Stability of the one-dimensional kink solution to a general Cahn-Hilliard equation

David Bettinson and George Rowlands
Department of Physics, University of Warwick, Coventry, CV4 7AL, England
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We give an analysis of the Cahn-Hilliard equation with a general potential, which admits a one-dimensional kink solution. We investigate the stability of this equilibrium solution to small perpendicular perturbations of variable wave number k. We develop a perturbation theory for small and large k and apply the general results to two commonly used forms for the potential. We go on and use a Padé approximant to describe the full dispersion relation, and for the particular potentials it is shown that the kink solution is stable for all k. [S1063-651X(96)05812-6]

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I. INTRODUCTION

Pattern formation resulting from a phase transition has been observed in alloys, glasses, polymer solutions, and binary liquid mixtures. We consider a two-component system where the phase transition is induced by rapidly decreasing the temperature (quenching) from some $T_0 > T_c$ to some $T_1 < T_c$. The continuum limit model, used to describe the dynamics of the subsequent concentration of one component, is that proposed by Cahn and Hilliard [1], namely,

$$u_t = \nabla^2 \left[\frac{dF}{du} - \nabla^2 u \right], \tag{1}$$

where the subscript denotes a partial differentiation with respect to t, while F is some general, nonlinear free energy expression that admits a stationary, one-dimensional kink solution to the equation above (a kink solution simply being any solution that describes a flip from one component to another over some finite distance). The equation arises from classical thermodynamic considerations for the interdiffusion of two components A and B. In the above equation $u = u_1$ denotes all components A and $u = u_2$ denotes all components B. For a kink solution we insist that

$$F(u_1) = F(u_2) = 0, \quad \frac{dF}{du}(u_1) = \frac{dF}{du}(u_2) = 0, \tag{2}$$

and $F(u) > 0$ for $u_1 < u < u_2$. Equation (1) has a stationary, one-dimensional kink solution $u_0(x)$ given by the solution of

$$\frac{dF}{du}(u_0) - \frac{d^2 u_0}{dx^2} = 0, \tag{3}$$

with $\lim_{x\to+\infty} u_0 = u_2$ and $\lim_{x\to-\infty} u_0 = u_1$. Also note that we can integrate Eq. (3) to obtain

$$\frac{1}{2}\left(\frac{du_0}{dx}\right)^2 = F(u_0) + K, \tag{4}$$

where K is the integration constant. In the limit $x \to +\infty$ this equation becomes

$$0 = F(u_2) + K = K, \tag{5}$$

and so Eq. (4) becomes

$$\left(\frac{du_0}{dx}\right)^2 = 2F(u_0). \tag{6}$$

To study perpendicular perturbations to this kink solution we write

$$u = u_0 + \varepsilon\, \delta u(x) e^{i(k_y y + k_z z)} e^{\gamma t}. \tag{7}$$

Inserting this into Eq. (1), we obtain the linear equation (having neglected products of δu), namely,

$$-\gamma \delta u = \left(\frac{d^2}{dx^2} - k^2\right)\left(\frac{d^2}{dx^2} - k^2 - \frac{d^2 F}{du^2}(u_0)\right)\delta u, \tag{8}$$

where $k^2 = k_y^2 + k_z^2$.

The plan of the paper is as follows. We use perturbation theory to determine the stability of the kink solution to small- and large-k perturbations in Secs. II and III, respectively. This analysis is performed for both a general and particular free energy. In Sec. IV we use a Padé approximant to derive a full dispersion relation for both the general and two particular cases. We draw conclusions in Sec. V.

II. SMALL-*k* ANALYSIS

A. General potential

We look for marginally stable modes ($\gamma = 0$), and so from Eq. (8) obtain

$$\left(\frac{d^2 F}{du^2}(u_0) - \frac{d^2}{dx^2} + k^2\right)\delta u = 0. \tag{9}$$

Differentiating Eq. (3) with respect to x gives us

$$\frac{d^2 F}{du^2}(u_0) = \frac{d^3 u_0}{dx^3} \Big/ \frac{du_0}{dx}, \tag{10}$$

and thus Eq. (9) has a solution $\delta u = du_0/dx$, when $k = 0$. We use this to find the stability of the kink solution for small k. Begin by expanding $\delta u(x)$ and γ in terms of the small parameter k,

$$\gamma=\gamma_1 k+\gamma_2 k^2+\gamma_3 k^3+\cdots,$$

$$\delta u=a_0\frac{du_0}{dx}+k\,\delta u_1+k^2\,\delta u_2+\cdots, \tag{11}$$

where a_0 is a constant coefficient.

