the above mean value satisfies the diffusion equation

$$\frac{\partial\langle u(x,t)\rangle}{\partial b}=\frac{\partial^2\langle u(x,t)\rangle}{\partial z^2}, \tag{2.10}$$

subject to the initial condition

$$\langle u(x,t)\rangle|_{b=0}=\frac{a}{(az)^2+1}. \tag{2.11}$$

Owing to this fact, we can obtain a different form of $\langle u\rangle$ as

$$\langle u(x,t)\rangle=\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}\frac{ae^{-\theta^2}}{a^2(z-2\sqrt{b}\theta)^2+1}\,d\theta. \tag{2.12}$$

Next, we shall evaluate the large time asymptotic of $\langle u(x,t)\rangle$. Using the expansion (A8), the leading term of the expansion is readily found to be as follows:

$$\langle u(x,t)\rangle\sim\left(\frac{\pi}{4Dt}\right)^{1/2}\exp\left[-\frac{(x+at-\xi_0)^2}{4Dt}\right]. \tag{2.13}$$

Thus, we can see that the profile of the averaged soliton tends to a Gaussian packet with velocity equal to the initial velocity of the soliton and the amplitude decreases as $t^{-1/2}$. Since the width of the averaged soliton grows like $t^{1/2}$, its area is conserved, i.e., $\int_{-\infty}^{\infty}\langle u(x,t)\rangle dx=\pi$ in the present case, which is obtained by integrating (2.9) with respect to x. The conservation of the area also follows directly from Eq. (2.1).

In the same way, we can calculate various mean values. We shall present only some of them:

$$\langle u^n(x,t)\rangle=\frac{(-1)^{n-1}}{(n-1)!}\,p^{n/2}\,\frac{\partial^{n-1}}{\partial p^{n-1}}\left[\frac{I(\sqrt{p},z,b)}{\sqrt{p}}\right]\quad (n=1,2,\ldots), \tag{2.14}$$

$$\langle uHu_x\rangle=-\left(p\,\frac{\partial^2}{\partial p^2}+\frac{1}{4p}\right)I(\sqrt{p},z,b), \tag{2.15}$$

$$\langle\Gamma(t)u^2(x,t)\rangle=2Dp\,\frac{\partial^2 J(\sqrt{p},z,b)}{\partial p^2}. \tag{2.16}$$

Here, the integrals I and J are defined by (A1) and (A2), respectively.

Lastly, we remark that Eq. (2.1) can be written in a Hamiltonian form as

$$u_t=\frac{\partial}{\partial x}\frac{\delta I_3}{\delta u}, \tag{2.17a}$$

$$I_3=\int_{-\infty}^{\infty}\left(\tfrac{2}{3}u^3+\tfrac{1}{2}uHu_x-\tfrac{1}{2}\Gamma u^2\right)dx. \tag{2.17b}$$

Hence, the mean value of the density of I_3 is easily evaluated with the use of (2.14)–(2.16) together with the formulas (A12)–(A15). The resulting expression is, however, not written down here.

2. Correlation function

The correlation function that we consider here is the following equal-time autocorrelation function:

$$C(x',t;x,t)=\langle u(x',t)u(x,t)\rangle-\langle u(x',t)\rangle\langle u(x,t)\rangle. \tag{2.18}$$

The second term on the right-hand side of (2.18) has already been given by (2.9) and hence we shall now evaluate the first term. We use the integral representation (2.7b) with (2.8) as well as the formula (2.3a) to obtain

$$\langle u(x',t)u(x,t)\rangle=\tfrac{1}{4}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\exp[ik'z'+ikz-Dt(k'+k)^2-(|k'|+|k|)/a]dk'\,dk, \tag{2.19a}$$

where

$$z=x+at-\xi_0, \quad z'=x'+at-\xi_0. \tag{2.19b}$$

After some manipulations, this double integral is transformed into the sum of two single integrals as

$$\langle u(x',t)u(x,t)\rangle = \frac{a}{a^2(z'-z)^2+4}\int_0^\infty (\cos kz'+\cos kz) \times e^{-Dtk^2-k/a}dk + \frac{2}{(z'-z)[a^2(z'-z)^2+4]} \times \int_0^\infty (\sin kz'-\sin kz)e^{-Dtk^2-k/a}dk. \tag{2.20}$$

When $x'=x$, this expression reduces to Eq. (2.14) with $n=2$. In terms of the integrals I and J defined in the Appendix, the autocorrelation function is expressed as follows:

$$C(x',t;x,t) = \frac{a}{a^2(z'-z)^2+4}[I(\sqrt{p},z',b)+I(\sqrt{p},z,b)] + \frac{2}{(z'-z)[a^2(z'-z)^2+4]} \times [J(\sqrt{p},z',b)-J(\sqrt{p},z,b)] - I(\sqrt{p},z',b)I(\sqrt{p},z,b). \tag{2.21}$$

