Dynamics of a system consisting of a van der Pol oscillator coupled to a Duffing oscillator

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This paper studies the dynamics of a system consisting of a van der Pol oscillator coupled to a Duffing oscillator. Analytic solutions are obtained in both the resonant and nonresonant cases. Chaotic behavior is observed using the Shilnikov theorem and from a direct numerical simulation of the coupled equations of motion. [S1063-651X(96)13610-2]

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I. INTRODUCTION

In recent years, particular interest has been devoted to the dynamics of coupled oscillators [1–20]. This is due to the fact that coupled oscillators provide fundamental models for the dynamics of various physical, electromechanical, chemical, and biological systems. Among these coupled oscillators, the most intensively studied are the coupled van der Pol or self-sustained oscillators [1–8] and the coupled Duffing oscillators [1,6,11,17–20]. To the best of our knowledge, less has been done in a system consisting of a self-excited oscillator coupled to an anharmonic oscillator of the Duffing type. Our aim in this paper is to consider the dynamics of such a system described by the following set of equations:

$$\ddot{x}-\varepsilon_1 F(x)\dot{x}+G(x)=c_1 y+c_2\dot{y}, \tag{1a}$$

$$\ddot{y}+\varepsilon_2\dot{y}+H(y)=c_1 x+c_2\dot{x}, \tag{1b}$$

where F, G, and H are polynomial functions of the form

$$F(x)=1-x^2, \tag{2a}$$

$$G(x)=W_1^2 x+cx^3, \tag{2b}$$

$$H(x)=W_2^2 x+c_0 x^3. \tag{2c}$$

ε_1 and ε_2 are positive parameters. c and c_0 are some nonlinearity coefficients. c_1 and c_2 are, respectively, the elastic and the dissipative coupling parameters. W_1 and W_2 are the natural frequencies of the oscillators.

The set of Eqs. (1) is a mathematical description of the time evolution of various coupled autonomous systems. Indeed when the coupling parameters are set equal to zero, the system (1) turns into two classical, rich, and well-studied oscillators. Namely, the first equation (1a) reduces to the van der Pol oscillator, which serves as a basic model for self-excited oscillators in physics, mechanics, and electronics [1,21–24]. The final state of this oscillator is a sinusoidal limit cycle when ε_1 is small, but leads to relaxation oscillations when ε_1 becomes large (see the above references). The second equation (1b) in its part reduces to the autonomous Duffing oscillator [25], which describes the motion of various physical systems such as the pendulum, electrical circuits, Josephson junctions, optical bistability, plasma oscillations, buckled beam, ship dynamics, vibration isolators, to name a few (see Refs. [1,21,25]). In this autonomous state, Eq. (1b) shows damped vibrations (ε_2 being the damping coefficient) (see Refs. [1,22]). An example of a physical system with practical interest that Eq. (1) can describe is a nonlinear oscillator functioning under the action of a self-sustained electrical oscillator.

Among problems related to the dynamics of systems described by Eq. (1), globally autonomous, we concentrate in this paper on the analysis of the oscillatory states in the resonant and nonresonant cases and on the question of a possible appearance of chaotic behavior.

The structure of the paper is as follows. In Sec. II, we give an analytic treatment of Eqs. (1). The method of multiple scales [1] is used to find approximate solutions of the oscillatory states. We end the section by giving the criterion of chaotic motion following Shilnikov's theorem [4,26]. Section III is concerned with a direct numerical integration of the coupled systems. We conclude in Sec. IV.