The method used relies heavily upon obtaining the correct form of the solution as $x\to\pm\infty$. It is shown in Ref. [2], for a different equation, how knowledge of the asymptotic solution is crucial in proving δu to be bounded. Here we show by demanding that the full solution to Eq. (8) has the correct slow spatial varying, asymptotic form, that we can obtain the value of γ to at least order k^4. The asymptotic solution is discussed in Appendix A, where it is shown that as $x\to\pm\infty$

$$\delta u\to A(k)[1-k|x|+O(k^2)], \tag{12}$$

We now insert Eqs. (11) into Eq. (8) and equate orders of k. To first order in k we obtain the equation

$$\left[-\frac{d^4}{dx^4}+\frac{d^2}{dx^2}\left(\frac{d^2F}{du^2}(u_0)\right)\right]\delta u_1=\gamma_1 a_0\frac{du_0}{dx}. \tag{13}$$

Integrating this twice gives

$$\left[\frac{d^2}{dx^2}-\frac{d^2F}{du^2}(u_0)\right]\delta u_1=c_0 x+c_1-\gamma_1 a_0\int_0^x u_0(x')dx'. \tag{14}$$

Note that the $c_0 x$ term will give rise to a contribution to δu_1 that is proportional to x as $x\to\pm\infty$. Terms proportional to x^n in this limit we call algebraically secular. From Eq. (12) we find that the solution has an algebraically secular term $|x|$, as $x\to\pm\infty$. Thus we must set $c_0=0$ at this and all subsequent orders of k. The constant c_1 can be determined using the consistency condition. This is imposed by multiplying Eq. (14) by du_0/dx and integrating over all x. Since the operator $d^2/dx^2-(d^2F/du^2)(u_0)$ is self adjoint, we find that

$$c_1=\frac{\gamma_1 a_0}{(u_2-u_1)}\left\langle\frac{du_0}{dx}\int_0^x u_0(x')dx'\right\rangle, \tag{15}$$

where $\langle\ \rangle$ denotes $\int_{-\infty}^{+\infty}dx$. For convenience we define I_1 such that

$$I_1=\left\langle\frac{du_0}{dx}\int_0^x u_0(x')dx'\right\rangle, \tag{16}$$

giving $c_1=\gamma_1 a_0 I_1/(u_2-u_1)$ (see Appendix B). Now consider Eq. (14) as $x\to+\infty$ and denote the value of δu_i as $x\to+\infty$ by $\overline{\delta u}_i$; then

$$\left[\frac{d^2}{dx^2}-\beta\right]\overline{\delta u}_1=c_1-\gamma_1 a_0(u_2 x+K_1), \tag{17}$$

where we have written $\int_0^x u_0(x')dx'=K_1+u_2 x$ as $x\to+\infty$, $(d^2F/du^2)(u_2)=\beta$, and we have neglected exponentially decaying terms. From Eq. (12) we see that to lowest order in k the asymptotic solution tends to a constant. In this limit, Eq. (17) requires the solution $\overline{\delta u}_1$ to have a contribution that is proportional to $u_2 x+K_1$. Clearly this is incompatible with the asymptotic form and so we set $\gamma_1=0$. This leaves $\delta u_1=a_1(du_0/dx)$, where a_1 is some constant coefficient.

To second order in k we obtain

$$\left[-\frac{d^2}{dx^2}+\frac{d^2F}{du^2}(u_0)\right]\delta u_2=c_2+\gamma_2 a_0\int_0^x u_0(x')dx'-a_0\frac{du_0}{dx}. \tag{18}$$

Again using Eq. (12), we see that the expression $\int_0^x u_0(x')dx'$ gives the wrong asymptotic form for δu and so we must set $\gamma_2=0$. We now apply the consistency condition to Eq. (18) to obtain

$$c_2=\frac{a_0 I_2}{u_2-u_1}, \tag{19}$$

where $I_2=\langle(du_0/dx)^2\rangle$. As $x\to+\infty$, Eq. (18) becomes

$$\left[\beta-\frac{d^2}{dx^2}\right]\overline{\delta u}_2=c_2, \tag{20}$$

which has the solution $\overline{\delta u}_2=(c_2/\beta)$ (we neglect exponentially decaying solutions).

