Temporal correlations in a one-dimensional sandpile model

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We investigate numerically temporal correlations in a one-dimensional critical-slope sandpile model with rules that on average conserve the number of particles. Our work is motivated by the existence of two well-separated time scales in self-organized sandpile models, one related to the spreading of avalanches and the other imposed by the external driving. We assume that avalanches are instantaneous events on the time scale imposed by the external deposition and study the autocorrelation function of the series of successive avalanche amplitudes. We find that the autocorrelation function has a log-normal form and for large system sizes tends to a constant, implying that the temporal correlations become stronger in the limit of large system size. We independently test this result by calculating the power spectrum of the series of successive avalanche lifetimes and sizes. For large system sizes L there is a frequency regime where the power spectrum tends to a $1/f$ type of noise, in agreement with the tendency of the autocorrelation function to approach a constant in large systems. [S1063-651X(96)07212-1]

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I. INTRODUCTION

The concept of self-organized criticality (SOC) has been introduced by Bak, Tang, and Wiesenfeld [1] in order to describe the tendency of complex dynamical systems to evolve into a critical state without fine tuning external parameters. Here the critical state refers to the absence of intrinsic length and time scales that reflects itself in the power-law distributions of relevant quantities. Bak, Tang, and Wiesenfeld [1] have illustrated the basic ingredients of SOC on a simple cellular automaton model, i.e., the sandpile model, defined on a discrete lattice in d dimensions, where each site is characterized by a scalar variable that represents a height. At each simulation step, the height of a randomly chosen site is increased by a fixed amount. Whenever the height exceeds a predefined threshold value the site relaxes, i.e., the particles are distributed to the nearest neighbors, according to rules that locally conserve the number of particles. In this manner, events (avalanches) of various sizes and durations occur. Except for the case of $d=1$, the model shows nontrivial SOC behavior with power-law distributions of avalanche amplitudes.

The basic motivation for introducing SOC was to explain the ubiquitously occurring long-range temporal correlations with a $1/f$-type power spectrum, which is characteristic of a variety of dynamical processes such as resistance fluctuations, the flow of sand in an hourglass, luminosity of stars [2], the dynamics of traffic and the stock market [3], and the rate of the human heartbeat [4,5]. Later analytical and numerical studies [6–8] on the sandpile model of Bak, Tang, and Wiesenfeld revealed, however, that the power spectrum $S(f)$ of the temporal activity scales as $1/f^2$, which corresponds to correlations characteristic of random walk-type processes, i.e., with no long-range correlations.

Different dynamical rules have been investigated that yield a variety of SOC universality classes. Kadanoff *et al.* [9] introduced a series of such models by modifying the threshold condition and local rules of relaxation. Among these models is a one-dimensional local-limited (LL) model with the local slope rules that trigger the avalanche dynamics. The LL model shows an unexpectedly complicated behavior [10], in contrast to the Bak-Tang-Wiesenfeld (BTW) model, which is trivial in $d=1$.

The sandpile models are originally defined in the limit of slow driving, meaning that each avalanche ends before an external perturbation starts a new one. Hwa and Kardar [11] generalized the LL model [9] in $d=1$ to a "running sandpile," which allows for different rates of the external driving. In the slow-driving regime where avalanches are separate events, one recovers a $1/f^2$-type of noise in the power spectrum of the sand flow, in agreement with the temporal behavior within the BTW critical-height model. In an intermediate-driving regime, where avalanches overlap, the power spectrum is found to be $1/f$ on a frequency interval that increases with the lattice size. In a fast-driving regime, the running sandpile exhibits system-wide discharge events that are anticorrelated in time. Thus the running sandpile reveals a rich temporal behavior, including long-range temporal correlations, but only in the regime where avalanches cannot be defined due to their mutual overlapping.

