# Theory of acoustic radiation pressure for actual fluids

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A body irradiated by a sound field is known to experience a steady force that is called the acoustic radiation pressure. This force plays an important role in many physical phenomena, such as cavitation, sonoluminescence, acoustic levitation, etc. The existing theory of acoustic radiation pressure neglects dissipative effects. The present paper develops a theory that takes these effects into account, both dissipative mechanisms, viscous and thermal, being considered. It is shown that, when they are no longer negligible, the dissipative effects drastically change the radiation pressure. As a result, its magnitude and sign become different from those predicted by the ''classical'' theory neglecting losses. [S1063-651X(96)14411-1]

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## I. INTRODUCTION

When a body is subjected to a sound field, it experiences a mean force which is due to the nonlinear properties of wave motion in fluids. This force, known as the acoustic radiation pressure, is an analog of the optical radiation pressure exerted by an electromagnetic wave on electrically or magnetically responsive objects, but the acoustic force is in general much larger than its electromagnetic counterpart [1]. For this reason, the acoustic radiation pressure is found to be useful in many applications. For example, it is used to levitate and position solid and liquid samples for the purpose of containerless processing, which keeps them from contamination by physical contact with the wall [2,3]. This force also plays an important role in the motion of bubbles undergoing acoustic cavitation and sonoluminescence [4].

The theory of acoustic radiation pressure was first proposed by Lord Rayleigh [5]. King [6] derived an expression for the radiation pressure on a rigid sphere, and Yosioka and Kawasima [7] extended his results to compressible spheres. Since then this subject has been investigated by many researchers (see reviews in [8–10]). The majority of them, following King and Yosioka and Kawasima, have assumed the surrounding fluid and the medium inside the sphere to be ideal, i.e., inviscid and non-heat-conducting. Those few papers in which attempts have been made to examine dissipative effects on the radiation pressure (see a review in [11]) are not satisfactory from the standpoint of their generality, rigor, and completeness but predict new interesting phenomena. This induced me to undertake a study that seeks (i) to obtain a general expression for the acoustic radiation pressure exerted on a spherical particle in an actual, i.e., viscous and heat-conducting, fluid under minimum limitations on parameters of this task, and (ii) to investigate by comparison with the theory for ideal fluids how the dissipative mechanisms affect the radiation pressure. In previous papers I [11–13] examined the viscous effects for various types of particles and sound fields. The present paper extends that work by taking into account both dissipative mechanisms and performing their comparative analysis. In deriving a general expression for the radiation pressure in Sec. II, the radius of the particle is assumed to be arbitrary compared with the sound, viscous, and thermal wavelengths which are also arbitrary relative to one another. The surrounding fluid is assumed to be either a liquid or a gas, and the internal structure of the particle to be any of a number of things, so that the particle may be a liquid drop, a gas bubble, a rigid or elastic sphere, a spherical shell, etc. The incident sound field is assumed to be axisymmetric (sound fields of most interest in applications fall into this category). In Sec. III, to demonstrate the dissipative effects clearly, the general theory is applied to the case of a rigid sphere in a plane traveling wave field. Finally, in Sec. IV a summary of the results obtained in this paper is given.

## II. GENERAL THEORY

### A. Problem formulation

Let us consider an arbitrary particle immersed in a viscous heat-conducting fluid and having the spherical shape at rest. The fluid motion is governed by the following equations [14]:

$$\frac{\partial}{\partial t}(\rho v_i)=\frac{\partial}{\partial x_k}(\sigma_{ik}-\rho v_i v_k), \tag{1}$$

$$\frac{\partial \rho}{\partial t}+\frac{\partial}{\partial x_i}(\rho v_i)=0, \tag{2}$$

$$\rho T\left(\frac{\partial s}{\partial t}+v_i\frac{\partial s}{\partial x_i}\right)=(\sigma_{ik}+p\,\delta_{ik})\frac{\partial v_i}{\partial x_k}+\frac{\partial}{\partial x_i}\left(\kappa\frac{\partial T}{\partial x_i}\right), \tag{3}$$

in which $\sigma_{ik}$ is the stress tensor, given by

$$\sigma_{ik}=-p\,\delta_{ik}+\eta\left(\frac{\partial v_i}{\partial x_k}+\frac{\partial v_k}{\partial x_i}-\frac{2}{3}\frac{\partial v_j}{\partial x_j}\delta_{ik}\right)+\xi\frac{\partial v_j}{\partial x_j}\delta_{ik}, \tag{4}$$

**v** is the fluid velocity, $\rho$ is the fluid density, $T$ is the absolute temperature, $s$ is the specific entropy, $p$ is the fluid pressure, $\kappa$ is the thermal conductivity, $\eta$ is the shear viscosity, $\xi$ is the bulk viscosity, $\delta_{ik}$ is the Kronecker delta, and as usual summation over repeated indices is implied. For this set of equations to be complete, we add to it two thermodynamic relations:

$$d\rho=(\gamma/c^2)dp-\alpha\rho dT, \tag{5}$$

 

$$ds=(c_p/T)dT-(\alpha/\rho)dp, \tag{6}$$

where $\gamma$ is the specific heat ratio, $c$ is the sound speed, $c_p$ is the specific heat at constant pressure, and $\alpha$ is the volume thermal expansion coefficient of the fluid defined as $\alpha=-(1/\rho)(\partial\rho/\partial T)_p$.