II. ANALYTIC TREATMENT

We seek approximate solutions of Eq. (1) by using the method of multiple scales described in Ref. [1]. In general, we consider $x(t)$ and $y(t)$ in the form

$$x(t,\varepsilon)=x_0(T_0,T_1)+\varepsilon x_1(T_0,T_1), \tag{3a}$$

$$y(t,\varepsilon)=y_0(T_0,T_1)+\varepsilon y_1(T_0,T_1), \tag{3b}$$

where $T_0=t$ is a fast scale and $T_1=\varepsilon t$ is a slow scale characterizing the modulation in the amplitudes and phases caused by nonlinearity, coupling, and resonances. The time derivatives (single overdot is d/dt and double overdot is d^2/dt^2) become

$$\frac{d}{dt}=D_0+\varepsilon D_1+\cdots , \tag{4a}$$

$$\frac{d^2}{dt^2}=D_0^2+\varepsilon(2D_0D_1)+\cdots , \tag{4b}$$

where $D_n=\partial/\partial T_n$.

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Substituting Eqs. (3) and (4) into Eqs. (1) with Eqs. (2) and equating coefficients of like powers of ε, and assuming that the system parameters are in the same order of ε, we obtain

$$D_0^2 x_0 + W_1^2 x_0 = 0, \quad (5a)$$

$$D_0^2 y_0 + W_2^2 y_0 = 0, \quad (5b)$$

$$D_0^2 x_1 + W_1^2 x_1 = \varepsilon_1 (1 - x_0^2) D_0 x_0 - c x_0^3 - 2 D_0 D_1 x_0 + c_1 y_0 + c_2 D_0 y_0, \quad (5c)$$

$$D_0^2 y_1 + W_2^2 y_1 = -\varepsilon_2 D_0 y_0 - c_0 y_0^3 - 2 D_0 D_1 y_0 + c_1 x_0 + c_2 D_0 x_0. \quad (5d)$$

The general solutions of Eqs. (5a) and (5b) can be written in the form

$$x_0(T_0,\varepsilon) = A_1(T_1) e^{iW_1 T_0} + \text{c.c.}, \quad (6a)$$

$$y_0(T_0,\varepsilon) = A_2(T_1) e^{iW_2 T_0} + \text{c.c.}, \quad (6b)$$

where c.c. stands for the complex conjugate of each preceding term. The quantities $A_1(T_1)$ and $A_2(T_1)$ are arbitrary, complex functions which are determined from Eqs. (5c) and (5d) by imposing solvability or secular conditions. Substituting Eqs. (6a) and (6b) into (5c) and (5d), we obtain

$$D_0^2 x_1 + W_1^2 x_1 = \{-2iW_1 D_1 A_1 + i\varepsilon_1 W_1 A_1 (1 - A_1 \overline{A_1}) - 3c\overline{A_1} A_1^2\} e^{iW_1 T_0} - A_1^3 \{c + i\varepsilon_1 W_1\} e^{i3W_1 T_0} + \{c_1 + ic_2 W_2\} A_2 e^{iW_2 T_0} + \text{c.c.}, \quad (7a)$$

$$D_0^2 y_1 + W_2^2 y_1 = \{-2iW_2 D_1 A_2 - i\varepsilon_2 W_2 A_2 - 3c_0 \overline{A_2} A_2^2\} e^{iW_2 T_0} - c_0 A_2^3 e^{i3W_2 T_0} + \{c_1 + ic_2 W_1\} A_1 e^{iW_1 T_0} + \text{c.c.}, \quad (7b)$$

where $\overline{A_1}$ and $\overline{A_2}$ are, respectively, the complex conjugates of A_1 and A_2.

A. The nonresonant case

Here we analyze the case $W_1 \neq W_2$. The conditions for elimination of the secular terms (solvability conditions) in Eqs. (7) are

$$2iW_1 D_1 A_1 - i\varepsilon_1 W_1 A_1 (1 - A_1 \overline{A_1}) + 3c\overline{A_1} A_1^2 = 0, \quad (8a)$$

$$2iW_2 D_1 A_2 + i\varepsilon_2 W_2 A_2 + 3c_0 \overline{A_2} A_2^2 = 0. \quad (8b)$$