This is a convenient form in evaluating the asymptotic behavior as shown below. If there are no correlations between $\Gamma(t)$ and $\Gamma(t')$, i.e., $D=0$, one can perform the integrals in (2.20) and obtain the relation $\langle u(x',t)u(x,t)\rangle = \langle u(x',t)\rangle\langle u(x,t)\rangle$. It then follows from (2.18) that $C=0$, which provides a check of the present analysis. We also remark that (2.19) can be expressed in a form analogous to (2.12) as

$$\langle u(x',t)u(x,t)\rangle = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} \frac{a^2 e^{-\theta^2}}{[a^2(z'-2\sqrt{b}\theta)^2+1][a^2(z-2\sqrt{b}\theta)^2+1]}d\theta. \tag{2.22}$$

It is worthwhile to mention here that the n-point equal-time correlation function is given by the integral representation

$$\langle u(x_1,t)u(x_2,t)\cdots u(x_n,t)\rangle = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} \frac{a^n e^{-\theta^2}}{\Pi_{j=1}^n[a^2(z_j-2\sqrt{b}\theta)^2+1]}d\theta, \tag{2.23}$$

where $z_j=x_j+at-\xi_0$ $(j=1,2,\ldots,n)$. One immediately sees that the expressions (2.12) and (2.22) are special cases of (2.23).

Next, we shall investigate the asymptotic behavior of C. We consider two interesting cases. The first case is the large time asymptotic of C while keeping z' and z finite values. It immediately follows from (2.21), (A8), and (A9) that

$$C(x',t;x,t) \sim \frac{a}{2}\left(\frac{\pi}{Dt}\right)^{1/2}\frac{e^{-z'^2/4Dt}+e^{-z^2/4Dt}}{a^2(z'-z)^2+4} + \frac{1}{2Dt}\frac{z'+z}{z'-z}\frac{e^{-z'^2/4Dt}-e^{-z^2/4Dt}}{(z'-z)[a^2(z'-z)^2+4]} - \frac{\pi}{4Dt}e^{-(z'^2+z^2)/4Dt}+O(t^{-3/2}), \tag{2.24}$$

showing that C decays like $t^{-1/2}$ as $t\to\infty$. This asymptotic behavior is the same as that of $\langle u(x,t)\rangle$ as seen from (2.13).

The second limiting case is when the distance between x' and x becomes large. If we keep x and t finite values and take the limit $x'\to\infty$, we find with the use of (A10) and (A11) the following expression:

$$C(x',t;x,t) \sim -\frac{2}{a^2}[azI(\sqrt{p},z,b)-J(\sqrt{p},z,b)]x'^{-3} + O(x'^{-4}). \tag{2.25}$$

Although we have dealt with the equal-time correlation functions, we can calculate the general correlation functions for different times in the same way. For instance, if we use Eqs. (2.3b) and (2.7), we obtain

$$\langle u(x',t')u(x,t)\rangle = \frac{1}{\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u_s(x-2\sqrt{Dt}y,t)u_s \times (x'-2\mu\sqrt{Dt}y-2\sqrt{\mu D|t'-t|}y',t') \times e^{-(y^2+y'^2)}dy\, dy', \tag{2.26a}$$

where

$$u_s(x,t) = \frac{a}{a^2(x+at-\xi_0)^2+1}, \tag{2.26b}$$

$$\mu = \frac{t+t'-|t-t'|}{2t}. \tag{2.26c}$$

Since the investigation of the asymptotic behavior of (2.26) is somewhat involved, the detailed analysis is not described here and will be reported elsewhere.

B. Periodic wave

1. Mean value

Here, we shall develop an analysis for a periodic solution of Eq. (2.1). One interesting feature of the periodic solution is that it can be expressed in terms of an elementary function unlike the periodic solution of the KdV equation, which is represented by Jacobi's elliptic function. Explicitly, it reads in the form [8]

$$u(x,t) = \frac{k}{2}\frac{\sinh\phi}{\cos\eta+\cosh\phi}, \tag{2.27a}$$

with

$$\eta=k(x-\xi),\quad \phi=\tanh^{-1}(k/a)\quad (\phi>0), \tag{2.27b}$$

where k represents the wave number and ξ is given by (2.6b). Note that the above periodic wave reduces to the solitary wave (2.6) in the long-wave limit $k\to 0$. Actually, if we replace ξ_0 by $\xi_0+\pi/k$ and take the limit $k\to 0$ while keeping a a finite value, this result follows immediately.

