Inelastic collapse of rotating spheres

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Inelastic hard spheres can undergo an infinite number of collisions in a finite time. This process is called inelastic collapse. We consider the effect of rotation in this process and find that in contrast to the case of nonrotating spheres, collapse is possible for even large values of the coefficient of restitution and the angle formed by the colliding particles can be large. The limit to the nonrotating case is discontinuous. The three-body problem is solved analytically under the flat surface approximation and many-particle simulations are carried out. [S1063-651X(96)15011-X]

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I. INTRODUCTION

Interest in granular materials has grown rapidly during the past several years. In addition to being a major engineering concern, the behavior of granular materials attracts the attention of physicists trying to understand the underlying physical mechanisms of such untraditional systems [1].

A granular system often exhibits long range correlations. Collective behavior like the formation of surface wave patterns has been observed [2,3], even though the grains interact only through collisions. The lack of long range interactions makes the occurrence of correlations nontrivial. One essential quality of granular systems is that they are dissipative. A character of dissipative dynamical systems is the appearance of attractors in phase space. If such attractors are strong enough, collective behavior can show up.

Inelastic hard spheres are a simple but effective model for grains of granular materials. Since inelastic collisions between particles reduce the velocity differences between them, correlations can be naturally built up. In previous simulational investigations of inelastic hard particles [4,7], it is observed that an initially homogeneous system will generally evolve into one with clusters of particles and voids. A shearing mode was identified [4,8], in which particles in a cluster have about the same velocities. This can be understood intuitively in the following way: the particles in a cluster collide very frequently with one another, so the velocity differences among them diminish due to inelasticity and the particles acquire about the same velocity which is just the average velocity of the cluster. Furthermore, since the cluster picks up a particular direction from this average velocity, the rotational symmetry of the space is broken, and consequently, the cluster forms a stringy structure [4]. All this suggests that there exist correlations among particles which are manifested as cluster formation in a spatial form, or as the shear mode described above in a velocity form.

Correlations are built up through inelastic collisions between particles. An extreme case is of great interest: the collapse state in which several particles collide an infinite number of times in a finite time interval [5–11]. Previous analytic work [11] showed that an attractor exists for a collapse state, so that spatial and velocity correlations are established for participating particles which were unrelated before the event. The method can be extended to arbitrary masses for which collapse is possible when $\cos\theta < 2\sqrt{r(1+m_0/m_1)(1+m_0/m_2)}/(1+r)$; where θ is the angle between the three particles and r the coefficient of restitution. Besides being a singularity, collapse also has the interesting feature that the spheres form a linear structure [7].

The previous calculation [11] was for nonrotating particles. We want to investigate the effects of rotation on the collapse process for an obvious reason: physical grains can rotate. Such consideration brings significant changes due to the coupling between velocity components. The question remains if a transverse coefficient of restitution makes the same important difference to real materials.

In some sense, inelastic collapse is a consequence of the hard sphere model. Generally, event driven computations used for hard particles require much less computer resources than molecular dynamics methods for soft particles. Unless the density is extremely high, the deformation of a particle is not an important factor, and hard spheres provide an effective model. Inelastic collapse requires an infinite number of collisions, but the model will break down when the particles get close enough to one another. Nevertheless, from the above discussion we see that the physical implications of inelastic collapse are not limited to the hard sphere model.

We discuss the friction law and set up the problem in Sec. II. In Sec. III we derive analytically the conditions on the parameters for which collapse of three rotating particles is possible. Section IV is a numerical study of the collapse probabilities. In Sec. V we carry out many-particle simulations and compare them to those with nonrotating particles by McNamara and Young [7,8].

II. THE COLLISION MODEL

The energy loss during a collision between two inelastic, rotating particles is usually attributed to two effects. The relative velocity of the colliders along the line of centers is multiplied by $-r$ as a result of the collision. The coefficient of restitution, r, is a number between zero and one. There is also a reduction of the relative velocity in the tangential direction due to friction.

