4/3 problem in classical electrodynamics

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We evaluate the self-force acting on an extended nonrelativistic charged particle, in the framework of classical electrodynamics with a cutoff at short distances. We show that in the regularized Maxwell theory, the point particle limit is finite and well defined. As a result, the electromagnetic mass of a point particle enters the equation of motion in a form consistent with the special theory of relativity. [S1063-651X(96)15111-4]

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I. INTRODUCTION

The nonrelativistic equation of motion of an extended charge was first derived by Abraham (1903) and Lorentz (1904), who considered a purely electromagnetic model of the electron's structure [1]. It was assumed that in the instantaneous rest frame of the particle, the charge distribution is rigid and spherically symmetric. If an external force is also applied to the electron, then the Abraham-Lorentz equation of motion which assumes no mechanical mass takes the form

$$\frac{4}{3}\frac{U(a)}{c^2}\dot{\mathbf{v}}-\frac{2}{3}\frac{e^2}{c^3}\ddot{\mathbf{v}}=\mathbf{F}_{\text{ext}}, \tag{1}$$

where $\mathbf{v}$ is the velocity of the particle and $U(a)$ is the electrostatic energy of a symmetric charge distribution $\rho(\mathbf{x})$, which is located within a sphere of radius a:

$$U(a)=\frac{1}{2}\int d^3x\int d^3x'\,\frac{\rho(\mathbf{x})\rho(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}. \tag{2}$$

The factor $(4/3)U/c^2$ in Eq. (1) is evidently the electromagnetic mass. If U were the total energy, special relativity would require $U=mc^2$, where m is the observed mass of the electron. For this reason, the factor 4/3 has been a source of considerable discussion. Poincaré [1] gave a solution to this problem, pointing out that the stability of the charge requires the existence of nonelectromagnetic, attractive forces acting on the particle. These internal forces yield a contribution m_0 to the mass of the particle, which would appear as an added coefficient of the acceleration in Eq. (1). Then the requirements of special relativity would apply only to the total self-energy and mass $m=m_{\text{el}}+m_0$. Nevertheless, such a solution has one puzzling aspect. Classical electrodynamics is a properly covariant theory, so we might expect that a correct calculation should not violate the requirements of Lorentz covariance. In fact, one can define a purely electromagnetic self-energy-momentum tensor having the correct Lorentz transformation properties (Rohrlich [2]; Jackson [3]). Comprehensive reviews of these and other relevant aspects of the above problem may be found in the literature (Erber [4]; Teitelboim, Villaroel, and Van Weert [5]; Pearle [6]).

The point electron has been of greatest interest because high-energy experiments indicate that the electron may be regarded as a point particle, at least down to distances of order of 10^{-15} cm. Using an approach where the electron is treated from the start as a point charge, Coleman [7] introduced a cutoff in the Maxwell theory, which enables an unambiguous derivation of the relativistic equation of motion comparable to Eq. (1), called the Lorentz-Dirac equation [8]. He considered the cutoff as a merely computational device. On the other hand, Moniz and Sharp [9] have shown in the context of the quantum theory of a nonrelativistic electron that a natural cutoff of order of the Compton wavelength of the electron may be effectively induced by the processes which occur in quantum electrodynamics.

In this work, we consider the point electron as a limiting case of an extended charged particle, which may lead to a useful insight. In particular, we would like to propose an alternative explanation for the factor 4/3 which appears in Eq. (1). To this end, we remark that the Abraham-Lorentz assumption of the existence of a rigid extended particle leads to a difficulty with regard to special relativity. In classical electrodynamics, the point particle limit cannot be taken because the electromagnetic mass becomes infinite. In order to circumvent this problem, we shall use a gauge and Lorentz invariant regularization of the Maxwell theory at short distances. This is done by introducing an appropriate cutoff at the threshold of the classical regime, which allows for the existence of a finite and well defined point particle limit. In this case, the calculation of the electromagnetic mass does not violate the requirements of Lorentz covariance, so that U/c^2 will enter the particle's equation of motion with the proper factor of unity, instead of 4/3.