Let us now assume that the host fluid is subjected to an axisymmetric sound field, such as a plane-traveling or -standing wave or a spherical wave, which varies with time as $\exp(-i\omega t)$, where $\omega$ is the angular frequency. Then the particle will experience the acoustic radiation force which is represented as follows [11,12]:

$$F_i=\int_{S_0}\langle\sigma_{ik}^{(2)}-\rho_0 v_i^{(1)}v_k^{(1)}\rangle n_k dS_0, \tag{7}$$

where $S_0$ is the particle surface at rest, $\rho_0$ is the equilibrium fluid density, $\mathbf{n}$ is the outward normal to $S_0$, the superscript $(j)$ $(j=1,2)$ is used to denote quantities of $j$th order in the incident wave amplitude, and $\langle\ \rangle$ means an average over the incident wave period.

In an ideal fluid, the tensor $\langle\sigma_{ik}^{(2)}\rangle$ is known to be expressed in terms of quantities of first order only, which considerably expedites calculating the radiation force. In order to find this tensor in an actual fluid, one must solve the time-averaged equations of the fluid motion with accuracy up to the second-order terms. Thus, to calculate the acoustic radiation force in a viscous heat-conducting fluid, we must first solve Eqs. (1)–(6) in the linear approximation and then, after averaging over time, find their solutions in the second approximation. These calculations are presented in the following two subsections.

### B. Linear equations

After linearizing and eliminating $\rho^{(1)}$ and $s^{(1)}$ by use of Eqs. (5) and (6), Eqs. (1)–(3), become

$$\rho_0\frac{\partial\mathbf{v}^{(1)}}{\partial t}=-\nabla p^{(1)}+\eta_0\Delta\mathbf{v}^{(1)}+(\xi_0+\eta_0/3)\nabla(\nabla\cdot\mathbf{v}^{(1)}), \tag{8}$$

$$\frac{\partial p^{(1)}}{\partial t}=\frac{c_0^2\rho_0}{\gamma_0}\left(\alpha_0\frac{\partial T^{(1)}}{\partial t}-\nabla\cdot\mathbf{v}^{(1)}\right), \tag{9}$$

$$\Delta T^{(1)}-\frac{1}{\chi_0}\frac{\partial T^{(1)}}{\partial t}=-\frac{\alpha_0 T_0}{\kappa_0}\frac{\partial p^{(1)}}{\partial t}, \tag{10}$$

where $\chi_0$ is the thermal diffusivity defined as $\chi_0=\kappa_0/(\rho_0 c_{p0})$, and the subscript 0 denotes the equilibrium conditions. Eliminating $p^{(1)}$ and considering that the first-order quantities vary with time as $\exp(-i\omega t)$, this system reduces to

$$\Delta\mathbf{v}^{(1)}+\left(\frac{1}{3}+\frac{\xi_0}{\eta_0}+\frac{ic_0^2}{\gamma_0\omega\nu_0}\right)\nabla(\nabla\cdot\mathbf{v}^{(1)})+\frac{i\omega}{\nu_0}\mathbf{v}^{(1)}=\frac{\alpha_0 c_0^2}{\gamma_0\nu_0}\nabla T^{(1)}, \tag{11}$$

$$\Delta T^{(1)}+\frac{i\omega}{\gamma_0\chi_0}T^{(1)}=\frac{\gamma_0-1}{\alpha_0\chi_0\gamma_0}\nabla\cdot\mathbf{v}^{(1)}, \tag{12}$$

where $\nu_0=\eta_0/\rho_0$ is the kinematic viscosity.

The velocity $\mathbf{v}^{(1)}$ can be represented as [15]

$$\mathbf{v}^{(1)}=\nabla\varphi^{(1)}+\nabla\times\boldsymbol{\psi}^{(1)}, \tag{13}$$

where $\varphi^{(1)}$ and $\boldsymbol{\psi}^{(1)}$ are the scalar and vorticity velocity potentials of first order, respectively. Substituting Eq. (13) into Eqs. (11) and (12), one obtains two combined equations for $\varphi^{(1)}$ and $T^{(1)}$:

$$(\Delta+\beta_1)\varphi^{(1)}=-\frac{i\omega\alpha_0}{\beta_3}T^{(1)}, \tag{14}$$

$$(\Delta+\beta_2)T^{(1)}=\frac{(1-\gamma_0)\beta_1}{\alpha_0\chi_0\gamma_0}\varphi^{(1)}, \tag{15}$$

in which

$$\beta_1=\gamma_0\omega^2/(c_0^2\beta_3), \tag{16}$$

$$\beta_2=i\omega(\gamma_0+\beta_3-1)/(\chi_0\beta_3\gamma_0), \tag{17}$$

$$\beta_3=1-i\omega\gamma_0(\xi_0+4\eta_0/3)/(\rho_0 c_0^2), \tag{18}$$

and an equation for $\boldsymbol{\psi}^{(1)}$,

$$(\Delta+k_3^2)\boldsymbol{\psi}^{(1)}=0, \tag{19}$$

where $k_3=(1+i)/\delta_v$ is the viscous wave number, and $\delta_v=\sqrt{2\nu_0/\omega}$ is the depth of penetration of the viscous wave. Note also that the quantity $\lambda_v=2\pi/\mathrm{Re}(k_3)=2\pi\delta_v$ (where Re denotes the real part) is called the viscous wavelength.