What sort of friction law should we assume for the pro-

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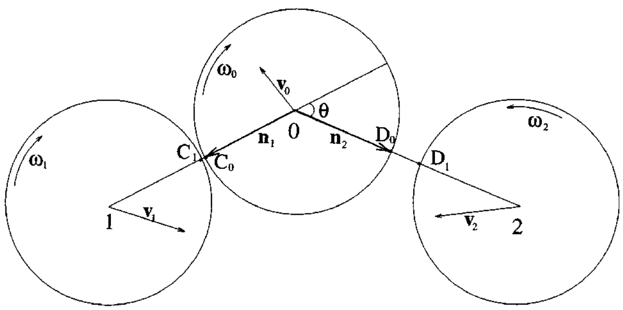

FIG. 1. Configuration of particles.

cess? Quite generally, the tangential frictional force can be expressed to first order as $f(v_r,v_t)=(\partial f/\partial v_r)v_r+(\partial f/\partial v_t)v_t$. Or rewritten, $\Delta v_t=\mu v_r+(\eta-1)v_t$ if $|\Delta v_t|<|v_t|$, since the change in tangential velocity may not exceed the tangential velocity itself. The case of nonrotating particles corresponds to $\eta=1$ and $\mu=0$, since the rotation then does not interfere with the collapse process. If $\eta=1$ the relative tangential velocity will not vanish after an infinite number of collisions which is not physical, so we will suppose $\eta<1$. For simplicity we set $\mu=0$, because the following calculation can be readily modified for nonzero μ. The collision rules are hence

$$v_r'=-rv_r \quad \text{and} \quad v_t'=\eta v_t, \tag{1}$$

with $0\leqslant r\leqslant 1$ and $-1<\eta<1$.

We look at a situation in which one particle (numbered zero) takes part in all collisions, the other two (numbered one and two) are alternatively collider and spectator. We assume all particles are identical, with unit radii and masses, and have a momentum of inertia I with respect to a tangential line ($I=1.5$ for a uniform disk, and $I=1.4$ for a uniform sphere).

Let $\vec{v}_j$, $\vec{x}_j$, and $\vec{\omega}_j$ be the velocity, the position, and the angular velocity of the jth particle at the instant before a collision occurs. Also assume $\vec{n}_j$ is a unit vector pointing to the center of particle j from the center of particle 0 (Fig. 1). Let particle 1 collide with particle 0. We denote the quantity after collision by a prime. Then from momentum conservation,

$$\vec{v}_1'+\vec{v}_0'=\vec{v}_1+\vec{v}_0. \tag{2}$$

Since we can suppose that the interaction force is acting on the contact point, the angular momentum of each particle is conserved with respect to that point:

$$I\vec{\omega}_1'+\vec{n}_1\times\vec{v}_1'=I\vec{\omega}_1+\vec{n}_1\times\vec{v}_1,$$

$$I\vec{\omega}_0'-\vec{n}_1\times\vec{v}_0'=I\vec{\omega}_0-\vec{n}_1\times\vec{v}_0. \tag{3}$$

And from Eqs. (1),

$$(\vec{v}_1'-\vec{v}_0')\cdot\vec{n}_1=-r(\vec{v}_1-\vec{v}_0)\cdot\vec{n}_1,$$

$$(\vec{v}_1'\times\vec{n}_1+\vec{\omega}_1'-\vec{v}_0'\times\vec{n}_1+\vec{\omega}_0')\times\vec{n}_1$$

$$=\eta(\vec{v}_1\times\vec{n}_1+\vec{\omega}_1-\vec{v}_0\times\vec{n}_1+\vec{\omega}_0)\times\vec{n}_1. \tag{4}$$

Certainly there are no changes for particle 2,

$$\vec{v}_2'=\vec{v}_2, \quad \vec{\omega}_2'=\vec{\omega}_2. \tag{5}$$

Equations (2)–(5) determine the configuration after a collision from the configuration before it. They are complemented by the equations corresponding to the positions of the particles at the next collision,

$$\vec{x}_0'=\vec{x}_0+\vec{v}_0't,$$

$$\vec{x}_1'=\vec{x}_1+\vec{v}_1't,$$

$$\vec{x}_2'=\vec{x}_2+\vec{v}_2't. \tag{6}$$

The time interval between the two collisions, t, is such that the magnitude of $\vec{x}_2'-\vec{x}_0'$ is 2. The above equations can then be applied to the next collision between particle 0 and particle 2, by interchanging the subscripts 1 and 2.

III. COLLAPSE OF THREE ROTATING PARTICLES

During inelastic collapse, the relative velocities of the contact points of the colliding particles approach zero as the collision number diverges. Suppose that just before a collision between particle 0 and particle 1, the contact point on particle 1, C_1, has a velocity $\vec{u}_1$ relative to the contact point on particle 0, C_0, and point D_1 has a velocity $\vec{u}_2$ relative to D_0 (Fig. 1). Then after this collision, from Eqs. (1),

$$\vec{u}_1'\cdot\vec{n}_1=-r(\vec{u}_1\cdot\vec{n}_1),$$

$$\vec{u}_1'\times\vec{n}_1=\eta(\vec{u}_1\times\vec{n}_1). \tag{7}$$

In order to calculate $\vec{u}_2'$, we need the change of the velocity of point D_0 during this collision since there is no change of the velocity of D_1. From Eqs. (2)–(5), we have

$$\Delta\vec{\omega}_0=-\frac{1-\eta}{2(I+1)}\vec{u}_1\times\vec{n}_1,$$

$$\Delta\vec{v}_0=\frac{1+r}{2}(\vec{u}_1\cdot\vec{n}_1)\vec{n}_1-\frac{I(1-\eta)}{2(I+1)}(\vec{u}_1\times\vec{n}_1)\times\vec{n}_1.$$