To calculate the mean value of u, we first modify Eq. (2.27) in the form of an infinite series as

$$u(x,t)=\frac{k}{2}+\frac{k}{2}\sum_{n=1}^{\infty}(-1)^n[e^{n(i\eta-\phi)}+e^{n(-i\eta-\phi)}]. \tag{2.28}$$

The convergence of this series is obvious because ϕ is a positive quantity. Taking the ensemble average with use of the formula (2.3a), we obtain the following result:

$$\langle u(x,t)\rangle=\frac{k}{2}+k\sum_{n=1}^{\infty}(-1)^n e^{-n^2k^2Dt-n\phi}\cos(nkz). \tag{2.29}$$

One can observe that the above average also obeys the diffusion equation (2.10) and reduces to the corresponding average (2.9) for the soliton in the long-wave limit. The asymptotic form of Eq. (2.29) for large time is readily derived as

$$\langle u(x,t)\rangle\sim\frac{k}{2}+k\sum_{n=1}^{\infty}(-1)^n e^{-n^2k^2Dt}\cos(nkz)$$
$$=\frac{k}{2}\,\theta_4\!\left(\frac{kz}{2},q\right)\quad (q=e^{-k^2Dt}), \tag{2.30}$$

where θ_4 is Jacobi's theta function defined by [15]

$$\theta_4(z,q)=1+2\sum_{n=1}^{\infty}(-1)^n q^{n^2}\cos(2nz)$$
$$(q=e^{i\pi\tau},\ \mathrm{Im}\,\tau>0). \tag{2.31}$$

We see from this expression that the average value of a periodic wave tends to a constant value $k/2$ in the limit of infinite time.

2. Correlation function

The calculation of the correlation function is performed by following the similar procedure to that for the soliton. Omitting the detail, we quote only the final result. It is given by an infinite series of the form

$$\langle u(x',t)u(x,t)\rangle=\frac{k^2}{4}\Bigg[1+2c_1+c_1\sum_{n=1}^{\infty}(-1)^n e^{-n^2k^2Dt-n\phi}\times(\cos nkz'+\cos nkz)+c_2\sum_{n=1}^{\infty}(-1)^n e^{-n^2k^2Dt-n\phi}\times(\sin nkz'-\sin nkz)\Bigg], \tag{2.32a}$$

with

$$c_1=\frac{2ak}{a^2+k^2-(a^2-k^2)\cos k(z'-z)}, \tag{2.32b}$$

$$c_2=\frac{2k^2\cot\frac{k}{2}(z'-z)}{a^2+k^2-(a^2-k^2)\cos k(z'-z)}. \tag{2.32c}$$

In the long-wave limit $k\to 0$, one can easily confirm that this expression reduces to the corresponding one for the soliton (2.20). Also the large time asymptotic of Eq. (2.32) with fixed z' and z is found to be as

$$\langle u(x',t)u(x,t)\rangle\sim\frac{k^2}{4}\Bigg[1+c_1+\frac{c_1}{2}\left\{\theta_4\!\left(\frac{kz'}{2},q\right)+\theta_4\!\left(\frac{kz}{2},q\right)\right\}+c_2\sum_{n=1}^{\infty}(-1)^n e^{-n^2k^2Dt}\times(\sin nkz'-\sin nkz)\Bigg]. \tag{2.33}$$

III. RANDOM BOTTOM TOPOGRAPHY

In this section, we shall study the propagation characteristic of an algebraic soliton under the influence of random bottom topography. For the sake of simplicity, however, the external flow field is assumed to be constant. The basic equation is now written in the form

$$u_t+\Gamma u_x-4uu_x-Hu_{xx}=\gamma B_x,\quad u=u(x,t), \tag{3.1}$$

where Γ is defined by (1.2) but in the present case it is independent of time. As seen from Eq. (3.1), the effect of random bottom topography acts as an external force where the parameter γ characterizes the magnitude of topography. In the following analytical treatment of the problem, we elucidate weak topographic effects on the dynamics of an algebraic soliton. In other words, we assume $\gamma\ll 1$. This enables us to apply a direct soliton perturbation theory for the perturbed BO equation developed recently [12,13].

In order to proceed further with the analysis, we must specify the nature of the random function $B(x)$. We shall consider the following two cases separately. The first case is that the bottom profile itself changes randomly while for the second case, the gradient of the bottom profile, i.e., $B_x(x)$ exhibits a random behavior. Both random functions are assumed to be Gaussian white noise whose characteristics are similar to those given by Eq. (2.2). Although it is possible to calculate various statistical quantities on the basis of the Gaussian stochastic process, we shall restrict our consideration to evaluate the mean value of u and examine its asymptotic behavior for large time.