Expressing $A_1(T_0,\varepsilon)$ and $A_2(T_0,\varepsilon)$ in the polar form

$$A_1(T_0,\varepsilon) = \tfrac{1}{2} a_1(T_0,\varepsilon) \exp[i\theta_1(T_0,\varepsilon)], \quad (9a)$$

$$A_2(T_0,\varepsilon) = \tfrac{1}{2} a_2(T_0,\varepsilon) \exp[i\theta_2(T_0,\varepsilon)], \quad (9b)$$

where $a_1(T_0,\varepsilon)$ and $\theta_1(T_0,\varepsilon)$, respectively, $a_2(T_0,\varepsilon)$ and $\theta_2(T_0,\varepsilon)$ are the amplitudes and the phases of the fundamental solutions. We thus obtain the following set of first-order differential equations for the amplitudes and phases:

$$\frac{da_1}{dT_1} = \frac{\varepsilon_1 a_1}{2} (1 - \tfrac{1}{4} a_1^2), \quad (10a)$$

$$\frac{da_2}{dT_1} = -\frac{\varepsilon_2 a_2}{2}, \quad (10b)$$

$$\frac{d\phi}{dT_1} = \frac{3}{8} \left(\frac{c}{W_1} a_1^2 - \frac{c_0}{W_2} a_2^2 \right), \quad (10c)$$

where $\phi = \theta_1 - \theta_2$.

In its general form, Eqs. (10) show that in the nonresonant case, both oscillators are uncoupled and the time evolution of the amplitudes is given by

$$a_1(T_0,\varepsilon) = \frac{2}{\sqrt{1 - \left(1 - \frac{4}{a_{01}^2}\right) \exp(-\varepsilon_1 T_1)}}, \quad (11a)$$

$$a_2(T_0,\varepsilon) = a_{02} \exp\left(-\frac{\varepsilon_2}{2} T_1 \right), \quad (11b)$$

corresponding to the amplitudes of the classical van der Pol oscillator and the autonomous Duffing oscillator (as t increases, $a_1 \rightarrow 2$ and $a_2 \rightarrow 0$).

B. The resonant case and the Shilnikov criterion for chaos

It follows from Eqs. (7) that only the primary resonance can be observed in our system. To express quantitatively the nearness of W_2 to W_1, we introduce a detuning parameter σ according to

$$W_2 = W_1 + \varepsilon\sigma. \quad (12)$$

The new solvability conditions in Eqs. (7) become

$$-2iW_1 D_1 A_1 + i\varepsilon_1 W_1 A_1 (1 - A_1 \overline{A_1}) - 3c\overline{A_1} A_1^2 + (c_1 + ic_2 W_2) A_2 e^{i\sigma T_1}, \quad (13a)$$

$$-2iW_2 D_1 A_2 - i\varepsilon_2 W_2 A_2 - 3c_0 \overline{A_2} A_2^2 + (c_1 + ic_2 W_1) A_1 e^{-i\sigma T_1}. \quad (13b)$$

Expressing $A_1(T_0,\varepsilon)$ and $A_2(T_0,\varepsilon)$ in the polar form as above defined and substituting into Eqs. (13), we obtain the following set of first-order differential equations for the amplitudes and phases:

$$\frac{da_1}{dT_1} = \frac{\varepsilon_1 a_1}{2} \left(1 - \frac{1}{4} a_1^2\right) + \frac{c_1}{2W_1} a_2 \sin\nu + \frac{c_2 W_2}{2W_1} a_2 \cos\nu, \quad (14a)$$

$$\frac{da_2}{dT_1} = -\frac{\varepsilon_2}{2} a_2 - \frac{c_1}{2W_2} a_1 \sin\nu + \frac{c_2 W_1}{2W_2} a_1 \cos\nu, \quad (14b)$$

$$\frac{d\nu}{dT_1} = \frac{3}{8} \left(\frac{c_0}{W_2} a_2^2 - \frac{c}{W_1} a_1^2 \right) + \frac{c_1}{2} \left(\frac{1}{W_1} \frac{a_2}{a_1} - \frac{1}{W_2} \frac{a_1}{a_2} \right) \cos\nu - \frac{c_2}{2} \left(\frac{W_1}{W_2} \frac{a_1}{a_2} + \frac{W_2}{W_1} \frac{a_2}{a_1} \right) \sin\nu + \sigma, \quad (14c)$$