In the present paper we study the one-dimensional LL model defined by Kadanoff *et al.* [9] and simulate a series of successive avalanches. We investigate the temporal behavior of this series by calculating the power spectrum in a way similar to that in Ref. [11], as well as the autocorrelation function. However, we are not interested in the sandpile activity on the microscopic time scale, i.e., the time scale of the spreading of the avalanche. Instead, we study the dynamics of the sandpile model on the time scale imposed by the external driving. Our idea is based on the fact that in sandpile models there are always two time scales, one related to the evolution of an avalanche and the other one imposed by the external driving. Moreover, it is believed that SOC behavior is possible only in the limit of slow driving, which is when the two time scales are well separated [12]. On the time scale of the external driving that we are investigating, avalanches are instantaneous events no matter how large the lifetime or the size of an avalanche. As a measure of the activity in one time step one can take either the lifetime of the avalanche or the number of different sites involved in the avalanche, i.e. the size of the avalanche, or some other quantity related to the avalanche amplitude.

In Sec. II we briefly describe the model and its basic ingredients. We also present numerical results for the avalanche lifetime and size distributions, which we show to exhibit multifractal scaling, consistent with earlier results on related quantities [9]. In Sec. III we present numerical results for the autocorrelation function and power spectrum, corresponding to the series of successive avalanche lifetimes and sizes. Our results show that the form of the correlation function is log-normal, tending to a constant for large system sizes, meaning that the temporal correlations become stronger in larger systems.

II. THE LOCAL-LIMITED MODEL

The one-dimensional LL model developed by Kadanoff *et al.* [9] is a non-Abelian sandpile model with evolution rules that depend on the local slope. An avalanche starts whenever the local slope increases beyond a preset threshold value. The boundary conditions consist of one reflecting and one absorbing wall, and produce an average flux of particles from the reflecting end to the absorbing end. The lack of translational invariance and the existence of the ''trapping sites'' make this model complicated. It has been shown that the multifractal scaling of the distribution of relaxation events and drop sizes might be more appropriate than finite-size scaling [9]. A scaling theory based on two diverging length scales has also been developed [10].

We consider the LL model defined on a one-dimensional lattice of length L. Each lattice site i is associated with a local slope σ_i, defined as the difference in height between two neighboring sites, $\sigma_i \equiv h_i - h_{i+1}$, where the variable h_i represents the height of the sandpile at the site i. One updating step in the simulation consists of two parts: (a) a site is chosen randomly and a particle is added to it and (b) a relaxing procedure is applied for sites with local slope larger than some threshold value σ_c until all the slopes in the system are smaller than or equal to σ_c. Adding a unit sand to site i increases σ_i and decreases σ_{i-1}. During the relaxation, n_f particles are transferred from site k to the neighboring site $k+1$, so that the local slope is ''distributed'' to nearest neighbors according to the rules

$$\sigma_i \rightarrow \sigma_i - 2n_f, \quad \sigma_{i+1} \rightarrow \sigma_{i+1} + n_f, \quad \sigma_{i-1} \rightarrow \sigma_{i-1} + n_f, \tag{1}$$

where $n_f \geqslant 2$ is needed in order to obtain nontrivial behavior [9]. The above rules can easily be translated into rules for local heights h_i by taking into account the local slope definition, $\sigma_i \equiv h_i - h_{i+1}$ and the boundary conditions $h_0 = h_1$ (reflecting wall) and $h_{L+1} = 0$ (absorbing wall). There are several conservation laws and sum rules that are associated with the LL model [9]. One of these laws is the conservation of mass (the total number of particles) which is found also in the critical-height sandpile models, such as the BTW model. The total mass of the sandpile fluctuates around a constant average value in the steady state, with the input flow of particles compensated by the output flow through the absorbing wall.

