Avalanches, scaling, and coherent noise

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We present a simple model of a dynamical system driven by externally imposed coherent noise. Although the system never becomes critical in the sense of possessing spatial correlations of arbitrarily long range, it does organize into a stationary state characterized by avalanches with a power-law size distribution. We explain the behavior of the model within a time-averaged approximation, and discuss its potential connection to the dynamics of earthquakes, the Gutenberg-Richter law, and to recent experiments on avalanches in rice piles. [S1063-651X(96)11512-9]

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I. INTRODUCTION

There has in the last few years been considerable interest in extended systems which self-organize into a state exhibiting large scale fluctuations and intermittent dynamics. One of the earliest attempts to model systems of this type was made in 1987 by Bak, Tang, and Wiesenfeld, who proposed a simple lattice model for the avalanches produced by depositing grains of sand on an ever-growing sand pile [1]. Despite having only short-range interactions and no tunable parameters, their model organizes itself into a state with long-range spatial correlations and avalanches of size not limited by any finite correlation length. It has been proposed that similar self-organized critical (SOC) behavior could lie behind a wide range of physical phenomena showing $1/f$ noise and scale-free fluctuation distributions. SOC models have been put forward to describe the dynamics of earthquakes [2], biological evolution [3] and extinction [4], interface depinning [5,6], forest fires [7], and many other systems [8]. The common features of these models are that (i) they are all driven very slowly (quasistatically), and (ii) they have perfect memory. In this paper we demonstrate that it is possible to produce intermittent dynamics with robust scale invariant distributions in systems which are not quasistatically driven.

A distinctive observable consequence of SOC dynamics is that the distribution of fluctuation or avalanche sizes takes a power-law form with characteristic exponent τ:

$$p_{\rm aval}(s) \propto s^{-\tau}. \quad (1)$$

The value of τ typically lies between 1 and $\frac{3}{2}$ with the value $\frac{3}{2}$, corresponding to a critical branching process, appearing if one makes the "random neighbor approximation" in which each site interacts with a randomly selected small number of other sites [9]. This approximation is equivalent to the limit of infinite dimension, and should give correct results for systems above their critical dimension. However, some systems display event size distributions with exponents considerably larger than this. Examples include the one-dimensional (1D) rice pile experiment of Frette *et al.* [10] which yields a value of $\tau = 2.1 \pm 0.1$, and terrestrial earthquakes. In the case of earthquakes, there is good evidence to support the contention that the areas of displacement S of events are distributed according to a power law $p(S) \propto S^{-\tau_s}$, with $\tau_s = 2.0 \pm 0.1$, at least when the earthquakes are not too big [11,12]. The distribution of released energies is less clear cut [13], and seems to depend more strongly on earthquake size, with measured exponents τ ranging from about 1.6 to 2.0 for smaller quakes [2,12,14,15], to somewhere between 2.0 and 2.5 for larger ones [2,14,15], the crossover occurring around magnitude 6.

A number of models have been proposed which offer explanations for these higher values of τ. Boundary driven sandpile models [16], for instance, can give power laws with exponents in the vicinity of $\tau = 2$, though they do so at the expense of initiating the avalanches only on the boundaries of the system. The model discussed by Christensen and Olami [2] also generates steeper power laws, at the price of having an entirely deterministic dynamics; if one introduces randomness into this model, the simple scaling behavior is destroyed. The reason is that when the exponent describing the distribution of avalanches' spatial extent becomes larger than 2, the mean avalanche size becomes finite and independent of system size, and the spatial overlap between subsequent avalanches becomes insignificant [17]. In the presence of randomness, this can prevent the system from building up any long-range correlations, and ultimately destroy the critical state. We conjecture that, in this regime, randomness in the positions of the nucleation centers of the avalanches will destroy self-organization of long-range spatial correlations.