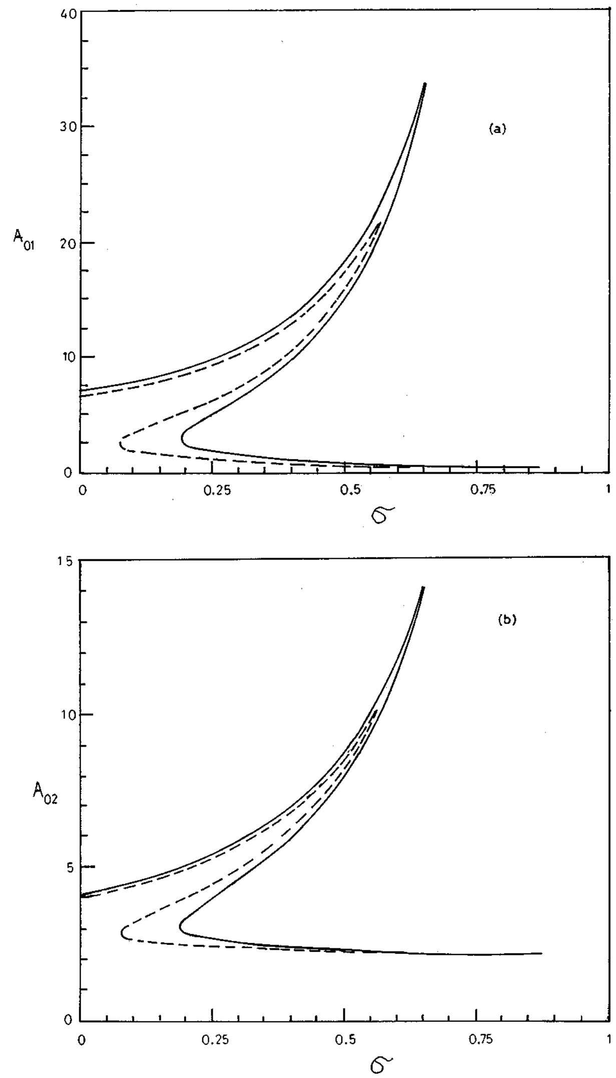

FIG. 1. (a) The stationary solutions A_{01} vs the detuning σ (dashed lines for $c_2=0.05$; full lines for $c_2=0.07$) with $W_2=1$, $\varepsilon_1=\varepsilon_2=0.01$, $c_0=0.0125$, $c=0.0250$, and $c_1=0$. (b) The stationary solutions A_{02} vs the detuning σ (dashed lines for $c_2=0.05$; full lines for $c_2=0.07$) with $W_2=1$, $\varepsilon_1=\varepsilon_2=0.01$, $c_0=0.0125$, $c=0.0250$, and $c_1=0$.

where $\nu=\theta_2-\theta_1+\sigma T_1$.

The equilibrium states $a_1=A_{01}$, $a_2=A_{02}$, and $\nu=\nu_0$ of Eqs. (14) are defined by the following set of nonlinear algebraic equations for A_{01} and A_{02} [ν_0 can be obtained after substitution of A_{01} and A_{02} into Eq. (14c)]:

$$16\varepsilon_2^2W_2^2A_{02}^2(W_1^2A_{01}^2+W_2^2A_{02}^2)^2+9W_1^2A_{01}^4A_{02}^2(W_2cA_{01}^2-W_1c_0A_{02}^2)^2-16c_2^2W_1^2A_{01}^2(W_1^2A_{01}^2+W_2^2A_{02}^2)^2=0, \quad (15a)$$

where

$$A_{02}^2=\frac{\varepsilon_1W_1^2}{4\varepsilon_2W_2^2}A_{01}^2(A_{01}^2-4), \quad (15b)$$