In our simulation, we start the sandpile dynamics with a random distribution of slopes, corresponding to an ''overloaded'' system, i.e., the initial mass is much larger compared to the mean stationary mass. We allow the system to relax and reach the stationary state, then we record successive avalanche lifetimes T and successive avalanche sizes S (the number of distinct sites involved in an avalanche). The two time series, avalanche lifetimes $T(t)$ and avalanche sizes $S(t)$, are then analyzed in two ways: (a) calculating the correlation function, defined as

$$C_A(\tau) \equiv \langle A(t) A(t+\tau) \rangle - \langle A(t) \rangle^2, \tag{2}$$

where $A(t)$ is either the avalanche lifetime $T(t)$ or size $S(t)$ at a time step t, and (b) using a Fourier transform to calculate the corresponding power spectrum $P_A(\Omega)$ as a function of the frequency Ω,

$$P_A(\Omega) = |\mathcal{F}[A(t)]|^2, \tag{3}$$

where $\mathcal{F}$ denotes a Fourier transform. According to the Wiener-Khinchin theorem [13], a power spectrum $P_A(\Omega)$ of a given time signal $A(t)$ is a Fourier transform of the autocorrelation function $C_A(\tau) = \langle A(t) A(t+\tau) \rangle$. The power spectra $P_T(\Omega)$ and $P_S(\Omega)$, if calculated directly from the time series, provide independent quantifications of temporal correlations.

III. RESULTS

We find that the distributions of avalanche lifetimes $D(T)$ and sizes $D(S)$ scale in a multifractal way in agreement with the scaling of drop sizes and general behavior of the model [9]. Figures 1(a) and 1(b) show multifractal scaling functions for avalanche amplitudes A, i.e., lifetimes and sizes, respectively, as found by rescaling the calculated distribution functions in a double logarithmic plot

$$f_A(\alpha) = \frac{\log_{10} D(A(\alpha))}{\log_{10}(L/L_0)}, \tag{4}$$

where $\alpha = \log_{10}(A/A_0)/\log_{10}(L/L_0)$ is an independent variable and A_0 and L_0 are the best fit parameters to the scaling. Figures 1(a) and 1(b) show rescaled distribution functions obtained for various system sizes. The two scaling functions $f_T(\alpha)$ and $f_S(\alpha)$ are fit to a cubic form (solid lines in Fig. 1) to obtain approximate analytical expressions for the scaling functions.

In accordance with the multifractal scaling of distribution functions, the moments $\langle A^q \rangle$ can be expressed as integrals weighted by the distribution function. Using a saddle-point approximation to the integrals we find that the moments $\langle A^q \rangle$ scale with L as

$$\langle A^q \rangle \sim \log_{10}^{1/2}(L/L_0) L^{\beta_A(q)}. \tag{5}$$

Here $\beta_A(q) = (1+q)\alpha_0(q) + f_A(\alpha_0(q))$ and $\alpha_0(q)$ represents the maximal value of a function $(1+q)\alpha + f_A(\alpha)$ with respect to an independent variable α for a given moment