Taking into account that the problem involved is axisymmetric about the direction of incident wave propagation, solutions to Eqs. (14) and (15) are written as follows [16]:

$$\varphi^{(1)}=e^{-i\omega t}\sum_{n=0}^{\infty}[A_{1n}z_n(k_1 r)+A_{2n}z_n(k_2 r)]P_n(\cos\theta), \tag{20}$$

$$T^{(1)}=\frac{i\beta_3}{\omega\alpha_0}e^{-i\omega t}\sum_{n=0}^{\infty}[(\beta_1-k_1^2)A_{1n}z_n(k_1 r)+(\beta_1-k_2^2)A_{2n}z_n(k_2 r)]P_n(\cos\theta). \tag{21}$$

Here

$$k_{1,2}=\left\{\frac{1}{2}(\beta_1+\beta_2)\left[1\mp\sqrt{1-\frac{4i\omega\beta_1}{\chi_0\gamma_0(\beta_1+\beta_2)^2}}\right]\right\}^{1/2}, \tag{22}$$

$k_1$ relating to the negative root and $k_2$ to the positive one, $z_n$ is a spherical cylindrical function of order $n$, $P_n$ is the Legendre polynomial of degree $n$, $r$ and $\theta$ are the coordinates of the spherical coordinate system $(r,\theta,\varepsilon)$ whose origin is at the equilibrium center of the particle and the $z$ axis (the zero direction of $\theta$) lies in the direction of incident wave propagation, and $A_{1n}$ and $A_{2n}$ are arbitrary constants.

Axisymmetric solution to Eq. (19) is given by [15]

$$\boldsymbol{\psi}^{(1)}=\hat{\mathbf{e}}_{\varepsilon}e^{-i\omega t}\sum_{n=1}^{\infty} B_n z_n(k_3 r)P_n^1(\cos\theta), \tag{23}$$

where $\hat{\mathbf{e}}_{\varepsilon}$ is the unit vector of the spherical coordinate system mentioned above, $P_n^1$ is the associated Legendre polynomial of the first order and degree $n$, and $B_n$ are arbitrary constants.

The total linear field consists of the incident field and the scattered field, so that we can write

$$\varphi^{(1)}=\varphi_I^{(1)}+\varphi_S^{(1)}, \tag{24}$$

$$T^{(1)}=T_I^{(1)}+T_S^{(1)}, \tag{25}$$

$$\boldsymbol{\psi}^{(1)}=\boldsymbol{\psi}_I^{(1)}+\boldsymbol{\psi}_S^{(1)}, \tag{26}$$

where the subscript $I$ refers to the incident field, and the subscript $S$ to the scattered field. Both fields are expressed by Eqs. (20)–(23) provided that $z_n$ is replaced by an appropriate specific function. For the incident field, we replace $z_n$ by the spherical Bessel function $j_n$. We also assume the incident field to be irrotational, so that $\boldsymbol{\psi}_I^{(1)}\equiv 0$. As a result, we obtain

$$\varphi_I^{(1)}=e^{-i\omega t}\sum_{n=0}^{\infty} [A_{1n}j_n(k_1 r)+A_{2n}j_n(k_2 r)]P_n(\cos\theta), \tag{27}$$

$$T_I^{(1)}=\frac{i\beta_3}{\omega\alpha_0}e^{-i\omega t}\sum_{n=0}^{\infty} [(\beta_1-k_1^2)A_{1n}j_n(k_1 r) +(\beta_1-k_2^2)A_{2n}j_n(k_2 r)]P_n(\cos\theta). \tag{28}$$

The constants $A_{1n}$ and $A_{2n}$ are determined by the type of incident wave and assumed to be the given quantities in this study. The first term between the square brackets in Eq. (27) defines the elastic sound waves, which are due to the fluid compressibility, and the second one defines the thermal sound waves, which occur through nonzero heat conduction of the fluid. It is known that the thermal sound waves die down on passage through a fluid much more than the elastic ones. Therefore, they can play a distinct role only within a thin layer with the thickness of the order of $\delta_t=\sqrt{2\chi_0/\omega}$ near the sound transducer. We will assume that the particle is placed at such a distance from the sound transducer that the thermal component of the incident sound field is negligible. This considerably simplifies calculations without the loss of their generality and practical significance. Then, setting $A_{2n}$ equal to zero and writing $A_n$ for $A_{1n}$, Eqs. (27) and (28) become

$$\varphi_I^{(1)}=e^{-i\omega t}\sum_{n=0}^{\infty} A_n j_n(k_1 r)P_n(\cos\theta), \tag{29}$$

$$T_I^{(1)}=\frac{i\beta_3(\beta_1-k_1^2)}{\omega\alpha_0}\varphi_I^{(1)}. \tag{30}$$

These expressions are what will be considered as a general form of the incident field in this paper.