A possible approach involves adding a new term to the Maxwell Lagrangian. Its form can be restricted by a few reasonable and simple properties, which leave the Maxwell theory as nearly unaltered as possible: (a) The Lagrangian must be gauge and Lorentz invariant. (b) It should lead to local field equations which are still linear in the field quantities. Then, the simplest possibility which includes a cutoff l leads to a Lagrangian containing second-order derivatives of the electromagnetic potentials $A_\alpha=(\mathbf{A},i\phi)$:

$$\mathcal{L}(l)=-\frac{1}{16\pi}F_{\alpha\beta}F_{\alpha\beta}-\frac{1}{8\pi}l^2\frac{\partial F_{\alpha\beta}}{\partial x_\beta}\frac{\partial F_{\alpha\gamma}}{\partial x_\gamma}+\frac{1}{c}j_\alpha A_\alpha, \tag{3}$$

where $F_{\alpha\beta}=\partial_\alpha A_\beta-\partial_\beta A_\alpha$ is the usual electromagnetic field tensor and $j_\alpha=(\mathbf{j},ic\rho)$ is the four-current. At distances much larger than the cutoff, the fields described by Eq. (3) become essentially equivalent to the Maxwell fields. We mention that

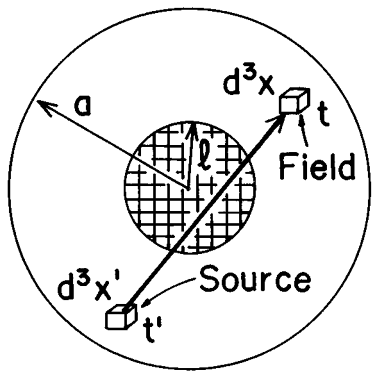

FIG. 1. The interaction between the elements of a spherically symmetric charged particle.

such a Lagrangian was proposed a long time ago by Podolsky and Schwed [10], in a somewhat different context.

In the Lorentz gauge $\partial_\alpha A_\alpha=0$, one obtains from Eq. (3) the following set of linear partial differential equations:

$$(1-l^2\Box)\Box A_\alpha=-\frac{4\pi}{c}j_\alpha. \tag{4}$$

In the static case, the scalar potential due to a charge distribution ρ turns out to be

$$\phi(\mathbf{x})=\int d^3x'\,\frac{\rho(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}\left[1-\exp\left(-\frac{|\mathbf{x}-\mathbf{x}'|}{l}\right)\right], \tag{5}$$

which, at distances much larger than l, becomes practically equal to the Coulomb potential. On the other hand, at short distances l behaves as an effective cutoff since for a point charge located at the origin, ϕ approaches the finite value e/l as $|\mathbf{x}|\to 0$.

In Sec. II, we present in this framework a direct calculation of the self-force acting on an extended spherically symmetric charged particle. We show in Sec. III that in the point particle limit, the electromagnetic mass is well defined and has a form consistent with the requirements of Lorentz covariance.

II. CALCULATION OF THE SELF-FORCE FROM THE RETARDED POTENTIALS

Let us evaluate, in the regularized Maxwell theory, the self-force on the rigid spherically symmetric charged particle shown in Fig. 1.

We calculate initially the force that a small volume element d^3x experiences from all other parts of the sphere. In terms of the electric and magnetic fields at this location, the force on the element d^3x is given by

$$d\mathbf{F}_s(\mathbf{x},t)=d^3x\rho(\mathbf{x})\left[\mathbf{E}(\mathbf{x},t)+\frac{1}{c}\mathbf{v}(t)\times\mathbf{B}(\mathbf{x},t)\right]. \tag{6}$$

Without loss of generality, we may consider the instantaneous rest frame of the particle. Then we need to evaluate only the electric field which can be derived from the retarded potentials as $\mathbf{E}=-\nabla\phi-\partial\mathbf{A}/c\,dt$. Integrating over all parts of the sphere, we find that the self-force acting on the extended charged particle is given in this frame by the expression

$$\mathbf{F}_s(t)=-\int\rho(\mathbf{x},t)\left[\nabla\phi(\mathbf{x},t)+\frac{1}{c}\frac{\partial\mathbf{A}}{\partial t}(\mathbf{x},t)\right]d^3x. \tag{7}$$

In order to determine the form of the retarded potentials, which satisfy the differential equations (4), it is useful to find the Green function for the equation

$$(1-l^2\Box)\Box G(\mathbf{x}-\mathbf{x}',t-t',l)=-4\pi\delta(\mathbf{x}-\mathbf{x}')\delta(t-t'). \tag{8}$$