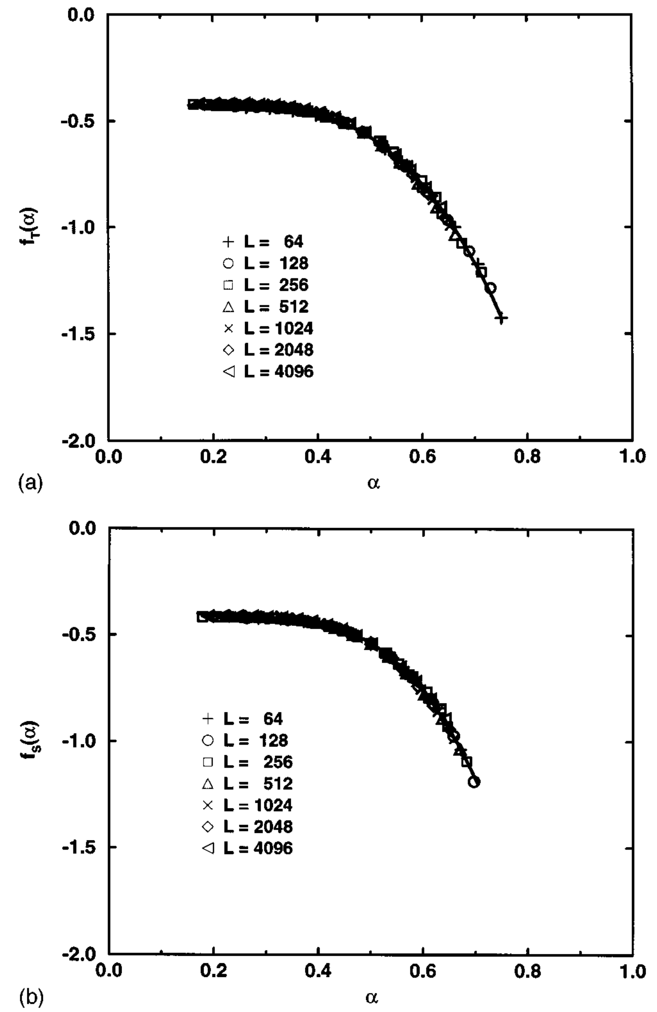

FIG. 1. Multifractal scaling functions (a) $f_T(\alpha)$ and (b) $f_S(\alpha)$ for the avalanche lifetimes $D(T)$ and the number of different sites participating in the avalanche $D(S)$, respectively. System sizes are $L=64$, 128, 256, 512, 1024, and 2048. The numerical data are binned logarithmically with a binning parameter $2^{1/2}$. The parameters from the fits are $L_0=0.025\pm0.002$ and $T_0=0.25\pm0.02$ for the case of avalanche lifetimes and $L_0=0.020\pm0.002$ and $S_0=0.20\pm0.02$ for the case of avalanche sizes. The solid lines are the fits to a cubic form, i.e., $f(\alpha)=a_0+a_1x+a_2x^2+a_3x^3$. The parameters of the fits are $a_0=-0.39$, $a_1=-0.67$, $a_2=3.64$, and $a_3=-6.09$ (lifetimes) and $a_0=-0.20$, $a_1=-2.28$, $a_2=8.00$, and $a_3=-9.62$ (sizes).

q. The result, Eq. (5), is a power law with a logarithmic correction. We tested this analytical scaling form for the moments $\langle T^q\rangle$ and $\langle S^q\rangle$ as a function of the system size L for $q=1/2$, $q=1$, and $q=2$. The results are plotted in Fig. 2 and the corresponding exponents $\beta(q)$ are given in Table I.

The distributions themselves do not provide information about correlations between successive avalanche amplitudes. The information on temporal correlations can be extracted from the correlation functions $C_T(t)$ and $C_S(t)$ and/or from the power spectra $P_T(\Omega)$ and $P_S(\Omega)$. The calculated correlation functions are depicted in Fig. 3 for different system sizes. The solid lines correspond to fits to the log-normal form