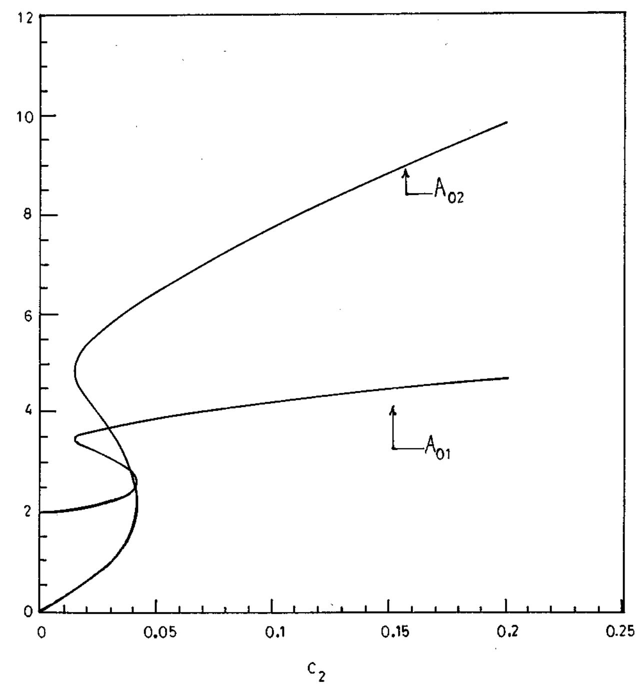

FIG. 2. The stationary solutions A_{01} and A_{02} vs c_2 (same value for the system parameters as in Fig. 1 but with $\sigma=0$).

where we have considered only the dissipative coupling. Equations (14) also have a trivial equilibrium point $A_{01}=A_{02}=0$ and ν_0 undefined.

The stability of each fixed point can be determined by calculating the eigenvalues of system (14), linearized about the steady values A_{01}, A_{02}, and ν_0. Substituting Eq. (15b) into Eq. (15a), we obtain the following tenth-order nonlinear algebraic equation:

$$b_{10}A_{01}^{10}+b_9A_{01}^9+b_6A_{01}^6+b_4A_{01}^4+b_2A_{02}^2+b_0=0, \quad (15c)$$

with the coefficients b_i defined in the Appendix.

Equation (15c) is solved using the Newton-Raphson algorithm. Figures 1 and 2 show the response curves, respectively, in terms of the detuning parameter σ and the dissipative coupling constant $c_2(\sigma=0)$ for some selected values of the system parameters ($W_2=1$; $\varepsilon_1=\varepsilon_2=0.01$; $c_0=0.0125$; $c=0.0250$; and $c_1=0$). Figures 1 and 2 show hysteresis domains and we have found that only the lower branches correspond to stable solutions. It can also be found that when the detuning increases, $a_1\to 2$ and $a_2\to 0$, leading to the nonresonant (or uncoupled) motion as analyzed above. To find the criterion for chaotic behavior in our model, we have used Shilnikov's theorem [4,26]. The theorem states that for chaos to occur within a third-order autonomous system such as that described by Eqs. (14), the eigenvalues of the 3×3 matrix, formed from the Jacobian derivative of the system at an equilibrium state (A_{01},A_{02},ν_0), must be $-\delta$, and $\gamma\pm i\omega$ with $\delta>\gamma>0$. By varying the coupling coefficient c_2, we have found after solving numerically the eigenvalue equation that Shilnikov's criterion is satisfied for $0.0417\leq c_2\leq 0.5100$. We have also analyzed the system by considering the elastic coupling c_1 and no range of chaos has appeared. In the next