A. Random change of bottom topography

When the profile of bottom topography changes according to the Gaussian white noise, the corresponding stochastic process may be characterized by the relations

$$\langle B(x)\rangle=0,\quad \langle B(x)B(y)\rangle=2D\delta(x-y), \tag{3.2}$$

together with the relation similar to (2.2a). The procedure for calculating the mean value $\langle u(x,t)\rangle$ is precisely the same as that which has been used in Sec. II for the case of random flow field. In the present situation, however, the corresponding calculation cannot be performed exactly because Eq. (3.1) is no longer an integrable NEE even if the parameter γ is very small. To overcome this difficulty, we shall here employ a direct soliton perturbation theory which can be applicable to a wide class of NEE's including the forced BO equation.

Let us first begin with a short summary of the perturbation theory that relies upon the multiple time scale expansion [12,13]. The problem under consideration is to solve the initial value problem of Eq. (3.1) under the initial condition

$$u(x,0)=\frac{a_0}{a_0^2(x-\xi_0)^2+1}, \tag{3.3}$$

where a_0 and ξ_0 are initial values of the amplitude and position of the soliton, respectively. We first expand u in powers of the small parameter γ as

$$u=u_0+\gamma u_1+\cdots\ . \tag{3.4}$$

Since the perturbation is very small, we can expect that the leading term u_0 would remain close to the soliton solution (2.6) of the BO equation. However, the amplitude a and the position ξ would suffer slow modulation due to the action of the perturbation. A direct substitution of Eq. (3.4) into (3.1) shows that u_0 yields a secular term proportional to t. In order to avoid this unphysical behavior, we demand nonsecular conditions, which in the present case turn out to be as follows [7]:

$$\frac{da}{dt}=-\frac{4\gamma}{\pi}\int_{-\infty}^{\infty}g_2B_xdx\equiv-\frac{4\gamma}{\pi}(g_2,B_x), \tag{3.5a}$$

$$\frac{d\xi}{dt}=\Gamma-a+\frac{4\gamma}{\pi}(g_1,B_x), \tag{3.5b}$$

where

$$g_1=\frac{x-\xi}{a^2(x-\xi)^2+1}, \tag{3.6a}$$

$$g_2=-\frac{a}{a^2(x-\xi)^2+1}. \tag{3.6b}$$

With the solution of Eq. (3.5), u_0 is now represented by

$$u_0(x,t)=\frac{a}{a^2(x-\xi)^2+1}. \tag{3.7}$$

The above expression is valid for large time up to order γ^{-1}. Beyond this time, one must take into account the next-order term u_1. In the following discussion, we shall confine ourselves to the leading-order analysis. Thus, the problem reduces to solve a system of equations (3.5). For a special functional form of B, for example, $B(x)=\lambda b/[(bx)^2+1]$ (b, λ are const), we have found that it becomes a completely integrable system [7]. Since in the present problem $B(x)$ is not specified except its ensemble average (3.2), one cannot solve Eq. (3.5) exactly. Therefore, we employ a successive approximation by expanding a and ξ in powers of γ as

$$a(t)=a_0+\gamma a_1, \tag{3.8a}$$

$$\xi(t)=\overline{\xi}+\gamma\xi_1, \tag{3.8b}$$

where $\overline{\xi}=(\Gamma-a_0)t+\xi_0$. Substituting Eq. (3.8) into (3.5), we obtain, after integrating the resultant equations with t, the explicit expressions of a_1 and ξ_1 as follows:

$$a_1=-\frac{4}{\pi}\int_0^t(g_2^{(0)},B_x)dt', \tag{3.9a}$$

$$\xi_1=\frac{4}{\pi}\int_0^t dt'\int_0^{t'}(g_2^{(0)},B_x)dt''+\frac{4}{\pi}\int_0^t(g_1^{(0)},B_x)dt'. \tag{3.9b}$$

Here

$$g_1^{(0)}=g_1|_{\gamma=0}=\frac{x-\overline{\xi}}{a_0^2(x-\overline{\xi})^2+1}, \tag{3.10a}$$

$$g_2^{(0)}=g_2|_{\gamma=0}=-\frac{a_0}{a_0^2(x-\overline{\xi})^2+1}. \tag{3.10b}$$