To third order in k we obtain

$$\left[-\frac{d^2}{dx^2}+\frac{d^2F}{du^2}(u_0)\right]\delta u_3=c_3+\gamma_3 a_0\int_0^x u_0(x')dx'-a_1\frac{du_0}{dx}, \tag{21}$$

and again we use the consistency condition to determine c_3, namely,

$$c_3=\frac{a_1 I_2-\gamma_3 a_0 I_1}{u_2-u_1}. \tag{22}$$

Since in the limit $x\to+\infty,\overline{\delta u}_2=\dfrac{c_2}{\beta}$, we can use Eq. (12) and require that as $x\to+\infty$,

$$\overline{\delta u}_3\propto 1+\alpha x, \tag{23}$$

where α is a constant. Knowing this, we do not set $\gamma_3=0$. As $x\to+\infty$ Eq. (21) becomes

$$\left[-\frac{d^2}{dx^2}+\beta\right]\overline{\delta u}_3=c_3+\gamma_3 a_0(K_1+u_2 x), \tag{24}$$

which has the algebraic solution

$$\overline{\delta u}_3=\frac{\gamma_3 a_0 u_2}{\beta}x+\frac{c_3+\gamma_3 a_0 K_1}{\beta}, \tag{25}$$

where again we have neglected exponentially decaying solutions. We now combine our asymptotic results to obtain

$$\overline{\delta u}=\frac{k^2 c_2}{\beta}+\frac{k^3(c_3+\gamma_3 a_0 K_1)}{\beta}+\frac{k^3\gamma_3 a_0 u_2}{\beta}x+O(k^4)$$
$$=\frac{k^2 c_2}{\beta}\left(1+\frac{k(c_3+\gamma_3 a_0 K_1)}{c_2}\right)\left(1+\frac{k\gamma_3 a_0 u_2 x}{c_2}\right)+O(k^4). \tag{26}$$

Comparing this with Eq. (A4) of Appendix A, we find

$$\gamma_3=-\frac{c_2}{a_0 u_2}=-\frac{I_2}{u_2(u_2-u_1)}. \tag{27}$$

Clearly since γ_3 is negative ($I_2>0$), to order k^3, the kink solution is stable. Note that this is identical to (2.14) in Ref. [3], obtained using the variational method.

The fourth-order equation is

$$\left[-\frac{d^2}{dx^2}+\frac{d^2F}{du^2}(u_0)\right]\delta u_4$$
$$=(\gamma_3 a_1+\gamma_4 a_0+a_0)\int_0^x u_0(x')dx'-2\,\delta u_2$$
$$+\frac{c_2}{\beta}\int_0^x\int_0^{x'}\frac{d^2F}{du^2}(u_0)dx''dx'-a_2\frac{du_0}{dx}-c_4, \tag{28}$$

where c_4 can be determined using the consistency condition. As $x\to+\infty$ Eq. (28) becomes

$$\left[-\frac{d^2}{dx^2}+\beta\right]\overline{\delta u}_4=(\gamma_3 a_1+\gamma_4 a_0+a_0)(K_1+u_2 x)-2\frac{c_2}{\beta}$$
$$+\frac{c_2}{\beta}\left(\frac{\beta x^2}{2}+K_2 x+K_3\right)-c_4, \tag{29}$$

where we have written $\int_0^x\int_0^{x'}(d^2F/du^2)(u_0)dx''dx' =\beta x^2/2+K_2 x+K_3$ in the limit $x\to+\infty$. Equation (29) has the algebraic solution

$$\overline{\delta u}_4=b_2 x^2+b_1 x+b_0, \tag{30}$$

where

$$b_0=\frac{1}{\beta}\left(K_1(\gamma_3 a_1+\gamma_4 a_0+a_0)-\frac{c_2}{\beta}-c_4+\frac{c_2 K_3}{\beta}\right), \tag{31}$$

$$b_1=\frac{1}{\beta}\left(u_2(\gamma_3 a_1+\gamma_4 a_0+a_0)+\frac{K_2 c_2}{\beta}\right), \tag{32}$$

$$b_2=\frac{c_2}{2\beta}. \tag{33}$$

We now collect the algebraic terms in our asymptotic form of δu to obtain

$$\overline{\delta u}=\frac{c_2}{\beta}k^2\left(1+k\frac{(c_3+\gamma_3 a_0 K_1)}{c_2}+k^2\frac{\beta b_0}{c_2}\right)\left[1+k\frac{a_0\gamma_3 u_2}{c_2}x\right.$$
$$\left.+k^2\left(\frac{\beta b_1 x}{c_2}-\frac{c_3 a_0\gamma_3 u_2 x}{c_2^2}-\frac{a_0^2 K_1\gamma_3^2 u_2 x}{c_2^2}+\frac{\beta b_2 x^2}{c_2}\right)\right]$$
$$+O(k^5). \tag{34}$$