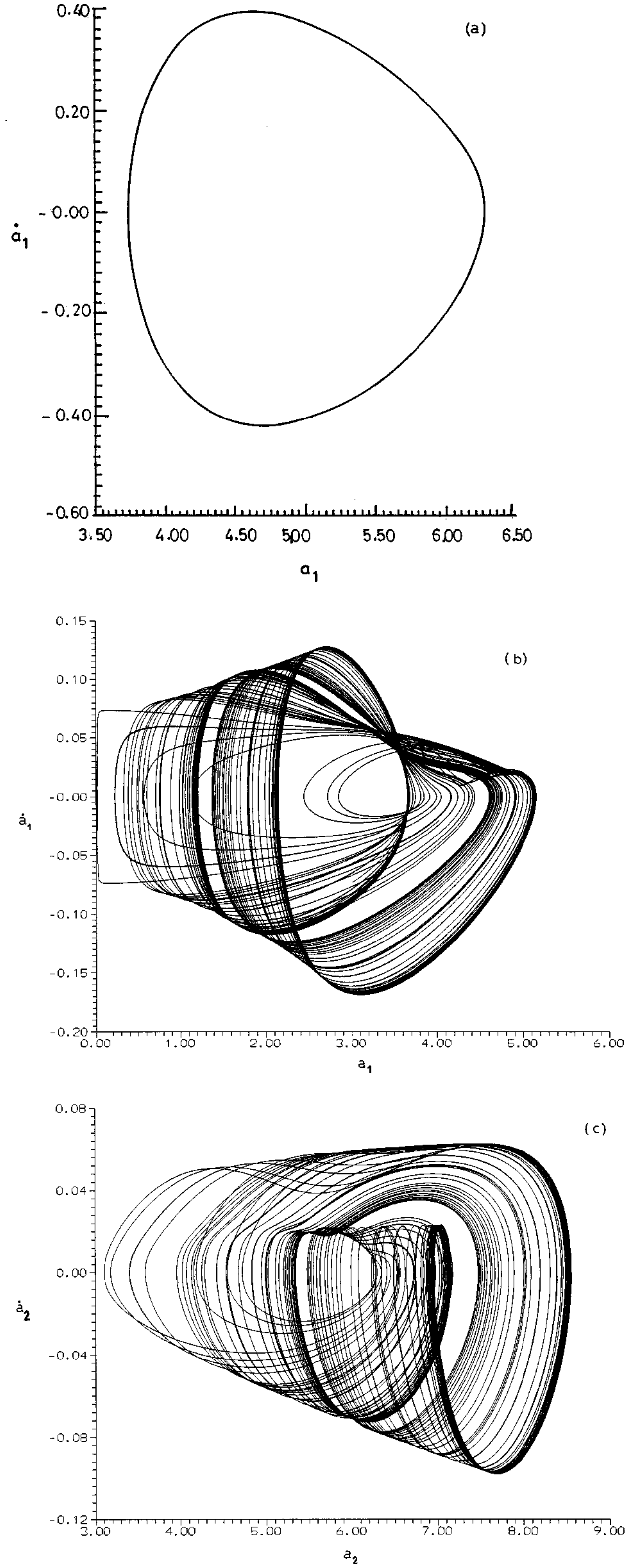

FIG. 3. (a) The phase space $(a_1, \dot{a}_1)$ for $c_2=0.4$. This corresponds to a limit cycle with a modulation of the amplitudes. ($W_1=W_2=1$, $\varepsilon_1=\varepsilon_2=0.01$, $c_0=0.0125$, $c=0.0250$, and $c_1=0$.) (b) The phase space $(a_1, \dot{a}_1)$ for $c_2=0.04$. This corresponds to a chaotic state. ($W_1=W_2=1$, $\varepsilon_1=\varepsilon_2=0.01$, $c_0=0.0125$, $c=0.0250$, and $c_1=0$.) (c) The phase space $(a_2, \dot{a}_2)$ for $c_2=0.04$. This corresponds to a chaotic state [same value for the system parameters as in (b)].