We are now in a position to calculate the mean value of u_0 that is given by Eq. (3.7) together with Eqs. (3.8) and (3.9). If we use Eqs. (2.7), (2.8), and (3.8), it can be written within the approximation correct up to $O(\gamma)$ as

$$\langle u_0(x,t)\rangle\sim\tfrac{1}{2}\left\langle(a_0+\gamma a_1)\int_{-\infty}^{\infty}\exp(i\tilde{k}[a_0(x-\overline{\xi})+\gamma\{a_1(x-\overline{\xi})-a_0\xi_1\}]-|\tilde{k}|)d\tilde{k}\right\rangle, \tag{3.11}$$

where we have introduced a new integration variable $\tilde{k}\equiv k/a$. To evaluate the above ensemble average, we note that a_1 and ξ_1 are linear functions of the Gaussian random function B with zero mean as seen from Eq. (3.9) and hence we can use the formulas (2.4) and (2.5). In view of the formula $(1+\gamma\partial/\partial x)f(x)=f(x+\gamma)+O(\gamma^2)$, the final result is expressed compactly in a form analogous to Eq. (2.9) as follows:

$$\langle u_0(x,t)\rangle\sim\int_0^{\infty}e^{-(1/2)\gamma^2b_1k^2-k/a_0}\cos k(\chi_0+\gamma^2\chi_1)dk, \tag{3.12a}$$

with

$$b_1=\frac{\langle a_1^2\rangle(x-\overline{\xi})^2}{a_0^2}-2\frac{\langle a_1\xi_1\rangle(x-\overline{\xi})}{a_0}+\langle\xi_1^2\rangle, \tag{3.12b}$$

$$\chi_0=x-\overline{\xi}, \tag{3.12c}$$

$$\chi_1 = \frac{\langle a_1^2\rangle}{a_0^2}(x - \overline{\xi}) - \frac{\langle a_1\xi_1\rangle}{a_0}. \tag{3.12d}$$

The averages $\langle a_1^2\rangle$, $\langle a_1\xi_1\rangle$, and $\langle \xi_1^2\rangle$ in the expressions of b_1 and χ_1 are easily computed with the use of Eqs. (3.2), (3.9), and (3.10). The results are given by

$$\langle a_1^2\rangle = \frac{32D}{\pi}\frac{a_0}{(\Gamma - a_0)^2}\frac{\tau^2}{\tau^2+4}, \tag{3.13a}$$

$$\langle a_1\xi_1\rangle = -\frac{16D}{\pi}\frac{1}{(\Gamma - a_0)^3}\frac{\tau^3}{\tau^2+4}, \tag{3.13b}$$

$$\langle \xi_1^2\rangle = \frac{32D}{\pi}\frac{1}{a_0(\Gamma - a_0)^2}\left[\frac{1}{2}\frac{\tau^2}{(\Gamma - a_0)^2} + \frac{2(\Gamma - 2a_0)}{a_0(\Gamma - a_0)^2} \times \ln\left(1+\frac{\tau^2}{4}\right) + \frac{\Gamma - 3a_0}{a_0^2(\Gamma - a_0)}\frac{\tau^2}{\tau^2+4}\right], \tag{3.13c}$$

where $\tau = a_0(\Gamma - a_0)t$.

We now investigate the limiting behavior of various averages obtained above in some detail. First, we consider the case $a_0 \neq \Gamma$. In the limit of $t \to 0$, the expressions (3.13) reduce to

$$\langle a_1^2\rangle \sim \frac{8Da_0^3}{\pi}t^2, \tag{3.14a}$$

$$\langle a_1\xi_1\rangle \sim -\frac{4Da_0^3}{\pi}t^3, \tag{3.14b}$$

$$\langle \xi_1^2\rangle \sim \frac{8D}{\pi a_0}t^2, \tag{3.14c}$$

whereas in the limit of $t \to \infty$, they behave like

$$\langle a_1^2\rangle \sim \frac{32D}{\pi}\frac{a_0}{(\Gamma - a_0)^2}, \tag{3.15a}$$

$$\langle a_1\xi_1\rangle \sim -\frac{16D}{\pi}\frac{a_0}{(\Gamma - a_0)^2}t, \tag{3.15b}$$

$$\langle \xi_1^2\rangle \sim \frac{16D}{\pi a_0(\Gamma - a_0)^2}t^2. \tag{3.15c}$$

The asymptotic behavior of $\langle u_0(x,t)\rangle$ for large $t(\sim\gamma^{-1})$ and large x such that χ_0 is fixed is now found from Eqs. (3.12) and (3.15) with the aid of Eq. (A8). The result is represented by a Gaussian packet of the form

$$\langle u_0(x,t)\rangle \sim \frac{1}{\gamma}\left(\frac{\pi}{2\tilde{b}_1}\right)^{1/2}\exp\left[-\frac{(\chi_0 + \gamma^2\tilde{\chi}_1)^2}{2\gamma^2\tilde{b}_1}\right], \tag{3.16a}$$

with

$$\tilde{b}_1 = \frac{16D}{\pi a_0(\Gamma - a_0)^2}t^2, \tag{3.16b}$$

$$\tilde{\chi}_1 = \frac{16D}{\pi(\Gamma - a_0)^2}t. \tag{3.16c}$$

Thus, the amplitude of the averaged soliton decays as t^{-1} and the width grows as t so that the area $\int_{-\infty}^{\infty}\langle u_0(x,t)\rangle dx$ is conserved. The latter result also follows directly from Eq. (3.1). The functional form of Eq. (3.16) is exactly the same as the asymptotic form (2.13) obtained for the random flow field. However, in the present case, the expression (3.16) exhibits a small amount of the phase shift $\gamma^2\tilde{\chi}_1$ caused by the interaction between the soliton and the random bottom topography.