Replacing γ_3 by $-c_2/a_0 u_2$ leaves

$$\overline{\delta u}=\frac{c_2}{\beta}k^2(1+\cdots)\left[1-kx+k^2\left(\frac{c_3 x}{c_2}+\frac{\beta b_1 x}{c_2}-\frac{K_1 x}{u_2}\right.\right.$$
$$\left.\left.+\frac{\beta b_2 x^2}{c_2}\right)\right]+O(k^5). \tag{35}$$

Using Eq. (A4), we equate the term above, proportional to $k^2 x$, to $-\gamma_3/2\beta$. This gives

$$\gamma_4=-1-\frac{I_1 I_2}{u_2^2(u_2-u_1)^2}+\frac{I_2^2}{2\beta u_2^2(u_2-u_1)^2}+\frac{I_2 I_3}{u_2(u_2-u_1)}, \tag{36}$$

where $I_3=K_1/u_2-K_2/\beta$. So, finally, we write

$$\gamma=\gamma_3 k^3+\gamma_4 k^4+O(k^5).$$
$$=-\frac{I_2}{u_2(u_2-u_1)}k^3-\left(1+\frac{I_1 I_2}{u_2^2(u_2-u_1)^2}-\frac{I_2^2}{2\beta u_2^2(u_2-u_1)^2}\right.$$
$$\left.-\frac{I_2 I_3}{u_2(u_2-u_1)}\right)k^4+O(k^5). \tag{37}$$

Note that to determine γ to this order in k, we do not need the full solution δu_2. Determination of c_4 (which uses the full solution δu_2) is not required, and thus we only need the asymptotic form of δu_2 in our analysis.

B. Particular potential

We now consider the particular case where the free energy is of the form

$$F(u)=\frac{1}{4}(1-u^2)^2. \tag{38}$$

This is a common approximate form for a binary system undergoing a phase transition, and then the Cahn-Hilliard equation (1) becomes

$$u_t=\nabla^2[u^3-u-\nabla^2 u]. \tag{39}$$

It is simple to see from Eq. (2) that $u_1=-1$ and $u_2=1$. Also in the limit $x\to+\infty$

$$F''(u_0)=\beta=2, \tag{40}$$

and using Appendix B we perform simple definite integrals to give

$$I_1=2\sqrt{2}(1-\ln 2),\quad I_2=\frac{2\sqrt{2}}{3},\quad I_3=\frac{1}{\sqrt{2}}(3-2\ln 2). \tag{41}$$