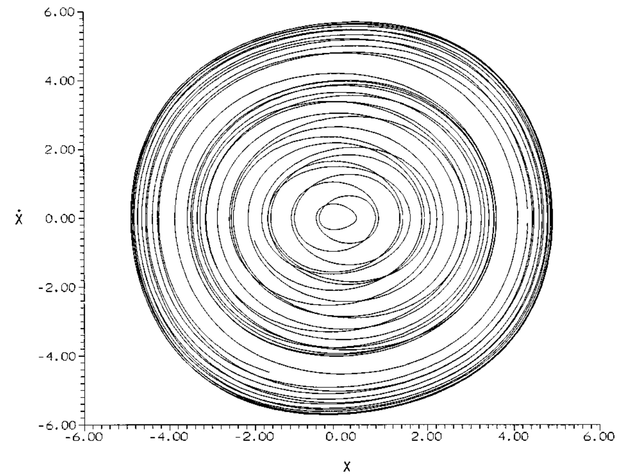

FIG. 4. Phase portrait of the first oscillator with the parameters of Fig. 3(b).

section, we verify Shilnikov's criterion by solving numerically the set of differential Eqs. (14) and the original Eqs. (1).

III. NUMERICAL COMPUTATION

We solve numerically Eqs. (1) and (14) to find the range of chaos. We use the fourth-order Runge-Kutta algorithm [27]. The time step is always $\Delta t=0.04$ and the calculations are performed using reals in extended mode. The integration time is always greater than $T=10^4$. To characterize the degree of chaos, we calculate the largest Lyapunov exponent and draw phase portraits. The Lyapunov exponent is defined as $\lambda=(1/t)\ln(|x_1|+|\dot{x}_1|+|y_1|+|\dot{y}_1|)$, where x_i and y_i are solutions of the variational equations [obtained from Eq. (1) by setting $x \rightarrow x+x_1$ and $y \rightarrow y+y_1$ and linearizing around the solutions x and y]. The same can be done for Eqs. (14).

Considering first the amplitude equation [see Eqs. (14)], it is seen that $a_i(t)$ always depend on time: a sort of beating oscillation is generated because of the coupling. The phase spaces $(a_i, \dot{a}_i)$ shows the existence of limit cycles with periodic variations of the amplitudes a_i and the phases ϕ_i [see Fig. 3(a)]. When $c_2 \in [0.0399, 0.2985]$, the amplitudes a_i vary chaotically as it appears in Figs. 3(b) and 3(c). For Figs. 3(b) and 3(c), the largest Lyapunov exponent computed from Eq. (14) is $\lambda_{max}=0.0014$.

In view of verifying the results from the analysis of Eqs. (14) obtained by the method of multiple scales, we have also computed numerically the original Eqs. (1). We first note that in general, the coupling generates, as mentioned here before, a modulation of the amplitudes a_i. For the parameters of Figs. 3(b) and 3(c), the phase portrait for the first oscillator is shown in Fig. 4. Here the largest Lyapunov exponent [computed from Eqs. (1)] is $\lambda_{max}=0.0012$. It follows from the numerical integration of Eqs. (1) that (for the values of the system parameters used for Figs. 3 and 4) the chaotic behavior occurs when $c_2 \in [0.0400, 0.2460]$.

The difference between the range of c_2 for the occurrence of chaos obtained from Shilnikov's theorem and those of the direct numerical simulations of Eqs. (1) and (14) can be explained by the fact that the application of Shilnikov's theorem required small values of the system parameters (the as-

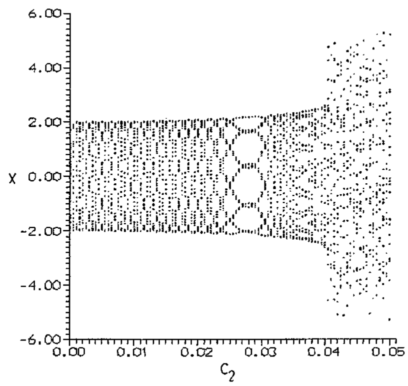

FIG. 5. Bifurcation diagram showing the first coordinate x of the attractor in the Poincaré cross section vs the coupling coefficient c_2. ($W_1=W_2=1$, $\varepsilon_1=\varepsilon_2=0.01$, $c_0=0.0125$, $c=0.0250$, and $c_1=0$).

sumption used to establish Eqs. (14) from the approximate method).