For the scattered field, we replace $z_n$ by the spherical Hankel functions of the first kind $h_n^{(1)}$. This results in

$$\varphi_S^{(1)}=e^{-i\omega t}\sum_{n=0}^{\infty} A_n[\alpha_{1n}h_n^{(1)}(k_1 r)+\alpha_{2n}h_n^{(1)}(k_2 r)]P_n(\cos\theta), \tag{31}$$

$$T_S^{(1)}=\frac{i\beta_3}{\omega\alpha_0}e^{-i\omega t}\sum_{n=0}^{\infty} A_n[(\beta_1-k_1^2)\alpha_{1n}h_n^{(1)}(k_1 r) +(\beta_1-k_2^2)\alpha_{2n}h_n^{(1)}(k_2 r)]P_n(\cos\theta), \tag{32}$$

$$\boldsymbol{\psi}_S^{(1)}=\hat{\mathbf{e}}_{\varepsilon}e^{-i\omega t}\sum_{n=1}^{\infty} \alpha_{3n}A_n h_n^{(1)}(k_3 r)P_n^1(\cos\theta), \tag{33}$$

where $\alpha_{1n}$, $\alpha_{2n}$, and $\alpha_{3n}$ are dimensionless constants (which are usually referred to as the linear scattering coefficients) to be determined by the boundary conditions at the particle surface, namely, the conditions of continuity of the velocity, stress, temperature, and heat flux. To apply these conditions we must point out a specific internal structure of the particle. However, the purpose of this section is to derive a general expression for the radiation pressure applicable to any type of dispersed particles. Therefore we now consider the linear scattering coefficients to be known and proceed to solve the equations of second order.

### C. Time-averaged equations of second order

By time-averaging Eqs. (1)–(3) and keeping up to the second order, we obtain

$$\eta_0\Delta\langle\mathbf{v}^{(2)}\rangle+(\xi_0+\eta_0/3)\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\langle\mathbf{v}^{(2)}\rangle)-\boldsymbol{\nabla}\langle p^{(2)}\rangle =\rho_0\langle\mathbf{v}^{(1)}(\boldsymbol{\nabla}\cdot\mathbf{v}^{(1)})+(\mathbf{v}^{(1)}\cdot\boldsymbol{\nabla})\mathbf{v}^{(1)}\rangle, \tag{34}$$

$$\boldsymbol{\nabla}\cdot\langle\mathbf{v}^{(2)}\rangle=-\frac{1}{\rho_0}\boldsymbol{\nabla}\cdot\langle\rho^{(1)}\mathbf{v}^{(1)}\rangle, \tag{35}$$

$$\kappa_0\Delta\langle T^{(2)}\rangle=\left\langle(\rho_0 T^{(1)}+T_0\rho^{(1)})\frac{\partial s^{(1)}}{\partial t}\right\rangle+\rho_0 T_0\langle\mathbf{v}^{(1)}\cdot\boldsymbol{\nabla}s^{(1)}\rangle -(\xi_0-2\eta_0/3)\langle(\boldsymbol{\nabla}\cdot\mathbf{v}^{(1)})^2\rangle -\frac{\eta_0}{2}\left\langle\left(\frac{\partial v_i^{(1)}}{\partial x_k}+\frac{\partial v_k^{(1)}}{\partial x_i}\right)^2\right\rangle, \tag{36}$$

where $\rho^{(1)}$ and $s^{(1)}$ are defined by Eqs. (5) and (6). One can see that Eq. (36), which describes a stationary temperature distribution in the fluid, is not related to Eqs. (34) and (35), which govern the stationary fluid motion, the so-called acoustic streaming. As the quantity $\langle T^{(2)}\rangle$ in itself does not appear in the expression for the radiation pressure, there is no necessity to solve Eq. (36). The procedure for solving Eqs. (34) and (35) is described in detail in [11] and therefore will not be repeated here.

### D. General expression for the radiation force

Substituting the first- and second-order solutions into Eq. (7), after laborious but straightforward calculations, one obtains

$$F=F_r+F_d, \tag{37}$$

where

$$F_r=\frac{3}{2}\pi\rho_0\sum_{n=0}^{\infty}\frac{n+1}{(2n+1)(2n+3)}(Z_nA_nA_{n+1}^*+Z_n^*A_n^*A_{n+1}),\tag{38}$$

$$Z_n=\sum_{j=1}^{3}\left(F_n^{(0j)}\alpha_{jn+1}^*+F_n^{(j0)}\alpha_{jn}+\sum_{k=1}^{3}F_n^{(jk)}\alpha_{jn}\alpha_{kn+1}^*\right),\tag{39}$$

$$\begin{aligned}F_d=&3\pi\eta_0R_0\int_0^{\pi}\langle v_{Iz}^{(2)}\rangle\Big|_{r=R_0}\sin\theta d\theta\\&-3\pi\rho_0|x_1|^2\sum_{n=0}^{\infty}\frac{n+1}{(2n+1)(2n+3)}\\&\times\mathrm{Re}\left\{\frac{A_nA_{n+1}^*}{x_3^2}[x_1^*|j_n'(x_1)|^2+x_1|j_{n+1}'(x_1)|^2]\right\}.\end{aligned}\tag{40}$$

Here $F$ is the total radiation force exerted on the particle, $F_r$ is the acoustic radiation pressure [it is Eq. (38) that gives the well-known expressions for the radiation pressure in ideal fluids [6,7] when the dissipative effects tend to zero], and $F_d$ is the drag force caused by the stationary fluid flow that is induced by the incident sound field in the absence of the particle (this force is an analog of Stokes' drag force and vanishes in an ideal fluid). The functions $F_n^{(jk)}$ are given in the Appendix, the asterisk indicates the complex conjugate, $R_0$ is the equilibrium radius of the particle, $\langle v_{Iz}^{(2)}\rangle$ is the projection of $\langle \mathbf{v}_I^{(2)}\rangle$ on the $z$ axis, $\langle \mathbf{v}_I^{(2)}\rangle$ is the stationary fluid velocity as if the particle were absent, $x_1=k_1R_0$, $x_3=k_3R_0$, the prime denotes differentiation, and Re denotes the real part. The velocity $\langle \mathbf{v}_I^{(2)}\rangle$ is found from Eqs. (34) and (35), keeping only the incident field in their right-hand sides. But it is much easier to calculate this quantity for a specific type of incident field than for the general case determined by Eq. (29). Expressions for $\langle \mathbf{v}_I^{(2)}\rangle$ in sound fields of most interest are given in [11–13].