In this paper, we present a different explanation to account for systems that have larger values of τ. We demonstrate that power-law event size distributions having τ around 2 or greater, are *typical* of extended systems with quenched memory if they are driven by coherent noise, and that in such systems they are present even in the absence of any interaction between the different parts of the system. (This is different from the situation in the SOC models, where the system is driven by a local driving force, coupled with interactions between the components of the system.) The simplest model demonstrating the phenomenon is defined as follows. Consider a system of N agents, such as grains on the surface of a sand pile or points of contact in a subterranean fault. With each agent i we associate a threshold for movement x_i which can take values falling in some specified range and represents the amount of stress that the agent will withstand before it moves. For convenience, we choose to measure x_i on a scale on which $0 \leqslant x_i < 1$, though none of the results given below depend on this choice. The dynamics of the model then consists of the repetition of two steps, a ''stress'' step and an ''aging'' step. (The stress step is the more important. Even in the absence of the aging step, the stress step would be sufficient to create large, avalanche-like events, though they would not then have a power-law size distribution.)

(i) We select a random number or ''stress level'' η from some distribution $p_{\text{stress}}(\eta)$. All x_i below η are exchanged with new random numbers selected uniformly from the interval $0 \leqslant x_i < 1$. The number of agents whose thresholds are changed in this fashion is the size s of the avalanche taking place in this time step.

(ii) A fixed fraction f of the agents is selected at random, and the values of their threshold variables x_i are also exchanged with new random numbers selected uniformly from the interval $0 \leqslant x_i < 1$.

The random selection of different values for η at each step may be thought of as imposing external stresses which coherently (in other words, simultaneously) influence all of the weaker agents—those having suitably low thresholds for stress—but leave unchanged the stronger ones. It seems physically reasonable to assume that smaller stresses should be more common than larger ones, and in the following discussion we make the assumption that $p_{\text{stress}}(\eta)$ is largest at $\eta = 0$ and falls off to zero as η becomes large. We denote the typical scale of the falloff by σ. The most interesting regime is when $\sigma \ll 1$ and $f \ll 1$.

II. RESULTS

We have examined the properties of this model both analytically and numerically. Instead of simulating the model directly, we have developed an algorithm which calculates the threshold distribution and avalanche sizes in a formally exact way for a system with $N = \infty$. Starting off with a uniform distribution of thresholds, the system evolves towards a statistically stationary state. In this state we record the mean threshold distribution and the frequency distribution of avalanches. The results are shown in Figs. 1 and 2 for a simulation using exponentially distributed stresses:

$$p_{\text{stress}}(\eta) = \frac{1}{\sigma} \exp(-\eta/\sigma). \tag{2}$$

As Fig. 1 shows, the distribution of avalanche sizes s is flat up to a certain point (whose position varies with σ and f) and then falls off as a power law according to Eq. (1) with $\tau \approx 2.0$. This power-law behavior appears to be robust in the regime of small f and σ. If, for example, instead of Eq. (2) we employ a Gaussian stress distribution

$$p_{\text{stress}}(\eta) = \sqrt{\frac{2}{\pi\sigma^2}}\, e^{-\eta/\sigma^2}, \tag{3}$$

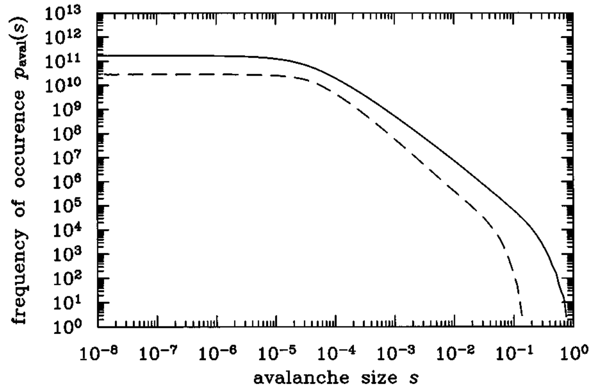

FIG. 1. Simulation results for the frequency distribution of avalanches with exponentially distributed stresses (solid line) and Gaussian ones (dashed line), with $f = 10^{-3}$ and $\sigma = \frac{1}{20}$ in each case.

then, although the average distribution of thresholds (Fig. 2) changes radically, the power-law form of the avalanche distribution remains. Notice, however, that the exponent τ changes slightly as the applied stresses are varied. For the Gaussian distribution, for example, we find $\tau = 2.02 \pm 0.02$, as opposed to $\tau = 1.84 \pm 0.03$ for the exponential. And for steeper distributions of stresses $(p(\eta) \propto \exp[-(\eta/\sigma)^q]$ with $q \geqslant 4)$ we find $\tau = 2.2$ or greater.