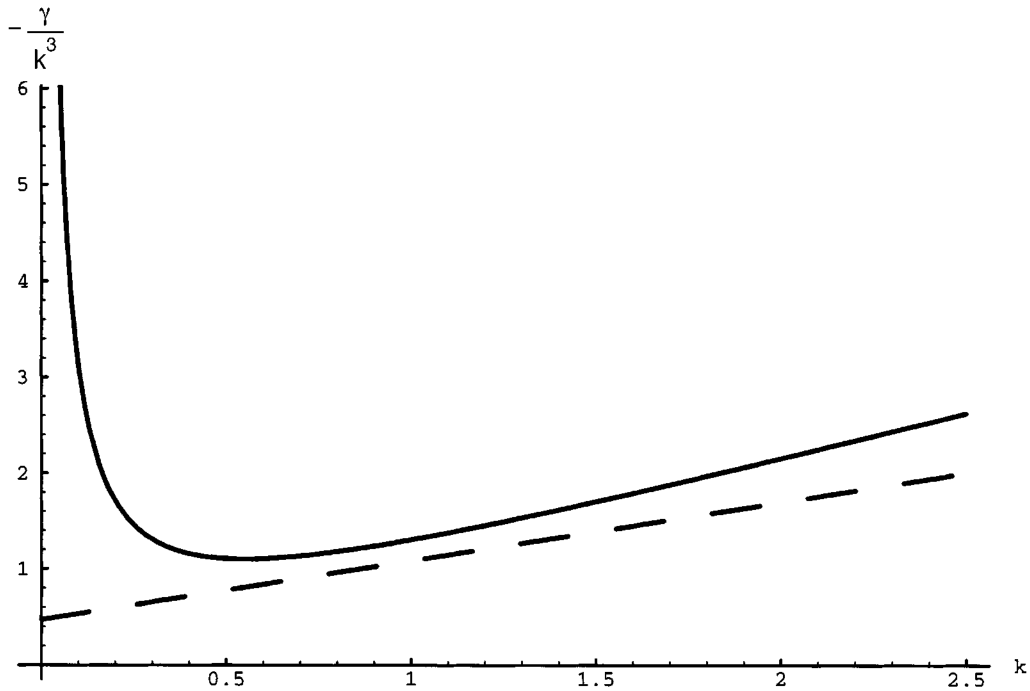

FIG. 1. Particular free energy: dashed line, small-k approximation to the growth rate; full line, large-k approximation to the growth rate.

We find for this potential that the stationary solution has a growth rate to perpendicular perturbations given by

$$\gamma=-\frac{\sqrt{2}}{3}k^3-\frac{11}{18}k^4+O(k^5). \tag{42}$$

To lowest order this agrees with that obtained using a variational method in Ref. [3]. To next order in k the result obtained in [3] is 8% greater than here. This is due to their use of the same trial eigenfunction for all orders of k.

III. LARGE-k ANALYSIS

A. General potential

In this section we consider the growth rate of perpendicular perturbations with small wavelength. Begin by dividing the linear equation (8) by k^4 to give

$$-\frac{\gamma}{k^4}\delta u=\left(\frac{1}{k^2}\frac{d^2}{dx^2}-1\right)^2\delta u-\frac{1}{k^2}\left(\frac{1}{k^2}\frac{d^2}{dx^2}-1\right)\frac{d^2F}{du^2}(u_0)\,\delta u. \tag{43}$$

Since $1/k$ is small we expand the variables as

$$\frac{\gamma}{k^4}=\gamma_a+\frac{\gamma_b}{k}+\frac{\gamma_c}{k^2}+\cdots, \tag{44}$$

$$\delta u=\delta u_a+\frac{\delta u_b}{k}+\frac{\delta u_c}{k^2}+\cdots. \tag{45}$$

The first two orders tell us that $\gamma_a=-1$ and $\gamma_b=0$. To order $1/k^2$ we obtain

$$\left(\frac{d^2}{dx^2}-\frac{1}{2}\frac{d^2F}{du^2}(u_0)\right)\delta u_a=\frac{\gamma_c\delta u_a}{2}, \tag{46}$$

which is an eigenvalue problem for γ_c. So for large k we have the following expression for the growth rate:

$$\frac{\gamma}{k^4}=-1+\frac{\gamma_c}{k^2}+\cdots. \tag{47}$$

B. Particular potential

For large k we have an expression for the growth rate given by Eq. (47). Since the stationary solution to Eq. (39) is $u_0=\tanh(x/\sqrt{2})$, γ_c is obtained by solving

$$\frac{d^2}{dx^2}\delta u_a+\left(\frac{3}{2}\operatorname{sech}^2\frac{x}{\sqrt{2}}-1-\frac{\gamma_c}{2}\right)\delta u_a=0. \tag{48}$$

Eigenvalue problems such as this have general solutions (given on p. 1651 of Ref. [4]). We find that

$$\gamma_c=\left(\frac{3-\sqrt{13}}{2}\right)\simeq-0.303 \tag{49}$$

and thus the growth rate of large-k perturbations is given by

$$\frac{\gamma}{k^4}=-1-\frac{0.303}{k^2}+\cdots. \tag{50}$$

To lowest order the kink solution is stable ($\gamma=-k^4$), which is in agreement with Ref. [3]. Figure 1 shows our two approximations for small and large k given by Eqs. (42) and (50), respectively.