To summarize, Eqs. (37)–(40) allow us to calculate the radiation force on an arbitrary spherical particle in an arbitrary axisymmetric sound field, and consequently the purpose of this section is achieved. But these equations do not allow us to see the dissipative effects on the force clearly. Therefore in the next section we apply the general theory to a specific case.

## III. EXAMPLE

As an example we consider a rigid sphere subjected to a plane traveling wave.

### A. Radiation force in a plane traveling wave

Let us apply Eqs. (38) and (40) to a plane traveling wave given by

$$\varphi_I^{(1)}=A\exp(i\mathbf{k}_1\cdot\mathbf{r}-i\omega t),\tag{41}$$

where $\mathbf{k}_1=k_1\hat{\mathbf{e}}_z$, $\hat{\mathbf{e}}_z$ is the unit vector in the $+z$ direction, and $\mathbf{r}$ is the position vector. By expanding Eq. (41) in the Legendre polynomials [17] and comparing the series obtained with Eq. (29), one finds

$$A_n=A(2n+1)i^n.\tag{42}$$

Substitution of Eq. (42) into Eq. (38) yields

$$F_r^{(tr)}=3\pi\rho_0|A|^2\sum_{n=0}^{\infty}(n+1)\mathrm{Im}(Z_n),\tag{43}$$

where Im denotes the imaginary part.

An expression for $\langle \mathbf{v}_I^{(2)}\rangle$ in a plane traveling wave is given by Eq. (5.11) of [11]. Substituting it along with Eq. (42) into Eq. (40), one obtains

$$\begin{aligned}F_d^{(tr)}=&-\frac{3}{2}\pi\rho_0|A|^2|x_1|^2(x_1+x_1^*)ix_3^{-2}\left[\frac{\sin(x_1-x_1^*)}{x_1-x_1^*}\right.\\&\left.-\sum_{n=0}^{\infty}(n+1)[|j_n'(x_1)|^2+|j_{n+1}'(x_1)|^2]\right].\end{aligned}\tag{44}$$

### B. Linear scattering coefficients for a rigid sphere

The general formula for the radiation pressure [see Eqs. (39) and (43)] is expressed in terms of the linear scattering coefficients $\alpha_{1n}$, $\alpha_{2n}$, and $\alpha_{3n}$ which are determined by the type of particle under consideration. For the rigid sphere, these coefficients are found from the following boundary conditions:

$$\mathbf{v}_I^{(1)}+\mathbf{v}_S^{(1)}=\mathbf{u}^{(1)}\quad\text{at}\quad r=R_0,\tag{45}$$

$$T_I^{(1)}+T_S^{(1)}=\tilde{T}^{(1)}\quad\text{at}\quad r=R_0,\tag{46}$$

$$\kappa_0\frac{\partial}{\partial r}(T_I^{(1)}+T_S^{(1)})=\tilde{\kappa}_0\frac{\partial\tilde{T}^{(1)}}{\partial r}\quad\text{at}\quad r=R_0,\tag{47}$$

where $\mathbf{u}^{(1)}$ is the first-order translational velocity of the sphere, and the tilde denotes quantities that concern the medium inside the sphere. Note also that the equilibrium temperature of the sphere $\tilde{T}_0$ is assumed equal to that of the host fluid $T_0$. The boundary conditions are supplemented with the equation of the translational motion of the sphere, which in linear approximation is written as

$$\frac{4}{3}\pi R_0^3\tilde{\rho}_0\frac{du_i^{(1)}}{dt}=\int_{S_0}\sigma_{ik}^{(1)}n_kdS\tag{48}$$

and an equation describing heat propagation inside the sphere for which we take the well-known Fourier equation,

$$\frac{\partial\tilde{T}^{(1)}}{\partial t}=\tilde{\chi}_0\Delta\tilde{T}^{(1)}.\tag{49}$$

Substituting Eqs. (29)–(33) into Eq. (48), one obtains for $\mathbf{u}^{(1)}$

$$\mathbf{u}^{(1)} = (A_1\lambda_\rho/R_0)\hat{\mathbf{e}}_z e^{-i\omega t}[j_1(x_1) + \alpha_{11}h_1^{(1)}(x_1) + \alpha_{21}h_1^{(1)}(x_2) + 2\alpha_{31}h_1^{(1)}(x_3)], \tag{50}$$

where $\lambda_\rho = \rho_0/\tilde{\rho}_0$ and $x_2 = k_2R_0$. Axisymmetric solution to Eq. (49), which is also required to be finite at $r=0$ and to oscillate at the driving frequency, is given by

$$\tilde{T}^{(1)} = e^{-i\omega t}\sum_{n=0}^{\infty} \tilde{\alpha}_n A_n j_n(\tilde{k}_2 r) P_n(\cos\theta), \tag{51}$$