Notice also that we measure the avalanche sizes in Fig. 1 as fractions of the total system size. Since the agents in our model do not interact directly with one another, there should be no finite size effects and we would expect the avalanche distribution to look exactly the same for all sizes of system when plotted in this way, except for a ''coarsening'' of the horizontal scale for smaller systems as the number of bins in the histogram decreases.

In order to investigate possible connections with spatially organized models, we have also implemented our model on a lattice and at each time step eliminated not only those agents whose thresholds for stress fall below the selected level, but also their neighbors. In all cases we observe a power-law distribution of avalanches with exponent in the vicinity of $\tau = 2$.

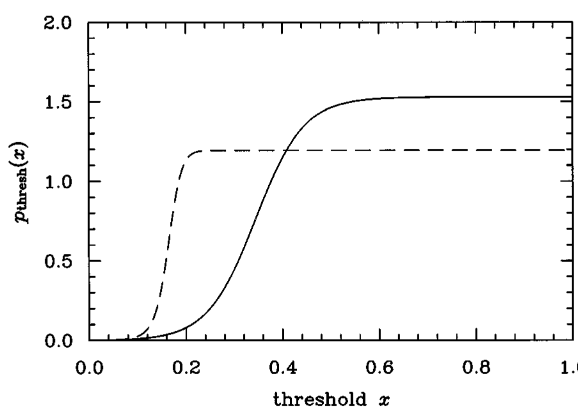

FIG. 2. Simulation results for the time-averaged distribution of thresholds x with exponentially distributed stresses (solid line) and Gaussian ones (dashed line). As in Fig. 1, $f = 10^{-3}$ and $\sigma = \frac{1}{20}$ in each case.

In order to understand the appearance of this power law, let us consider the time-averaged behavior of the model. The statistically stationary state arises as a competition between the two processes comprising the dynamics: the stresses which tend to remove lower thresholds from the distribution and thus shift the weight of the threshold distribution to higher values of x, and the aging, which tends to move weight back down again. The result is that the average threshold distribution $p_{\text{thresh}}(x)$ is a highly nonhomogeneous, monotonic increasing function of x which, for small σ, tends to have a plateau as x approaches unity (see Fig. 2). By balancing the two competing processes, we can calculate $p_{\text{thresh}}(x)$ and hence the avalanche distribution. For concreteness, we perform the calculation here for the exponentially distributed stresses of Eq. (2).

The probability of an agent possessing a threshold x lying below the stress level η at any given time step (and hence of it moving during this time step) is

$$P_{\text{move}}(x)=\int_x^\infty p_{\text{stress}}(\eta)\,d\eta=e^{-x/\sigma}. \tag{4}$$

The total time-averaged rate at which agents move in the interval between x and $x+dx$ is then

$$P_{\text{move}}(x)p_{\text{thresh}}(x)dx+fp_{\text{thresh}}(x)dx=Wdx, \tag{5}$$

where the x-independent constant W on the right-hand side is the time-averaged rate at which probability is added to p_{thresh}. Rearranging we have

$$p_{\text{thresh}}(x)=\frac{W}{f+e^{-x/\sigma}}. \tag{6}$$

The constant is easily fixed by requiring that $p_{\text{thresh}}(x)$ integrate to unity, giving

$$W=\frac{f}{\sigma}\left[\log\frac{fe^{1/\sigma}+1}{f+1}\right]^{-1}. \tag{7}$$

For small f and σ, $p_{\text{thresh}}(x)$ rises exponentially near $x=0$ and then levels off in a plateau around $x=-\sigma\log f$. Physically, this arises because agents possessing thresholds above this point are affected only by the aging process, which treats them all equally. Below this level, the stress process is important too, and it preferentially moves those with lower thresholds.