Next, we shall briefly discuss the behavior of $\langle u_0(x,t)\rangle$ when the condition $a_0 = \Gamma$ holds. This situation occurs provided that the fluctuating component of the external flow from the phase velocity of the linear wave just coincides with the initial velocity of the soliton. It then turns out that Eqs. (3.13) reduce to the expressions

$$\langle a_1^2\rangle = \frac{8Da_0^3}{\pi}t^2, \tag{3.17a}$$

$$\langle a_1\xi_1\rangle = -\frac{4Da_0^3}{\pi}t^3, \tag{3.17b}$$

$$\langle \xi_1^2\rangle = \frac{8D}{\pi a_0}\left(t^2 + \frac{a_0^4t^4}{4}\right). \tag{3.17c}$$

The large time asymptotic of $\langle u_0\rangle$ corresponding to Eq. (3.16) takes the form

$$\langle u_0(x,t)\rangle \sim \frac{1}{\gamma}\left(\frac{\pi}{2\tilde{b}_1}\right)^{1/2}\exp\left[-\frac{(\tilde{\chi}_0 + \gamma^2\tilde{\chi}_1)^2}{2\gamma^2\tilde{b}_1}\right], \tag{3.18a}$$

with

$$\tilde{b}_1 = \frac{2Da_0^3}{\pi}t^4, \tag{3.18b}$$

$$\tilde{\chi}_0 = x - \xi_0, \tag{3.18c}$$

$$\tilde{\chi}_1 = \frac{4Da_0^2}{\pi}t^3. \tag{3.18d}$$

This profile shows a Gaussian packet decaying as t^{-2}. However, when compared with the first case (3.16), the center position of the packet initially located at $x = \xi_0$ moves to the left direction more slowly with a velocity proportional to $(\gamma t)^2$.

B. Random change of the gradient of bottom topography

Finally, we shall consider the evolution of an algebraic soliton when the source of randomness lies in the gradient of bottom topography. Its statistical properties corresponding to (3.2) are now characterized by the relations

$$\langle B_x(x)\rangle = 0, \quad \langle B_x(x)B_y(y)\rangle = 2D\delta(x-y). \tag{3.19}$$

Since the calculation of the mean value of u_0 parallels that of the previous one for the random bottom topography, we shall describe only the main results.

The mean value $\langle u_0(x,t)\rangle$ takes the same form as Eq. (3.12) where the averages corresponding to Eq. (3.13) are now written as

$$\langle a_1^2\rangle = \frac{32D}{\pi a_0(\Gamma - a_0)^2}\left[\tau \tan^{-1}\frac{\tau}{2} - \ln\left(1+\frac{\tau^2}{4}\right)\right], \tag{3.20a}$$

$$\langle a_1\xi_1\rangle = -\frac{32D}{\pi a_0^2(\Gamma - a_0)^3}\left[\tau^2 \tan^{-1}\frac{\tau}{2} - \tau\ln\left(1+\frac{\tau^2}{4}\right)\right], \tag{3.20b}$$

$$\langle \xi_1^2\rangle = \frac{32D}{\pi a_0^3(\Gamma - a_0)^4}\left[\frac{4}{3}\tau^2 + \frac{2}{3}(\tau^3 - 3\tau)\tan^{-1}\frac{\tau}{2} - (\tau^2 - \tfrac{2}{3})\ln\left(1+\frac{\tau^2}{4}\right) - \frac{2a_0-\Gamma}{a_0}\tau^2 + 2\left(\frac{2a_0-\Gamma}{a_0}\right)^2\left\{\tau\tan^{-1}\frac{\tau}{2} - \ln\left(1+\frac{\tau^2}{4}\right)\right\}\right]. \tag{3.20c}$$