An interesting question related to the problem of chaos is the way the chaos appears in the system. It follows from our numerical simulation that the transition to chaos is abrupt. We have drawn in Fig. 5 the bifurcation diagram showing transition to chaos as the coupling coefficient increases. The bifurcation diagram shows the first coordinate x of the attractor in the Poincaré cross section versus the coupling coefficient c_2 that has been increased in small steps. After each step the last solution has been used as new initial conditions.

IV. CONCLUSION

This paper has dealt with the study of a self-sustained oscillator (van der Pol) coupled to a nonlinear oscillator of the Duffing type. The coupling is elastic and dissipative. The multiple scale technique has been used to derive analytic solutions both in the resonant and nonresonant cases. It is seen that the coupling generates the modulation of the amplitudes and a sort of beating oscillation is obtained. Shilnikov's theorem is used to define the range of the coupling coefficient leading to chaotic behavior. In view of verifying the results obtained from Shilnikov's theorem, we have carried out a direct numerical simulation of the coupled equations of motion. A difference between the range of chaos following Shilnikov's theorem and that obtained from the numerical simulation is found. This can be explained by the fact that the analytic prediction required small values of the system parameters and is obtained from the approximate method. Our numerical computation has also shown that in our model, chaos arises suddenly.

An interesting question under investigation is that of finding the analytic solution of the system described by Eqs. (1) in the case where the coupling coefficients are not small. Another problem under consideration is that of analyzing the oscillatory states and chaotic behavior of our system when an external sinusoidal force is added to Eqs. (1).

APPENDIX

$$b_{10}=9c_0^2\varepsilon_1^3W_1^{10},$$

$$b_9=-108c_0^2\varepsilon_1^3W_1^{10}-72c_0c\varepsilon_1^2\varepsilon_2W_1^7W_2^3,$$

$$b_6=16\varepsilon_1^3\varepsilon_2^2W_1^6W_2^6+432\varepsilon_1^3c_0^2W_1^{10}+576\varepsilon_1^2\varepsilon_2c_0cW_1^7W_2^3+144\varepsilon_1\varepsilon_2^2c^2W_1^4W_2^6,$$

$$b_4=128\varepsilon_1^2\varepsilon_2^3W_1^6W_2^6-64c_2^2\varepsilon_1^2\varepsilon_2W_1^6W_2^6-1152c_0c\varepsilon_1^2\varepsilon_2W_1^7W_2^3-192\varepsilon_1^3\varepsilon_2^2W_1^6W_2^6-576c_0^2\varepsilon_1^3W_1^{10}-576c^2\varepsilon_1\varepsilon_2^2W_1^4W_2^6,$$

$$b_2=512\varepsilon_1^2\varepsilon_2c_2^2W_1^6W_2^6-1024\varepsilon_1^2\varepsilon_2^3W_1^6W_2^6+768\varepsilon_1^3\varepsilon_2^2W_1^6W_2^6+256\varepsilon_1\varepsilon_2^4W_1^6W_2^6-512\varepsilon_1\varepsilon_2^2c_2^2W_1^6W_2^6,$$

$$b_0=-1024\varepsilon_2^3c_2^2W_1^6W_2^6-1024\varepsilon_2^2\varepsilon_1^3W_1^6W_2^6+2048\varepsilon_1^2\varepsilon_2^3W_1^6W_2^6-1024\varepsilon_1^2\varepsilon_2c_2^2W_1^6W_2^6+2048\varepsilon_1\varepsilon_2^2c_2^2W_1^6W_2^6-1024\varepsilon_1\varepsilon_2^4W_1^6W_2^6.$$

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