The avalanche size distribution is given by

$$p_{\text{aval}}(s)=\int_0^\infty p(s|\eta)p_{\text{stress}}(\eta)\,d\eta. \tag{8}$$

The probability $p(s|\eta)$ of getting an avalanche of a certain size given a certain stress level depends on the distribution of thresholds, which will in general vary from one time step to another. However, if we make the ''time-averaged approximation'' (TAA) whereby one assumes that at each time step the threshold distribution can be approximated by its time-averaged value, then $p(s|\eta)=\delta(s(\eta)-s)$ where $s(\eta)$ is just

$$s(\eta)=\int_0^\eta p_{\text{thresh}}(x)\,dx. \tag{9}$$

The avalanche size distribution then becomes

$$p_{\text{aval}}(s)=\int_0^\infty \delta(s(\eta)-s)p_{\text{stress}}(\eta)\,d\eta=\frac{p_{\text{stress}}(\eta(s))}{p_{\text{thresh}}(\eta(s))}$$
$$=\frac{1}{W\sigma}e^{-\eta(s)/\sigma}(f+e^{-\eta(s)/\sigma}), \tag{10}$$

where we have used Eqs. (6) and (9). We can calculate the stress level $\eta(s)$ corresponding to an avalanche of size s from the same two equations, which give

$$s=\left[\log\frac{1+fe^{\eta/\sigma}}{1+f}\right]\Bigg/\left[\log\frac{1+fe^{1/\sigma}}{1+f}\right]\approx\sigma\log(1+fe^{\eta/\sigma})-\sigma f \tag{11}$$

for $e^{-1/\sigma}\ll f\ll 1$ and $\sigma\ll 1$. We can now distinguish a number of different regimes. For small avalanches, such that $s\ll\sigma$, the logarithm on the right-hand side can be expanded, giving $s+\sigma f\approx\sigma fe^{\eta/\sigma}$. Substituting into Eq. (10)

$$p_{\text{aval}}(s)\propto[s+\sigma f]^{-2}\quad\text{for } s\ll\sigma. \tag{12}$$

This gives a flat avalanche distribution for small s up to about $s=\sigma f$, and then a power-law distribution for larger s with exponent $\tau=2$. The approximation breaks down when $s\approx\sigma$, giving way to a regime in which $e^{\eta/\sigma}\sim e^s$, and hence the avalanche distribution falls off exponentially with s. The various regimes can clearly be seen in the numerical results presented in Fig. 1, and the predicted crossover points between them agree well with the theory.

When f decreases below $e^{-1/\sigma}$, the approximations in Eq. (11) break down and instead it becomes valid to write $e^{\eta/\sigma}\approx 1+se^{1/\sigma}$. In this regime the theory predicts a breakdown in the scaling, a phenomenon which is also seen in the simulations. Thus the aging process, whose scale is set by f, must be small but necessarily nonzero if we are to see power-law behavior in the avalanche distribution. Notice, however, that at precisely $f=0$ the theory predicts a return to $\tau=2$ scaling, which is not seen in the simulations, implying that the TAA breaks down in this regime because the distribution $p(s|\eta)$ becomes too broad to be well approximated by a δ function.

The physical principle behind the appearance of a power-law distribution here is the interdependence of the avalanche and threshold distributions; the avalanche distribution is a function of the particular distribution of thresholds at any time, but the threshold distribution is itself produced by the action of the avalanches.

III. CONNECTION WITH OTHER MODELS

There are clear similarities between our model and SOC models. SOC models with extremal dynamics [3,8] have agents which possess thresholds for withstanding stress in a way similar to the model described here. Furthermore, our model has a source term, the aging or reloading fraction f of agents which at each time step lose memory of their previously assigned thresholds. This source term is similar in effect to the addition of the single grains of sand in sand pile models [1]. There are however also some crucial differences between our model and the SOC models. First, the stresses in our model act coherently, rather than on one site at a time, and second, the agents are, at least in the simplest versions of the model, entirely noninteracting. In SOC models, it is the interactions which give rise to avalanches. In our model on the other hand the avalanches of simultaneously moving agents arise because all the agents feel the same externally imposed stresses. There is no causal connection between the events which comprise an avalanche; each agent moves independently of the others.