We now investigate the asymptotic behavior of $\langle u_0\rangle$ for two cases, $a_0 \neq \Gamma$ and $a_0 = \Gamma$, separately. For the first case $a_0 \neq \Gamma$, Eqs. (3.20) reduce, in the limit of $t \to 0$, to

$$\langle a_1^2\rangle \sim \frac{16a_0D}{\pi}t^2, \tag{3.21a}$$

$$\langle a_1\xi_1\rangle \sim -\frac{8a_0D}{\pi}t^3, \tag{3.21b}$$

$$\langle \xi_1^2\rangle \sim \frac{16D}{\pi a_0^3}t^2, \tag{3.21c}$$

and in the limit of $t \to \infty$, they behave like

$$\langle a_1^2\rangle \sim \frac{16D}{|\Gamma - a_0|}t, \tag{3.22a}$$

$$\langle a_1\xi_1\rangle \sim -\frac{16D}{|\Gamma - a_0|}t^2, \tag{3.22b}$$

$$\langle \xi_1^2\rangle \sim \frac{32D}{3|\Gamma - a_0|}t^3, \tag{3.22c}$$

Hence, the large time asymptotic is exactly the same as (3.16a) with $\tilde{b}_1$ and $\tilde{\chi}_1$ given by

$$\tilde{b}_1 = \frac{32Da_0^2}{3|\Gamma - a_0|}t^3, \tag{3.23a}$$

$$\tilde{\chi}_1 = \frac{16D}{a_0|\Gamma - a_0|}t^2. \tag{3.23b}$$

Thus, the initial profile of algebraic type approaches asymptotically a Gaussian packet whose amplitude decays as $t^{-3/2}$.

For the second case $a_0 = \Gamma$, on the other hand, one obtains from Eq. (3.18) the exact expressions

$$\langle a_1^2\rangle = \frac{16Da_0}{\pi}t^2, \tag{3.24a}$$

$$\langle a_1\xi_1\rangle = -\frac{4Da_0}{\pi}t^3, \tag{3.24b}$$

$$\langle \xi_1^2\rangle = \frac{16D}{\pi a_0^3}\left(t^2 + \frac{a_0^4 t^4}{8}\right). \tag{3.24c}$$

The asymptotic form of $\langle u_0(x,t)\rangle$ for large time is found to coincide with Eq. (3.18) except b_1 and x_1, which are now given by

$$\tilde{b}_1 = \frac{2Da_0}{\pi}t^4, \tag{3.25a}$$

$$\tilde{\chi}_1 = \frac{8D}{\pi}t^3. \tag{3.25b}$$

The above analysis implies that the profile of $\langle u_0(x,t)\rangle$ approaches a Gaussian packet with the decreasing amplitude as t^{-2}.

IV. SUMMARY AND OUTLOOK

In this paper, we have introduced the stochastic BO equation that models the propagation of nonlinear random waves in a simple two-layer fluid system with uneven bottom topography and studied the effect of randomness on the dynamics of soliton and periodic wave. Under the assumption of the Gaussian stochastic process for the random field, analytical calculations have been performed for obtaining various statistical quantities.

In the case of the flat bottom topography, the basic equation turns out to be completely integrable. Thanks to this fact, we were able to evaluate exactly various mean values as well as the correlation functions for both soliton and periodic wave. It was found that the large time asymptotic of an algebraic soliton approaches a Gaussian wave packet with decaying amplitude and growing width and that of a periodic wave is represented by Jacobi's theta function. The latter is of particular interest because the existing literatures are mainly concerned with the calculation of the mean value of the soliton field. See, for example, Ref. [16] for the analysis based on the stochastic KdV equations.

When the bottom topography changes randomly, on the other hand, we have employed a direct soliton perturbation theory to obtain the mean value of an algebraic soliton under the assumption of small topographic effect. The asymptotic behavior of the soliton was found to be the same as that of the integrable case, but in the present situation the Gaussian packet suffers a small amount of the phase shift caused by the interaction between the soliton and the random bottom topography.

The generalization of the present work to the case in which the field is composed of many solitons is an interesting problem. In particular, the effect of randomness on the interaction process of solitons is worth studying. In this respect, we remark that the effect of higher-order nonlinearity and dispersion on the interaction of two algebraic solitons was investigated recently while applying a direct multisoliton perturbation theory to a higher-order BO equation [12,13]. In the present research, we have performed the perturbation analysis only for a soliton. However, the corresponding one for a periodic wave is important in connection with the modulation phenomenon of the wave.

Although the stochastic BO equation seems to be rather model specific, we may introduce more general stochastic NEE's. One example is the stochastic version of the intermediate long-wave equation [17] that reduces to the stochastic BO equation in deep-water limit and to the stochastic KdV equation in shallow-water limits. The approach described in the present paper can be applied to the above equation.