$$C_A(\tau)=C_A(T_A)\exp[-\gamma_A\log_{10}^2(\tau/T_A)], \qquad (6)$$

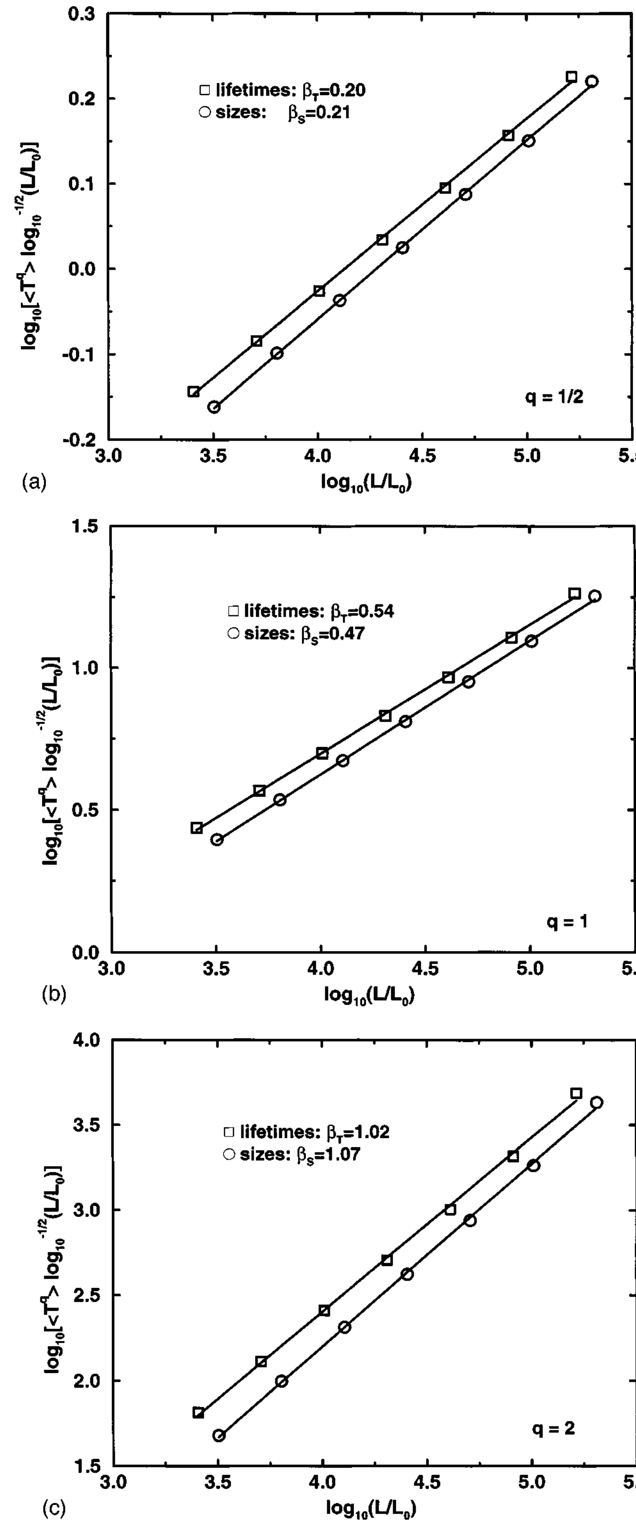

FIG. 2. Scaling of the moments of the lifetimes $\langle T^q\rangle$ and sizes $\langle S^q\rangle$ for (a) $q=1/2$, (b) $q=1$, and (c) $q=2$, calculated from the numerically obtained distribution functions. The results are fit to a power law with a logarithmic correction in accordance with the multifractal scaling. The corresponding scaling exponents are presented in Table I.

where the coefficient γ_A is expected to scale with the system size L, while the characteristic time T_A is expected to be independent of the system size. The scaling of γ_T and γ_S with the system size L is presented in Fig. 4. Both coeffi-

TABLE I. Summary of exponents $\beta_A(q)$ defined in Eq. (5).

q	β_T	β_S
1/2	0.20 ± 0.02	0.21 ± 0.02
1	0.45 ± 0.02	0.47 ± 0.02
2	1.02 ± 0.02	1.07 ± 0.02.

cients decrease with the system size L roughly as $L^{-1/2}$. Thus, for large L the correlations become stronger, i.e., $C_A(\tau)$ tends to a constant, independent of τ.

To determine the type of temporal correlations in another way, we calculate also the power spectra of the two time series $T(t)$ and $S(t)$. The results are presented in Fig. 5. The structure of both calculated power spectra is as follows: (i) in the large frequency regime, which corresponds to short time lags, the power spectra seem to be ''white,'' indicating an absence of any correlations between successive avalanches; (ii) the white spectrum crosses over to a $1/f$-type noise in an