Unlike other model systems for large scale fluctuations, such as the Burridge-Knopoff (BK) model [18] and the recycled version of the democratic fiber bundle model (DFBM) [19], the model presented here does not make a distinction between small, finite-sized events, and large ones whose size scales like the size of the system. In the BK model, for instance, the spectrum of event sizes contains two separate parts, one composed of small events which scales as s^{-2}, and another composed of the big events, which occur quasiperiodically. The BK and DFBM models are not statistically stationary, by contrast with our model whose dynamics rapidly reaches a statistically stationary state. Models such as BK and DFBM also show ''foreshock'' events in which large avalanches are preceded by smaller ones. Our dynamics does not have foreshocks but does display aftershock events, a phenomenon which we discuss in greater detail in the next section.

IV. DISCUSSION

We would like to examine the potential relationship of our model to processes occurring in real physical systems. First we consider earthquakes. To begin with, we ignore spatial correlations and consider the variables x_i to be thresholds for movement at various points along a fault. The coherent stress η is provided by long-wavelength background noise from some external source, such as other distant tremors, or movements in the deeper regions of the earth, and the aging f is due to slow plastic deformation from tectonic movements of the crust. As we have seen, these elements alone lead directly to a power-law distribution of earthquake sizes very close to the observed Gutenberg-Richter law, without needing to invoke interactions between neighboring parts of the fault. That is not to say that such interactions do not exist, only that they are not necessary to produce the observed power law. (Kagan [20] has presented evidence of a fractal pattern in the spatial distribution of earthquake activity, which is an indication that interactions are a feature of the dynamics. This however need not lead us to conclude that these interactions are necessary for producing the observed distribution of events.)

An interesting feature of our model is that it shows clear aftershocks. The mechanism for these is straightforward. When a large avalanche occurs the thresholds of all the agents involved are assigned new, uniformly distributed random values. Among other things, this has the effect of increasing the number which have thresholds close to zero, such thresholds being rare under normal circumstances (see Fig. 2). The result is that subsequent stresses on the system have a larger-than-normal effect, and we see an amplification of the usual level of ''background'' avalanches in the aftermath of a particularly large event. In Fig. 3, we show a section of a time series of avalanches from one of our simulations, which clearly displays this aftershock effect. Notice that if we apply the argument iteratively, we would also expect to see sequences of ''after aftershocks'' following each of the aftershocks, a behavior which is indeed evident in Fig. 3.

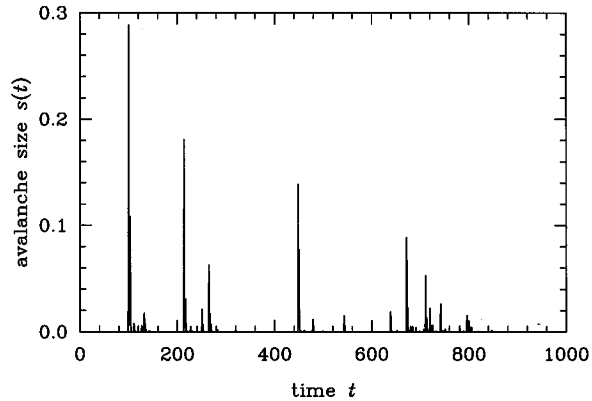

FIG. 3. Time series plot of avalanche sizes during a portion of a simulation showing clear aftershocks.

We have also measured the average probability of getting an event of a certain size $s \geqslant s_1$ a certain time t after a large event $s \geqslant s_2 > s_1$, and found that for small times its distribution goes approximately as t^{-1} (Fig. 4), regardless of the thresholds s_1, s_2 chosen to define these large events. A similar result is seen in the data from real earthquakes, and is commonly referred to as Omori's law [21]. The t^{-1} distribution can be understood as follows. Suppose a large avalanche occurs, redistributing a fraction s of the agents in the system uniformly across the allowed interval of threshold values ($0 \leqslant x < 1$ in this case). A subsequent stress of magnitude η_1 will remove all those agents with $x < \eta_1$, and produce an aftershock event of size $\eta_1 s$. In order to get another significant aftershock we now need a stress $\eta_2 > \eta_1$ in order to reach those agents which were not affected by the first aftershock. In general, if it took a time t_1 to get the first stress, then on average it will take as long again to get another of the same size or larger, or a total time of $t_2 = 2t_1$ until a stress of size $\eta_2 > \eta_1$ comes along [and produces an event of