Then the solution of Eq. (4) will be

$$A_\alpha(\mathbf{x},t,l)=\frac{1}{c}\int G(\mathbf{x}-\mathbf{x}',t-t',l)j_\alpha(\mathbf{x}',t')d^3x'dt'. \tag{9}$$

The appropriate Green function must satisfy the causality condition that $G=0$ for $t<t'$. With the help of the invariant functions described, for example, in [11], we find that the proper solution of the differential equation (8) is given by

$$G(R,\tau,l)=\frac{1}{R}\delta\left(\tau-\frac{R}{c}\right)+\frac{c}{R}\frac{\partial}{\partial R}\left[\theta\left(\tau-\frac{R}{c}\right)J_0\left(\frac{\sqrt{c^2\tau^2-R^2}}{l}\right)\right], \tag{10}$$

where $R=|\mathbf{x}-\mathbf{x}'|$, $\tau=t-t'$, and J_0 is the Bessel function of order zero. Note that in the limit $l\to 0$, the second term vanishes, so that Eq. (10) reduces to the usual retarded Green function of classical electrodynamics. Combining the two terms in Eq. (10), and using the fact that G is an invariant function of $c^2\tau^2-R^2$, we arrive at the expression

$$G(R,\tau,l)=-\frac{\theta\left(\tau-\frac{R}{c}\right)}{c\tau}\frac{\partial}{\partial\tau}\left[J_0\left(\frac{\sqrt{c^2\tau^2-R^2}}{l}\right)\right]. \tag{11}$$

We recall at this point that the distinctive feature of the Abraham-Lorentz calculation in electrodynamics is a series expansion in powers of R/c, for R/c small, of the four-current j_α, which must be evaluated at the retarded time $t'=t-R/c$. This is essentially equivalent to an expansion of the usual Green function as

$$\frac{1}{R}\delta\left(\tau-\frac{R}{c}\right)=\sum_{n=0}^{\infty}\frac{(-1)^n}{n!c^n}R^{n-1}\frac{d^n}{d\tau^n}\delta(\tau), \tag{12}$$

followed by the t' integration in Eq. (9).

Proceeding in a similar way, we may expand the retarded Green function $G(R,\tau,l)$ as

$$G(R,\tau,l)=\sum_{n=0}^{\infty}\frac{(-1)^n}{n!c^n}R^{n-1}f_n\left(\frac{l}{R}\right)\frac{d^n}{d\tau^n}\delta(\tau), \tag{13}$$

where $f_n(l/R)$ are dimensionless functions which should reduce to unity in the limit $l\to 0$. These can be determined by multiplying G by τ^n and performing the τ integration using the properties of the Bessel functions [12]. We record here the expressions for the lowest order functions f_n:

$$f_0\left(\frac{l}{R}\right)=1-e^{-R/l},\quad f_1=1$$

$$f_2\left(\frac{l}{R}\right)=1+\frac{l}{R}\,e^{-R/l},\quad f_3=1 \tag{14}$$

which will be used subsequently.

Substituting series (13) in Eq. (9), the self-force (7) becomes

$$\mathbf{F}_s(t)=-\sum_{n=0}^{\infty}\frac{(-1)^n}{n!c^n}\int d^3x\rho(\mathbf{x},t)\int d^3x'\times\int dt'\left\{\left[\rho(\mathbf{x}',t')\boldsymbol{\nabla}(R^{n-1}f_n)+\frac{1}{c^2}\mathbf{j}(\mathbf{x}',t')R^{n-1}f_n\frac{d}{d\tau}\right]\frac{d^n}{d\tau^n}\delta(\tau)\right\}. \tag{15}$$

We consider the first two terms arising from the scalar potential separately. Using Eq. (14), we see that the $n=1$ term vanishes identically. The $n=0$ term leads to the electrostatic self-force

$$\mathbf{F}_s^{\mathrm{el}}=-\int d^3x\rho(\mathbf{x},t)\int d^3x'\rho(\mathbf{x}',t)\boldsymbol{\nabla}\left[\frac{1}{R}(1-e^{-R/l})\right], \tag{16}$$

which vanishes by symmetry. Eliminating these two terms and increasing by two the summation index on the terms contributing by the scalar potential, we find after performing the t' integration that the sum (15) can be written as follows:

$$\mathbf{F}_s(t)=-\sum_{n=0}^{\infty}\frac{(-1)^n}{n!c^{n+2}}\int d^3x\rho(\mathbf{x},t)\times\int d^3x'\frac{\partial^{n+1}}{\partial t^{n+1}}\left\{\frac{\partial\rho(\mathbf{x}',t)}{\partial t}\frac{\boldsymbol{\nabla}(R^{n+1}f_{n+2})}{(n+1)(n+2)}+R^{n-1}f_n\mathbf{j}(\mathbf{x}',t)\right\}. \tag{17}$$