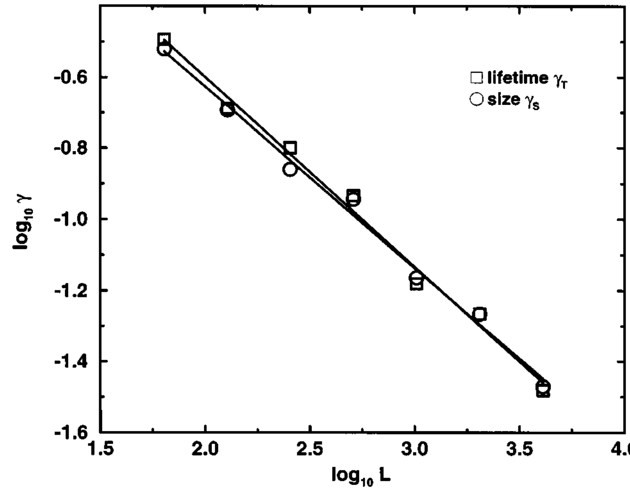

FIG. 4. Dependence of the parameters γ_T and γ_S from the log-normal fit given by Eq. (6) on L. The slopes are $\gamma_T=-0.53\pm0.03$ and $\gamma_S=-0.51\pm0.03$.

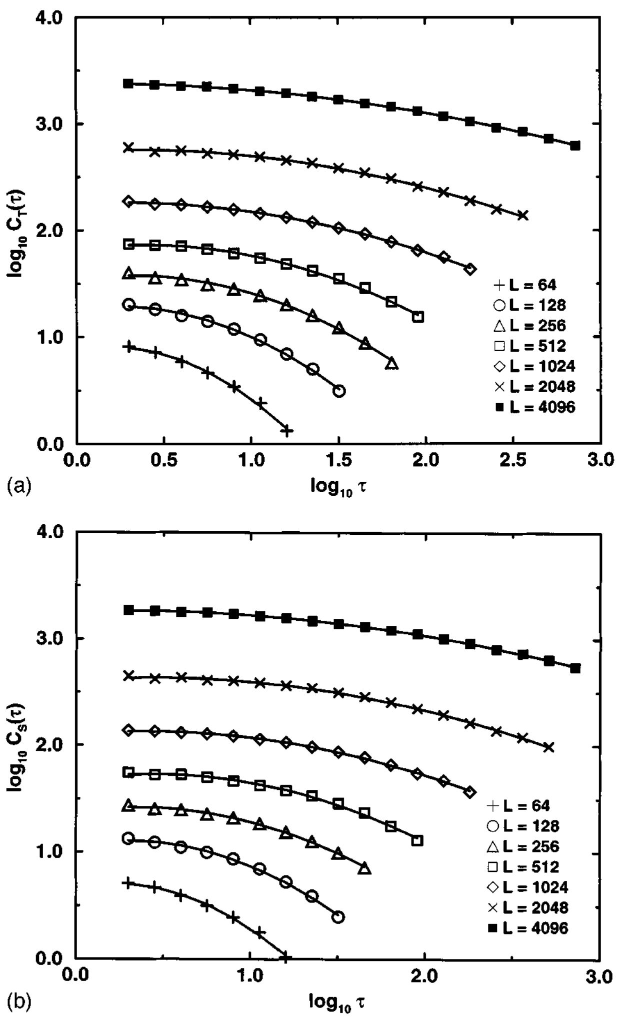

FIG. 3. Autocorrelation functions for the time series of (a) avalanche lifetimes $C_T(\tau)$ and (b) sizes $C_S(\tau)$ calculated for system sizes $L=64$, 128, 256, 512, 1024, 2048, and 4096. The data are binned with the binning parameter $2^{1/2}$. Solid lines are log-normal fits with the parameters γ_T and γ_S that scale with L as shown in Fig. 4.