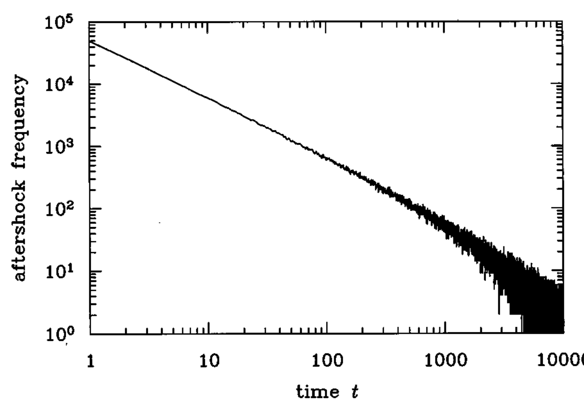

FIG. 4. Histogram of the time distribution of aftershocks following a major avalanche. The histogram follows a power law with an exponent close to 1 (Omori's law).

size approximately $(\eta_2-\eta_1)s$]. Repeating the argument, it will take as long again, or a total time of $t_3=2t_2=4t_1$ to get the third aftershock, and so forth. Thus the average time it takes to produce n aftershocks is

$$t=2^n t_1. \tag{13}$$

The number of aftershocks occurring in any interval dt following the initial large event is therefore

$$dn=\frac{dt}{n2^{n-1}t_1}\sim\frac{dt}{t}. \tag{14}$$

In other words, the exponential increase in the time intervals between aftershock events implies a histogram with a t^{-1} power-law form, regardless of the precise distribution of stresses applied to the system.

Note that the mechanism proposed here is by no means the only way to obtain aftershocks. One alternative mechanism has been proposed by Nakanishi [22] using a Burridge-Knopoff-like model in which relaxation processes are introduced by considering the geometry of stress redistribution following large quakes. As with the BK model, Nakanishi's model has quasiperiodic dynamics, but unlike the BK model it displays aftershocks, whereas the BK model displays foreshocks. Also of interest is the work of Ito and Matsuzaki [23] in which behavior approximating Omori's law is obtained in a sandpile model by introducing an extra step into the dynamics in which thresholds are randomly reshuffled after each avalanche has taken place. This has the effect of perturbing the system away from its normal critical attractor and aftershocks are generated as it evolves back to the steady state. For further discussions of aftershocks and Omori's law, see Ref. [24].

Next, let us compare our model with the results of recent studies of one-dimensional rice piles by Frette *et al.* [10]. In these studies the experimenters found a frequency distribution of avalanche sizes s which was flat up to a certain fraction of the total size of the pile, and then fell off as a power of s for larger avalanches according to Eq. (1), with a measured exponent of $\tau=2.1\pm0.1$. Although the exponent measured in these experiments describes the energy released in the avalanche, whereas our coherent noise scenario deals with the number of sites taking part, it is striking that the statistics of our simulations (Fig. 1) also display a flat distribution of avalanches up to a certain fraction $\sim\sigma f$ of the total system size, and then a power-law fall in avalanche frequency for larger sizes with an exponent close to 2. A possible interpretation of the experimental data then is that the dynamics of the rice pile is one of avalanches produced by the interplay of reloading with coherent stress. The reloading f could arise as a result of newly added grains of rice, which tend to randomize the thresholds for grains on the surface, and the stresses might come from the tumbling of new grains as they are added to the pile. The plateau in the avalanche distribution for small sizes s is then caused by rice grains which tumble past a number of sites before coming to rest, but have only enough energy to disturb the most unstable of those sites, and the larger events which form the $\tau=2$ power law are the result of occasional larger stresses in the tail of the distribution. The one-dimensional nature of the system ensures that all input disturbances propagate through a large portion of the system, and thus may be treated as coherent. In a two-dimensional system this would not be the case, and the pile might well show entirely different dynamics, either possessing a shallower power-law distribution $\tau<2$, indicating perhaps that a true SOC dynamics is at work, or not possessing a power-law distribution at all, indicating that coherent driving forces are the only mechanism responsible for power laws in this system.