The continuity equation may now be used to replace $\partial\rho/\partial t$ by $-\boldsymbol{\nabla}'\cdot\mathbf{j}(\mathbf{x}',t)$. In the d^3x' integration we can integrate these ρ-dependent terms by parts, so that the bracket in Eq. (17) is equivalent to

$$\{\ \}=R^{n-1}f_n\mathbf{j}(\mathbf{x}',t)-\frac{[\mathbf{j}(\mathbf{x}',t)\cdot\boldsymbol{\nabla}]\boldsymbol{\nabla}(R^{n+1}f_{n+2})}{(n+1)(n+2)}. \tag{18}$$

The first two terms in Eq. (17) may now be evaluated explicitly with the help of Eq. (14), to give

$$\mathbf{F}_s^0=-\frac{1}{2c^2}\int d^3x\rho(\mathbf{x},t)\int d^3x'\frac{\partial}{\partial t}\left[\frac{1-e^{-R/l}}{R}\mathbf{j}(\mathbf{x}',t)+\left(1-e^{-R/l}-\frac{R}{l}e^{-R/l}\right)\frac{\mathbf{j}\cdot\mathbf{R}}{R^3}\mathbf{R}\right] \tag{19}$$

and

$$\mathbf{F}_s^1=\frac{2}{3c^3}\int d^3x\rho(\mathbf{x},t)\int d^3x'\frac{\partial^2}{\partial t^2}\mathbf{j}(\mathbf{x}',t). \tag{20}$$

For a rigid charge distribution the current density is $\mathbf{j}(\mathbf{x}',t)=\rho(\mathbf{x}',t)\mathbf{v}(t)$; therefore, $\mathbf{j}\cdot\mathbf{R}=\rho\mathbf{v}\cdot\mathbf{R}$. Furthermore, for a spherically symmetric charge distribution only the component of $\mathbf{R}$ parallel to $\mathbf{v}$ can survive the integration. In this case $(\mathbf{v}\cdot\mathbf{R})^2$ can be replaced by its mean value $v^2R^2/3$. Since at low velocities we can neglect terms nonlinear in v, the expressions for $\mathbf{F}_s^0$ and $\mathbf{F}_s^1$ become, respectively,

$$\mathbf{F}_s^0=-\frac{2}{3}\frac{\dot{\mathbf{v}}}{c^2}\int d^3x\rho(\mathbf{x},t)\int d^3x'\rho(\mathbf{x}',t)\times\left[\frac{1}{R}(1-e^{-R/l})-\frac{1}{4l}e^{-R/l}\right] \tag{21}$$

and

$$\mathbf{F}_s^1=\frac{2}{3}\frac{\ddot{\mathbf{v}}}{c^3}\int d^3x\rho(\mathbf{x},t)\int d^3x'\rho(\mathbf{x}',t)=\frac{2e^2}{3c^3}\ddot{\mathbf{v}}, \tag{22}$$

where e is the total charge of the particle. The $\mathbf{F}_s^1$ term is independent of the size of the particle and reproduces the radiation reaction force found in classical electrodynamics.

III. LORENTZ COVARIANCE PROPERTIES OF THE MODEL

Let us consider now the structure of the $\mathbf{F}_s^0$ term. It depends on the radius a of the charged particle as well as on the cutoff l. If we first take the limit $l\to 0$, we see that the double integral becomes proportional to the electrostatic self-energy (2) of the charge distribution. Then, $-\mathbf{F}_s^0$ will yield the Abraham-Lorentz term which appears in Eq. (1).