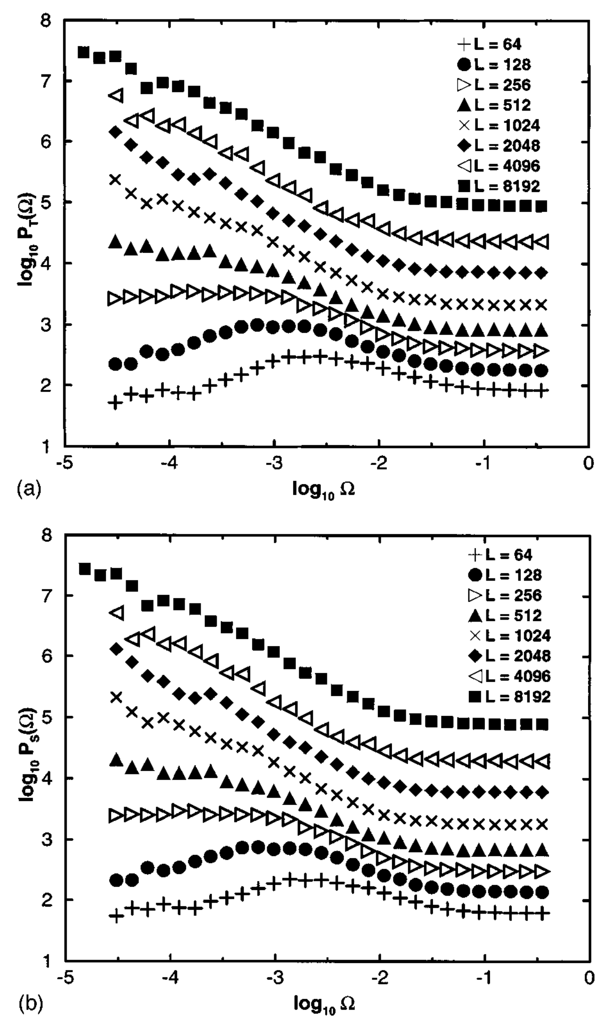

FIG. 5. Power spectra of series of avalanche lifetimes and sizes (a) $P_T(\Omega)$ and (b) $P_S(\Omega)$, calculated numerically for system sizes $L=64$, 128, . . . , 8192. The data are binned with the binning parameter $2^{1/2}$.

intermediate frequency regime; (iii) for relatively small system sizes L, there is another regime of the spectrum at low frequencies, where the spectrum indicates the presence of temporal anticorrelations. Feature (iii) seems not to be present in the power spectra that correspond to large system sizes. However, this may be due to insufficient maximum observation time t_{max}, which in our simulations is $t_{max}=2^{18}$. The structure of the spectra described above is consistent with a log-normal form of the correlation functions. We verified, using the Wiener-Khinchin theorem, that this specific form of the power spectrum is obtained by a Fourier transform of the log-normal functional dependence of the correlation function, given by Eq. (6).

IV. SUMMARY

In summary, we study temporal behavior of a one-dimensional sandpile model. In our numerical analysis we assume two well-separated time scales, one associated with the microscopic avalanche dynamics and the other imposed by the external driving. On the time scale of the external driving, avalanches occur instantaneously. At each time step we quantify the avalanche by its duration (lifetime) and its spatial extention (size). We investigate temporal correlations in the series of successive avalanche amplitudes, i.e., lifetimes or sizes. The calculated correlation function and the power spectrum indicate the presence of nontrivial temporal correlations of the log-normal form.

The local-limited model that we study belongs to the family of directed critical slope-type models [9]. It is not a typical sandpile model, since the distributions of avalanche lifetimes and sizes are not simple power laws. Rather, the distributions of avalanche amplitudes exhibit multifractal scaling and the average avalanche amplitude scales with the system size as a power law with a logarithmic correction.

A natural question that arises from the above results is whether such temporal correlations found in the one-dimensional local-limited model are characteristic also for other sandpile-type models, such as the undirected critical-height model [1]. Another question is also whether these correlations persist in higher dimensions. Unfortunately, the local-limited model displays trivial behavior in two dimensions. To answer these questions more systematic numerical work on different types of sandpile models should be done.

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