IV. FULL DISPERSION RELATION

A. General case

For the particular potential given by Eq. (38), the only bounded solution to Eq. (9) with $k^2\geqslant 0$ is $\delta u=du_0/dx$

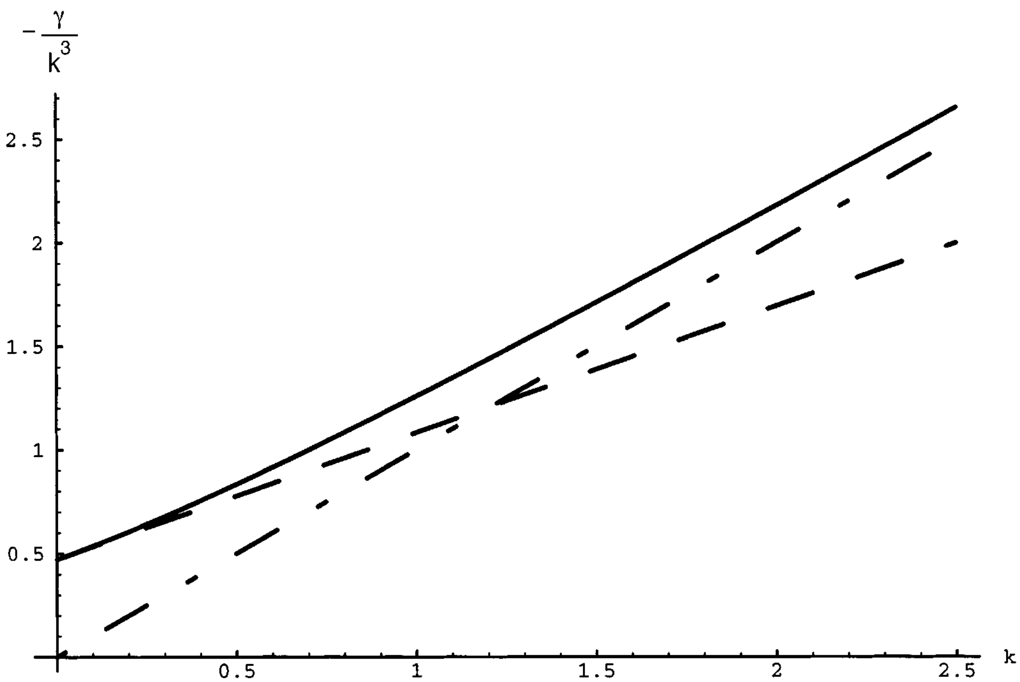

FIG. 2. Particular free energy (a): dashed line, small-k approximation to the growth rate; dot dashed line, large-k approximation to the growth rate; full line, Padé approximation to the growth rate.

$(k=0)$, and so γ does not cross the k axis for all $k>0$. We assume that this applies in general, and so combine growth rate results for small and large k. This is done using a simple Padé approximant to obtain an expression for all k. We assume a general form for the growth rate as

$$\frac{\gamma}{k^3}=a_1\left(\frac{1+a_2k+a_3k^2}{1+a_4k}\right). \tag{51}$$

For small k Eq. (51) becomes

$$\frac{\gamma}{k^3}=a_1. \tag{52}$$

A comparison of this with Eq. (37) gives us $a_1=\gamma_3$. To next order for small k we have

$$\frac{\gamma}{k^3}=\gamma_3[1+(a_2-a_4)k]. \tag{53}$$

Again, a comparison with Eq. (37) gives us $a_2=\gamma_4/\gamma_3+a_4$. We now divide Eq. (51) by k and obtain

$$\frac{\gamma}{k^4}=\gamma_3\left(\frac{\frac{1}{k^2}+\frac{a_2}{k}+a_3}{\frac{1}{k}+a_4}\right). \tag{54}$$

Thus, for large k,

$$\frac{\gamma}{k^4}=\gamma_3\frac{a_3}{a_4}, \tag{55}$$

which when compared to Eq. (47) gives us $a_3=-a_4/\gamma_3$. We go to next order in $1/k$ and find $a_4=a_3/a_2$, so that our final Padé approximant for the growth rate is

$$-\frac{\gamma}{k^3}=\frac{\gamma_3^2}{(1+\gamma_4)k-\gamma_3}+k. \tag{56}$$

B. Particular cases

We now consider the case (a), where the free energy is given by Eq. (38). It is found that $\gamma_3=-\sqrt{2}/3$ and $\gamma_4=-\frac{11}{18}$, and thus the growth rate given by Eq. (56) becomes

$$-\frac{\gamma}{k^3}=\frac{4}{6\sqrt{2}+7k}+k. \tag{57}$$

This is plotted, along with approximations for small and large k, in Fig. 2. This appears to be in good agreement with Fig. 6 of Ref. [3].

We can calculate another Padé approximation for the growth rate in this particular case. Here, instead of using the fourth-order result for small k, we use the $(1/k^2)$-order result for large k given in Eq. (50). This gives

$$-\frac{\gamma}{k^3}=\frac{4}{6\sqrt{2}+\frac{8k}{\sqrt{13}-3}}+k, \tag{58}$$

which is at most 8.5% different from Eq. (57).