where $\tilde{k}_2 = (1+i)/\tilde{\delta}_t$, $\tilde{\delta}_t = \sqrt{2\tilde{\chi}_0/\omega}$, and $\tilde{\alpha}_n$ are dimensionless constants to be determined by the boundary conditions. Substituting Eqs. (50) and (51) along with Eqs. (29)–(33) into Eqs. (45)–(47) and eliminating $\tilde{\alpha}_n$, one obtains a set of three simultaneous equations in $\alpha_{1n}$, $\alpha_{2n}$, and $\alpha_{3n}$,

$$a_{i1}\alpha_{1n} + a_{i2}\alpha_{2n} + a_{i3}\alpha_{3n} = b_i, \quad i = 1,2,3, \tag{52}$$

where

$$a_{11} = (1-\lambda_\rho\delta_{1n})nh_n^{(1)}(x_1) - x_1h_{n+1}^{(1)}(x_1),$$

$$a_{12} = (1-\lambda_\rho\delta_{1n})nh_n^{(1)}(x_2) - x_2h_{n+1}^{(1)}(x_2),$$

$$a_{13} = n(n+1)(1-\lambda_\rho\delta_{1n})h_n^{(1)}(x_3),$$

$$a_{21} = (1-\lambda_\rho\delta_{1n})h_n^{(1)}(x_1),$$

$$a_{22} = (1-\lambda_\rho\delta_{1n})h_n^{(1)}(x_2),$$

$$a_{23} = (1-\lambda_\rho\delta_{1n})(n+1)h_n^{(1)}(x_3) - x_3h_{n+1}^{(1)}(x_3),$$

$$a_{31} = (1-nq_n)h_n^{(1)}(x_1) + q_nx_1h_{n+1}^{(1)}(x_1),$$

$$a_{32} = \frac{\beta_1 - k_2^2}{\beta_1 - k_1^2}[(1-nq_n)h_n^{(1)}(x_2) + q_nx_2h_{n+1}^{(1)}(x_2)], \quad a_{33} = 0,$$

$$b_1 = x_1j_{n+1}(x_1) - (1-\lambda_\rho\delta_{1n})nj_n(x_1),$$

$$b_2 = (\lambda_\rho\delta_{1n} - 1)j_n(x_1),$$

$$b_3 = (nq_n - 1)j_n(x_1) - q_nx_1j_{n+1}(x_1),$$

$$q_n = \lambda_\kappa j_n(\tilde{x}_2)/[nj_n(\tilde{x}_2) - \tilde{x}_2j_{n+1}(\tilde{x}_2)], \tag{53}$$

$\lambda_\kappa = \kappa_0/\tilde{\kappa}_0$, and $\tilde{x}_2 = \tilde{k}_2R_0$. Equations (52) can be easily solved by any of the conventional methods. Explicit expressions for $\alpha_{jn}$ are, however, extremely cumbersome and therefore are not presented here.

### C. Radiation force in the limits of weak and strong dissipative effects

Substituting $F_n^{(jk)}$ from the Appendix and the linear scattering coefficients from the preceding subsection into Eq. (43), we can obtain the radiation pressure for the most general case. The expression is, however, too complicated to see dissipative effects clearly. Therefore we consider below two opposite limiting cases that allow this expression to be substantially simplified. First we assume the sound wavelength to be much larger than the viscous and thermal wavelengths ($|x_1| \ll |x_2|, |x_3|, |\tilde{x}_2|$) and the radius of the sphere ($|x_1| \ll 1$). This limit, known as the long sound wavelength limit, is of most interest in applications. Further, remaining within the above limit, we consider two opposite limiting cases. In the first of these, the dissipative effects are assumed weak, so that the following inequality holds: $|x_1| \ll 1 \ll |x_2|, |x_3|, |\tilde{x}_2|$. In the second case, the dissipative effects are assumed strong, so that $|x_1| \ll |x_2|$, $|x_3|, |\tilde{x}_2| \ll 1$.

#### 1. Weak dissipative effects: $|x_1| \ll 1 \ll |x_2|, |x_3|, |\tilde{x}_2|$

Passing to this limit in Eq. (44), the expressions for $F_n^{(jk)}$ (see the Appendix), and Eqs. (52), solving the latter for $\alpha_{jn}$ and substituting the obtained approximate expressions for $F_n^{(jk)}$ and $\alpha_{jn}$ into Eq. (43), one finds the leading term of the total radiation force to be

$$F = 2\pi\rho_0|A|^2x_{10}^3\left[3\left(\frac{1-\lambda_\rho}{2+\lambda_\rho}\right)^2\frac{\delta_v}{R_0} + \frac{\gamma_0 - 1}{2(1+\lambda_\kappa\tilde{\delta}_t/\delta_t)}\frac{\delta_t}{R_0}\right], \tag{54}$$

where $x_{10} = \omega R_0/c_0$ and $\delta_v/R_0$, $\delta_t/R_0 \ll 1$. Let us compare Eq. (54) with the well-known King's formula [6] which was derived without considering the dissipative mechanisms:

$$F_K = 2\pi\rho_0|A|^2x_{10}^6\frac{9+2(1-\lambda_\rho)^2}{9(2+\lambda_\rho)^2}. \tag{55}$$