Our model also makes quantitative predictions about the scaling of the line between the two regimes in the avalanche distribution: the position of the line should go like $N\sigma f$, the factor of the system size N appearing when we shift from measuring avalanches as fractions of the system size to measuring the total energy they release. Scaling of precisely this form with N is indeed seen in the experiments. Frette *et al.* also mention that simple scaling disappears when the experiment is repeated with ''rounder'' rice. We can explain this result in terms of the narrower distribution of thresholds that round rice can support, which corresponds to larger values of both σ and f.

The results of these experiments have also been modeled by Christensen *et al.* [25] and by Amarel and Lauritsen [26] using a SOC model with interacting elements. There are aspects of the dynamics captured by their model which are missing from ours, particularly geometrical effects concerned with the spatial distributions of avalanches and the corresponding transport properties of rice in the pile [27]. However, because the exponent τ is greater than 2, making $\langle s\rangle$ independent of system size, we can expect these properties to be independent of the largest avalanche events (though, on the other hand, they should now depend strongly on the position and nature of the crossover between the two regimes of the avalanche distribution). We suggest that the reverse is also true, i.e., that the observed large avalanches could appear even in the absence of long-range spatial correlations.

One characteristic which does seem to distinguish our model from the SOC alternatives is the existence of aftershock events. It might therefore might be profitable to investigate the existence of aftershock avalanches in the experimental data, in order to make a quantitative distinction between the two classes of dynamics.

Finally we would like to point out that models of the type introduced here do not constrain τ to values close to 2. Although the values found with the simple version of the model outlined in Sec. II all lie approximately in the range $1.8<\tau<2.4$, we have investigated other variants on the model which produce values outside this range. One particularly interesting version is one in which we allow for the possibility of there being many different kinds of stress on an agent. We suppose that agent i is subject to M independent types of stress, and that it has a separate threshold for yielding to each one, making our threshold parameter an M-dimensional vector quantity $\mathbf{x}_i$. One then assumes that all M components of $\mathbf{x}_i$ are to be replaced with new values every time any one of the types of stress exceeds the corresponding threshold value. In the limit $M=1$, this model is clearly just the same as the version discussed above, and for higher values of M we continue to see a power-law distribution of avalanche sizes, regardless of the nature of the applied stresses. However, the exponent of the power law becomes steeper as the value of M increases, and appears to approach 3 as M becomes large. (We have investigated the model numerically up to $M=50$.) It is interesting to note that stock market fluctuations show power-law fluctuation distributions with exponents close to $\tau=3$ [28]. One may speculate whether these so-called ''fat tails'' in the distribution are the natural response to the action of external stresses on the market (of which there are indeed many).

V. CONCLUSION

To summarize, we have demonstrated that coherent noise in large systems can give rise to intermittent behavior with an ''avalanche'' type dynamics characterized by a power-law distribution of avalanche sizes with exponent in the vicinity of $\tau=2$. This value is similar to that seen in a number of real systems, including rice piles and earthquakes, suggesting that these systems may in fact be driven by external noise, rather than self-organizing under the influence of short-range internal interactions. If one allows more elaborate types of stress on the system one can obtain power laws with exponents as high as $\tau=3$.

We believe that the study of systems driven in this fashion by coherent external noise may offer alternative interpretations of intermittent dynamics in a variety of extended nonequilibrium systems in terms of a direct interplay between small scale structures and long-wavelength fluctuations in the system. Such systems might include not only the rice piles and earthquakes considered here, but possibly also extended chaotic systems such as economies [28] and turbulence [29].

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[17] In fact it could be argued that $\tau=2$ is the upper ''critical'' value for the exponent governing the event size distribution, in the case of a dynamics in which each agent has only a limited memory and there is some randomness in the driving force on the system. To see this, consider a simple 1D system. We observe that if $\tau>2$, then the mean avalanche size is finite and independent of system size. Consequently, on large scales consecutive small avalanches behave like annealed noise in the system; the only source of long-range spatial order on a length scale L comes from large avalanches of size L or greater. The probability of such an avalanche occurring is on the order of $1/L^{\tau-1}$, and, since it affects L sites, its weighted contribution to ordering on length scale L is $L/L^{\tau-1}$. This average organization effect should be compared with the disorganization caused by the annealed noise from nonoverlapping smaller avalanches appearing with frequency of order 1. Criticality demands that organization should be larger than disorganization for $L\rightarrow\infty$, implying that $\tau<2$.

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