We now look at a particular case (b) where the free energy is given by

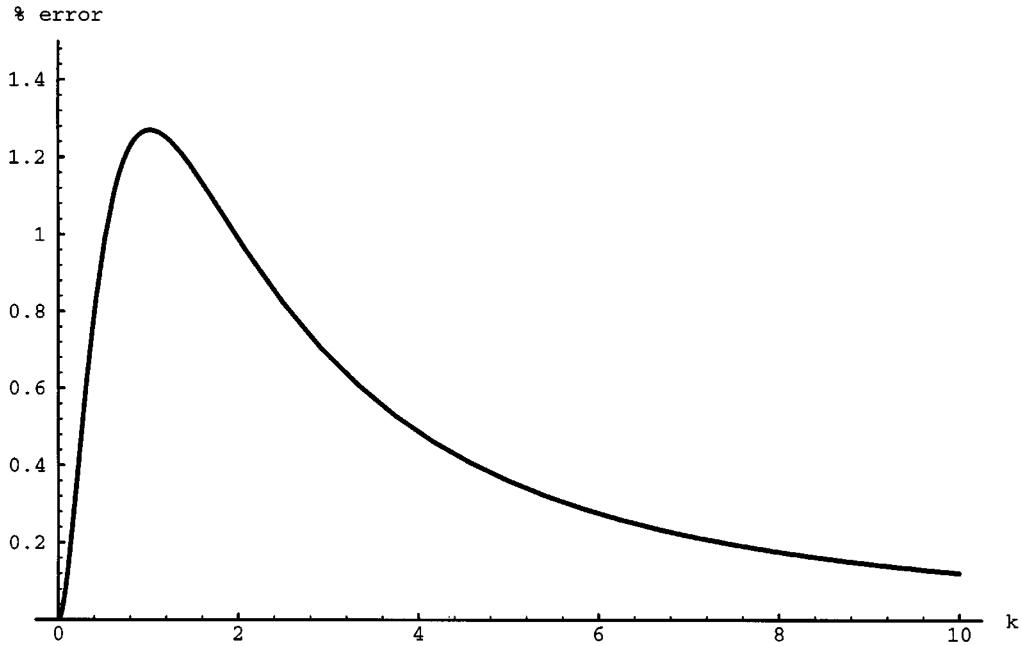

FIG. 3. Particular free energy (b): percentage error in growth rate given by the Padé approximant.

$$F(u)=\begin{cases}\frac{1}{2}(1-u)^2, & u>0\\ \frac{1}{2}(1+u)^2, & u<0,\end{cases} \tag{59}$$

which is the so-called double Gaussian potential. It is a commonly used approximation because an exact growth rate relation for the linear equation (8) can be obtained [see Eq. (2.16) of [5]]. Using this potential we find $d^2F/du^2=1$ for all x, $u_2=1$, $u_1=-1$, and using Appendix B we calculate the values of the definite integrals, namely,

$$I_1=I_2=I_3=1. \tag{60}$$

This gives $\gamma_3=-\frac{1}{2}$ and $\gamma_4=-\frac{5}{8}$, which are substituted into Eq. (56) to give a Padé approximant form for the growth rate as

$$-\frac{\gamma}{k^3}=\frac{2}{4+3k}+k. \tag{61}$$

When compared to the exact growth rate relation, given by Eq. (2.16) of Ref. [5], this approximation has a maximum error of 1.3%, as shown in Fig. 3.

V. CONCLUSIONS

We have found expressions for the growth rate of perpendicular perturbations to the kink solution of a general Cahn-Hilliard equation, at small and large values of the wave number k. This is done using ordinary perturbation analysis combined with knowledge of the asymptotic form of the linear equation. We derive a Padé approximant to the growth rate for all k. We apply our results to the Cahn-Hilliard equation for two particular potentials. In both cases, it is found that the kink solution is stable for all k, with large wave numbers decaying quickest, and $k=0$ (infinite wavelength perturbations) being marginally stable ($\gamma=0$). For the case of the double Gaussian potential, our approximation is within 1.3% of the exact result. This leads us to believe that Eq. (56) is a good approximation to the growth rate of perturbations for all wavelengths, for *any* potential admitting a stationary kink solution.