Equation (54) is seen to be quite different from Eq. (55). This is explained as follows. In reality the total expansion of the radiation force in the small parameter $x_{10}$ involves both these equations. If we have assumed the dissipative effects to be zero, then we would obtain King's formula, Eq. (55), for the leading term of the radiation force. We, however, have considered the dissipative effects to be small but still finite. In this case, the leading term in $x_{10}$ is found to be given by Eq. (54), which is of lower order in $x_{10}$ and accordingly of higher order in $\delta_v/R_0$ and $\delta_t/R_0$ than King's formula. Each of the above equations can be dominant depending on the magnitude of the dissipative effects. By comparing Eq. (54) with Eq. (55), we can roughly estimate that the former begins to predominate when

$$x_{10}^3 \ll \delta_v/R_0 \quad \text{and/or} \quad x_{10}^3 \ll \delta_t/R_0. \tag{56}$$

Specific examples show that in the majority of cases of interest in applications conditions (56) are well satisfied. It follows that the radiation force due to a plane traveling wave should be much larger than that predicted by King's theory. This fact has not been detected because King's formula, Eq. (55), has not been verified by experiment. The only experiments in this field which have dealt with plane traveling waves are due to Hasegawa and collaborators [18], who, however, studied solid spheres of large size ($x_{10} \geqslant 1$). In addition, $\delta_v/R_0$ and $\delta_t/R_0$ were very small (about $10^{-4}$) in those experiments.

Let us now analyze Eq. (54) itself. It consists of two terms. The first is caused by viscosity, and the second by heat conduction. Both terms are positive, so that the sphere will be urged away from the sound transducer. To compare contributions from viscosity and heat conduction in magnitude, consider a typical example, say, a dust particle in air.

Setting $R_0=100$ $\mu$m and $f=40$ kHz, one obtains $x_{10}\approx 0.076$ ($x_{10}^3\approx 4.4\times 10^{-4}$), $\delta_v/R_0\approx 0.11$, and $\delta_t/R_0\approx 0.15$. It is seen that Eqs. (56) are satisfied. The ratio of the thermal term of Eq. (54) to the viscous term is about 0.36, showing that the thermal effects give a rather substantial correction. The ratio of $F$ to $F_K$ is about 837, supporting the fact that Eq. (54) is dominant.

#### 2. Strong dissipative effects: $|x_1|\ll|x_2|,|x_3|,|\tilde{x}_2|\ll 1$

In this limit, the total radiation force becomes

$$F=-\frac{2}{3}\pi\rho_0|A|^2x_{10}^3\left[\frac{11(1-\lambda_\rho)}{5\lambda_\rho}\frac{R_0}{\delta_v}+\frac{(\gamma_0-1)\delta_t}{\lambda_\kappa\tilde{\delta}_t}\frac{R_0}{\tilde{\delta}_t}\right]. \tag{57}$$

It should be emphasized that Eq. (57) cannot be compared with King's formula, Eq. (55), since these equations describe fundamentally different limiting cases. Let us analyze Eq. (57). It shows that heavy particles, i.e., ones for which $\lambda_\rho<1$, will be urged away towards the sound transducer. Recall that in the reverse limiting case the force is directed from the sound transducer. To compare the viscous and thermal contributions, consider, for example, a steel sphere in glycerin. To satisfy the required limiting conditions, we take $R_0=1$ $\mu$m and $f=1$ kHz. Then the ratio of the thermal term of Eq. (57) to the viscous term is about 1.2. This shows that both dissipative mechanisms are of equal importance.

Let us now consider light particles for which $\lambda_\rho>1$. It is seen that these can be urged away both from the sound transducer and towards it depending on whether the viscous or thermal term is dominant in Eq. (57). For example, for a sawdust particle in glycerin at the same values of $R_0$ and $f$ the thermal term predominates, causing the particle to move to the sound transducer.

## IV. CONCLUSIONS

In this paper, a general theory has been developed for the acoustic radiation pressure exerted by a sound field on a spherical particle, which takes account of the viscous and thermal effects in the surrounding fluid. Restrictions imposed on the incident field are as follows: (i) the incident field is axisymmetric (a plane-traveling and -standing wave and a spherical wave, which are of most interest in applications, fall into this category); (ii) the thermal component of the incident field is negligible (this can result from an appropriate method of sound generation or, as is usually the case in practice, because the particle is far enough from the sound transducer). No restrictions have been imposed on the size and the internal structure of the particle. This means that the particle can have an arbitrary radius with respect to the sound, viscous, and thermal wavelengths and be any of the following types: a liquid drop, a gas bubble, a rigid or elastic sphere, a spherical shell, etc. The general formula for the radiation pressure is expressed in terms of the linear scattering coefficients which are determined by a specific type of particle under consideration. To demonstrate the dissipative effects on the radiation pressure clearly, the general theory has been applied to the case of a rigid sphere subjected to a plane traveling wave field. It has been shown that in many cases of interest the dissipative mechanisms are not negligible and radically modify the expression for the radiation force from that given by the theory for ideal fluids: the radiation force is substantially increased, and when the dissipative effects are strong it can also reverse its direction. Comparative analysis of the viscous and thermal effects has found that both mechanisms are in general of equal importance.