In order to obtain the general expression for the electrostatic self-energy when $l\neq 0$, we recall that the corresponding energy can be written in a positive definite form as [10]

$$U(a,l)=\frac{1}{8\pi}\int d^3x[\mathbf{E}\cdot\mathbf{E}+l^2(\boldsymbol{\nabla}\cdot\mathbf{E})^2], \tag{23}$$

where $\mathbf{E}=-\boldsymbol{\nabla}\phi$. Making use of the relations (4) and (5), Eq. (23) can easily be shown to be equivalent to the expression

$$U(a,l)=\frac{1}{2}\int d^3x\int d^3x'\rho(\mathbf{x})\rho(\mathbf{x}')\frac{1}{R}[1-e^{-R/l}], \tag{24}$$

which reduces to Eq. (2) in the limit $l\to 0$. On the other hand, if the point particle limit $a\to 0$ is taken first, U approaches the finite value $e^2/2l$. In this case, the second part of the double integral in Eq. (21) removes the factor 4/3, so that we obtain

$$-\mathbf{F}_s^0(0,l)=\frac{4}{3}\left(1-\tfrac{1}{4}\right)U(0,l)\frac{\dot{\mathbf{v}}}{c^2}=\frac{U(0,l)}{c^2}\dot{\mathbf{v}}. \tag{25}$$

This has the correct form required by special relativity, since the electrostatic self-energy divided by c^2 can be identified with the regularized electromagnetic mass of a point charge, $m_{\mathrm{el}}=e^2/2lc^2$.

To see more clearly the origin of this behavior, we write Eq. (21) in the form

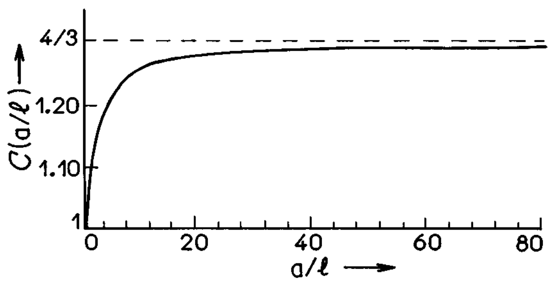

FIG. 2. Behavior of the coefficient $\mathcal{C}(a/l)$ as a function of the radius of the charged particle.

$$-\mathbf{F}_s^0(a,l)=\mathcal{C}\left(\frac{a}{l}\right)\frac{U(a,l)}{c^2}\dot{\mathbf{v}}, \tag{26}$$

where the coefficient $\mathcal{C}$ is a dimensionless function of a/l. It may be evaluated explicitly given a specific model for the charge distribution. For example, if the charged particle is regarded as a spherical shell of charge of radius a, then we find from Eq. (21) that

$$\mathcal{C}\left(\frac{a}{l}\right)=\frac{5}{3}+\frac{1/3}{[\exp(-2a/l)-1]^{-1}+l/2a}. \tag{27}$$

This monotonic function is illustrated in Fig. 2.

We see that the coefficient $\mathcal{C}(a/l)$ interpolates smoothly between the point particle limit in the regularized theory, when $\mathcal{C}(0)=1$, and the result obtained for an extended charged particle in the Maxwell theory, when $\mathcal{C}(\infty)=4/3$. It is important to note that unless the radius of the particle vanishes, U/c^2 does not appear in Eq. (26) with the correct coefficient required by Lorentz covariance. This occurs in consequence of the assumed existence of a rigid extended particle, which is incompatible with the special theory of relativity.

Higher-order terms in the expansion (17) give for a point charged particle contributions of the form

$$\mathbf{F}_s^n=\frac{b_n e^2}{n!c^{n+2}}l^{n-1}\frac{d^{n+1}}{dt^{n+1}}\mathbf{v}, \tag{28}$$

where b_n are constants. These terms become relevant only when significant changes in motion occur in very short times of order l/c. In particular, the higher-order terms are essential in suppressing, when the electron bare mass is positive, the unphysical runaway solutions characterized by an exponential growth of the acceleration with time [9,13].

In the situations when such terms can be neglected, the nonrelativistic equation of motion of a point charged particle can be approximated as

$$\left[\frac{U(0,l)}{c^2}+m_0\right]\dot{\mathbf{v}}-\frac{2e^2}{3c^3}\ddot{\mathbf{v}}\simeq\mathbf{F}_{\text{ext}}, \tag{29}$$

where the electromagnetic mass m_{el} occurs in accord with the special theory of relativity. Since in practice the electromagnetic field cannot be separated from the bare particle, we combine m_{el} with the bare mass m_0 to get the empirical rest mass m. The equation of motion then takes the form

$$m\dot{\mathbf{v}}\simeq\frac{2e^2}{3c^3}\ddot{\mathbf{v}}+\mathbf{F}_{\text{ext}}, \tag{30}$$

which describes correctly the radiation damping observed in the classical domain.

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