APPENDIX

In this appendix, we shall describe some properties of the integrals often used in evaluating various statistical quantities. Let us first define the integrals I and J by

$$I(\alpha,\beta,\gamma)=\int_0^\infty e^{-\gamma k^2-\alpha k}\cos\beta k\,dk, \tag{A1}$$

$$J(\alpha,\beta,\gamma)=\int_0^\infty e^{-\gamma k^2-\alpha k}\sin\beta k\,dk, \tag{A2}$$

where α and γ are positive parameters and β is a real parameter. These integrals are expressed in terms of the error function as [18]

$$I(\alpha,\beta,\gamma)=\frac{1}{4}\left(\frac{\pi}{\gamma}\right)^{1/2}\left[\exp\left[\frac{(\alpha-i\beta)^2}{4\gamma}\right]\left\{1-\mathrm{erf}\left(\frac{\alpha-i\beta}{2\sqrt{\gamma}}\right)\right\}+\exp\left[\frac{(\alpha+i\beta)^2}{4\gamma}\right]\left\{1-\mathrm{erf}\left(\frac{\alpha+i\beta}{2\sqrt{\gamma}}\right)\right\}\right], \tag{A3}$$

$$J(\alpha,\beta,\gamma)=-\frac{i}{4}\left(\frac{\pi}{\gamma}\right)^{1/2}\left[\exp\left[\frac{(\alpha-i\beta)^2}{4\gamma}\right]\left\{1-\mathrm{erf}\left(\frac{\alpha-i\beta}{2\sqrt{\gamma}}\right)\right\}-\exp\left[\frac{(\alpha+i\beta)^2}{4\gamma}\right]\left\{1-\mathrm{erf}\left(\frac{\alpha+i\beta}{2\sqrt{\gamma}}\right)\right\}\right]. \tag{A4}$$

The error function defined by

$$\mathrm{erf}(z)=\frac{2}{\sqrt{\pi}}\int_0^z e^{-s^2}ds, \tag{A5}$$

has a Taylor series expansion for $z\to 0$

$$\mathrm{erf}(z)=\frac{2}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{(-1)^n z^{2n+1}}{n!(2n+1)}, \tag{A6}$$

and an asymptotic expansion for $z\to\infty$

$$\mathrm{erf}(z)\sim 1-\frac{1}{\sqrt{\pi}}\frac{e^{-z^2}}{z}\left[1+\sum_{n=1}^{\infty}(-1)^n\frac{(2n-1)!!}{(2z^2)^n}\right]. \tag{A7}$$

When the conditions $\alpha\ll\sqrt{\gamma}$ and $|\beta|\ll\sqrt{\gamma}$ hold, I and J are expanded as

$$I(\alpha,\beta,\gamma)\sim\frac{1}{2}\left(\frac{\pi}{\gamma}\right)^{1/2}e^{-(\beta^2-\alpha^2)/4\gamma}\left[1-\frac{\alpha}{\sqrt{\pi\gamma}}+O(\gamma^{-1})\right], \tag{A8}$$

$$J(\alpha,\beta,\gamma)\sim\frac{1}{2}\left(\frac{\pi}{\gamma}\right)^{1/2}e^{-(\beta^2-\alpha^2)/4\gamma}\times\left[\frac{\beta}{\sqrt{\pi\gamma}}-\frac{\alpha\beta}{2\gamma}+O(\gamma^{-3/2})\right], \tag{A9}$$

whereas for $\beta\gg\sqrt{\gamma}$, they take the forms

$$I(\alpha,\beta,\gamma)\sim\frac{\alpha}{\alpha^2+\beta^2}\left[1-\frac{2(\alpha^2-3\beta^2)}{(\alpha^2+\beta^2)^2}\gamma+O(\gamma^2)\right], \tag{A10}$$

$$J(\alpha,\beta,\gamma)\sim\frac{\alpha}{\alpha^2+\beta^2}\left[1-\frac{2(3\alpha^2-\beta^2)}{(\alpha^2+\beta^2)^2}\gamma+O(\gamma^2)\right]. \tag{A11}$$

The following formulas are also useful for calculating various mean values of u:

$$\frac{\partial I}{\partial\alpha}=\frac{\alpha}{2\gamma}I+\frac{\beta}{2\gamma}J-\frac{1}{2\gamma}, \tag{A12}$$

$$\frac{\partial J}{\partial\alpha}=\frac{\alpha}{2\gamma}J-\frac{\beta}{2\gamma}I, \tag{A13}$$

$$\frac{\partial^2 I}{\partial\alpha^2}=\frac{1}{4\gamma^2}(\alpha^2-\beta^2+2\gamma)I+\frac{\alpha\beta}{2\gamma^2}J-\frac{\alpha}{4\gamma^2}, \tag{A14}$$

$$\frac{\partial^2 J}{\partial\alpha^2}=-\frac{1}{4\gamma^2}(\alpha^2+\beta^2-2\gamma)J+\frac{\beta}{4\gamma^2}. \tag{A15}$$

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