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## APPENDIX

$$F_n^{(01)}=\frac{1}{2}[X_n^{(11)}(x_1,x_1)+X_n^{(21)}(x_1,x_1)],$$

$$F_n^{(02)}=\frac{1}{2}[X_n^{(11)}(x_1,x_2)+X_n^{(21)}(x_1,x_2)],$$

$$F_n^{(03)}=\frac{1}{2}[Y_n^{(110)}(x_1)+Y_n^{(210)}(x_1)]^*,$$

$$F_n^{(10)}=\frac{1}{2}[X_n^{(11)}(x_1,x_1)+X_n^{(12)}(x_1,x_1)],$$

$$F_n^{(20)}=\frac{1}{2}[X_n^{(11)}(x_2,x_1)+X_n^{(12)}(x_2,x_1)],$$

$$F_n^{(30)}=\frac{1}{2}[Y_n^{(101)}(x_1)+Y_n^{(201)}(x_1)],$$

$$F_n^{(11)}=X_n^{(11)}(x_1,x_1),\quad F_n^{(21)}=X_n^{(11)}(x_2,x_1),$$

$$F_n^{(12)}=X_n^{(11)}(x_1,x_2),\quad F_n^{(22)}=X_n^{(11)}(x_2,x_2),$$

$$F_n^{(13)}=[Y_n^{(110)}(x_1)]^*,\quad F_n^{(23)}=[Y_n^{(110)}(x_2)]^*,$$

$$F_n^{(31)}=Y_n^{(101)}(x_1),\quad F_n^{(32)}=Y_n^{(101)}(x_2),$$

$$\begin{aligned}F_n^{(33)}=&\,n(n+2)\{(n+1)x_3^2[H_{nn+1}^{(011)}(x_3,x_3)-H_{nn+1}^{(211)}(x_3,x_3)]\\&+ix_3^2[H_{n+1\,n}^{(011)}(x_3,x_3)-H_{n+1\,n}^{(211)}(x_3,x_3)]+(n+1)\\&\times[h_n^{(1)}(x_3)(h_n^{(1)\prime}(x_3))^*+(h_{n+1}^{(1)}(x_3))^*h_{n+1}^{(1)\prime}(x_3)]\}.\end{aligned}$$

Here

$$X_n^{(kl)}(x,y)=\frac{1}{2}[ny^{2*}-(n+2)x^2][H_{nn+1}^{(0kl)}(x,y)-H_{nn+1}^{(2kl)}(x,y)]+xy^*[H_{n+1\,n}^{(0kl)}(x,y)-H_{n+1\,n}^{(2kl)}(x,y)]$$
$$+\frac{y^{2*}-x^2}{x_3^2}[ny^*H_{nn}^{(1kl)}(x,y)-xy^*H_{n+1\,n}^{(0kl)}(x,y)+(n+2)xH_{n+1\,n+1}^{(1kl)}(x,y)]$$
$$-\frac{xy^*}{x_3^2}[y^*h_n^{(k)\prime}(x)(h_n^{(l)\prime}(y))^*+xh_{n+1}^{(k)\prime}(x)(h_{n+1}^{(l)\prime}(y))^*],\quad k,l=1,2,$$

and

$$Y_n^{(jkl)}(x)=\frac{n+2k}{2}\{[(n+1)x^{2*}-(n+2l)x_3^2][H_{n+kn+l}^{(01j)}(x_3,x)-H_{n+kn+l}^{(21j)}(x_3,x)]+x^*x_3^2[H_{n+kn+k}^{(-11j)}(x_3,x)-H_{n+kn+k}^{(11j)}(x_3,x)]$$
$$-x^{2*}x_3[H_{n+ln+l}^{(-11j)}(x_3,x)-H_{n+ln+l}^{(11j)}(x_3,x)]+2x^*h_{n+l}^{(1)\prime}(x_3)(h_{n+l}^{(j)\prime}(x))^*\}+(n+2k)(-1)^l$$
$$\times\Bigg\{x^*x_3[H_{n+ln+k}^{(01j)}(x_3,x)-H_{n+ln+k}^{(21j)}(x_3,x)]+\frac{(n+1)x^{2*}}{x_3^2}h_{n+k}^{(1)}(x_3)(h_{n+k}^{(j)\prime}(x))^*$$
$$-\frac{x^{2*}}{x_3^2}[(n+1)x^*H_{n+kn+k}^{(11j)}(x_3,x)-(n+2l)x_3H_{n+ln+l}^{(11j)}(x_3,x)]\Bigg\},\quad j=1,2;\quad k,l=0,1$$

in which

$$H_{nm}^{(jkl)}(x,y)=\int_1^\infty z^{-j}h_n^{(k)}(xz)[h_m^{(l)}(yz)]^*dz$$
$$=\sum_{p=0}^{n}\sum_{q=0}^{m}D_{pq}^{(nm)}x^{-(p+1)}(y^*)^{-(q+1)}E_{p+q+j+2}[i(-1)^kx-i(-1)^ly^*]\delta_{pq}^{(nm)}(k,l),\quad j=-1,0,1,2;k,l=1,2;$$

$$D_{pq}^{(nm)}=\frac{-i^{n+m}(n+p)!(m+q)!}{(2i)^{p+q}p!(n-p)!q!(m-q)!},$$
$$\delta_{pq}^{(nm)}(1,1)=(-1)^{n+p+1},\quad \delta_{pq}^{(nm)}(1,2)=(-1)^{n+m+p+q},$$
$$\delta_{pq}^{(nm)}(2,1)=1,\quad \delta_{pq}^{(nm)}(2,2)=(-1)^{m+q+1},$$

$h_n^{(k)}$ is the spherical Hankel function of the $k$th kind, and $E_n[x]$ is the integral exponent of order $n$ [17].

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