APPENDIX A: ASYMPTOTIC BEHAVIOR OF SOLUTIONS TO THE LINEAR CAHN-HILLIARD EQUATION

As $x\rightarrow+\infty$, the linear equation (8) can be written as

$$\left[\left(\frac{d^2}{dx^2}-k^2\right)^2-\beta\left(\frac{d^2}{dx^2}-k^2\right)+\gamma\right]\delta u=0, \tag{A1}$$

where terms such as $e^{-\sqrt{\beta}x}$ have been neglected. This has solutions of the form $\delta u\propto e^{\lambda x}$. Combining this with Eq. (A1) gives

$$\lambda=-\left(\frac{\beta\pm\beta\sqrt{1-\frac{4\gamma}{\beta^2}}}{2}+k^2\right)^{1/2}. \tag{A2}$$

Since we are only interested in the slow behavior (the algebraic terms discussed in Sec. II), we need only consider

$$\lambda=-\left(\frac{\beta-\beta\sqrt{1-\frac{4\gamma}{\beta^2}}}{2}+k^2\right)^{1/2}. \tag{A3}$$

If $\gamma=\gamma_3k^3+\gamma_4k^4+\cdots$ and k is small, then we can expand our solution $e^{\lambda x}$ to obtain

$$\delta u=A\left[1-kx+k^2\left(\frac{x^2}{2}-\frac{\gamma_3x}{2\beta}\right)+O(k^3)\right], \tag{A4}$$

where A is a function of k. Note how this expansion contains algebraically secular terms that appear to be unbounded as $x \rightarrow +\infty$, but are in fact simply parts of a slowly decaying exponential term. Also we have not included γ_1 or γ_2 in our expansion of γ. Including such terms makes equating asymptotic results here to those obtained using perturbative techniques impossible.

APPENDIX B: UNKNOWN INTEGRALS

We have determined the growth rate of small-k perturbations to fourth order in k. The exact values are dependent upon three unknown integrals I_1, I_2, I_3. Here we redefine them from integrals over all x to integrals over all u_0.

From the definition given by Eq. (16),

$$I_1 = \left\langle \frac{du_0}{dx} \int_0^x u_0(x')dx' \right\rangle \equiv \int_{-\infty}^{+\infty} \frac{du_0}{dx} \int_0^x u_0(x')dx'\,dx, \tag{B1}$$

which on integrating by parts becomes

$$\begin{aligned} I_1 &= \left[u_0 \int_0^x u_0(x')dx' \right]_{-\infty}^{+\infty} - \int_{-\infty}^{+\infty} u_0^2 dx \\ &= u_2 \int_0^{+\infty} u_0(x)dx - u_1 \int_0^{-\infty} u_0(x)dx - \int_{-\infty}^{+\infty} u_0^2(x)dx, \end{aligned} \tag{B2}$$

and using Eq. (6) leaves

$$\begin{aligned} I_1 &= u_2 \int_0^{u_2} \frac{u_0}{\sqrt{2F(u_0)}} du_0 - u_1 \int_0^{u_1} \frac{u_0}{\sqrt{2F(u_0)}} du_0 \\ &\quad - \int_{u_1}^{u_2} \frac{u_0^2}{\sqrt{2F(u_0)}} du_0. \end{aligned} \tag{B3}$$

Similarly,

$$\begin{aligned} I_2 &= \left\langle \left(\frac{du_0}{dx} \right)^2 \right\rangle = \int_{-\infty}^{+\infty} \left(\frac{du_0}{dx} \right)^2 dx \\ &= \int_{u_1}^{u_2} \frac{du_0}{dx} du_0 = \int_{u_1}^{u_2} \sqrt{2F(u_0)}du_0 \end{aligned} \tag{B4}$$

and

$$\begin{aligned} I_3 &= \frac{K_1}{u_2} - \frac{K_2}{\beta} = \lim_{x \rightarrow +\infty} \int_0^x \left[\frac{u_0}{u_2} - \frac{1}{\beta} \frac{d^2F}{du^2}(u_0) \right] dx \\ &= \frac{1}{u_2 \beta} \int_0^{u_2} \left[\beta u_0 - u_2 \frac{d^2F}{du^2}(u_0) \right] \frac{du_0}{u_{0x}} \\ &= \frac{1}{\sqrt{2} u_2 \beta} \int_0^{u_2} \frac{[\beta u_0 - u_2 F''(u_0)]}{\sqrt{F(u_0)}} du_0. \end{aligned} \tag{B5}$$

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