These yield

$$\vec{u}_2'=\vec{u}_2-\Delta\vec{v}_0-\Delta\vec{\omega}_0\times\vec{n}_2$$

$$=\vec{u}_2-\frac{1+r}{2}(\vec{u}_1\cdot\vec{n}_1)\vec{n}_1+\frac{I(1-\eta)}{2(I+1)}(\vec{u}_1\times\vec{n}_1)\times\vec{n}_1$$

$$+\frac{1-\eta}{2(I+1)}(\vec{u}_1\times\vec{n}_1)\times\vec{n}_2. \tag{8}$$

We have established the recursion relation for the $\vec{u}_j$ values. In order to find linearized expressions, we introduce the ''flat surface approximation,'' following the procedure of the calculation for the nonrotating case [11]. Introduce the superscripts, c, denoting the collider, and s, denoting the spectator particle, as well as a subscript, i, to denote the instant

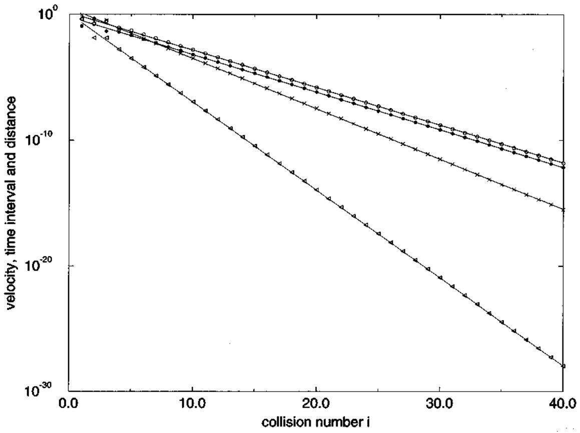

FIG. 2. Comparison of simulational results (● for $|V_i^c|$, ○ for $|\vec{u}_i^c \times \vec{n}_i^c|$, $\times$ for t_i, $\triangleleft$ for d_i) with the theoretical predictions (solid lines) for the case $r=0.1$, $\eta=0.05$, $\theta=84.8°$, and $I=1.5$.

before the ith collision occurs. Notice that a collider becomes a spectator immediately following a collision.

Since the collapse process corresponds to a very short time interval, we can neglect the t terms in Eqs. (6) and then $\vec{n}_j' = \vec{n}_j$, or

$$\vec{n}_{i+1}^c = \vec{n}_i^s, \quad \vec{n}_{i+1}^s = \vec{n}_i^c .$$

If we define

$$V_i^c \equiv \vec{u}_i^c \cdot \vec{n}_i^c , \quad U_i^c \equiv \vec{u}_i^c \cdot \vec{n}_i^s ,$$

$$V_i^s \equiv \vec{u}_i^s \cdot \vec{n}_i^s , \quad U_i^s \equiv \vec{u}_i^s \cdot \vec{n}_i^c ,$$

then relations (7) and (8) imply the following recursion relation (θ is the angle indicated in Fig. 1):

$$\begin{bmatrix} V_{i+1}^c \\ U_{i+1}^c \\ V_{i+1}^s \\ U_{i+1}^s \end{bmatrix} = \begin{bmatrix} \left(\frac{1+r}{2} - \frac{I(1-\eta)}{2(I+1)}\right)\cos\theta & \frac{I(\eta-1)}{2(I+1)} & 1 & 0 \\ -\frac{1+r}{2} + \frac{1-\eta}{2(I+1)}\cos\theta & \frac{1-\eta}{2(I+1)} & 0 & 1 \\ -r & 0 & 0 & 0 \\ (r+\eta)\cos\theta & \eta & 0 & 0 \end{bmatrix} \begin{bmatrix} V_i^c \\ U_i^c \\ V_i^s \\ U_i^s \end{bmatrix} .$$

The characteristic polynomial of the above matrix is

$$\lambda^4 - \left[\frac{1+r}{2}\cos\theta + \frac{(1-\eta)(1-I\cos\theta)}{2(I+1)}\right]\lambda^3 + \left[r-\eta+\frac{(1+r)(1-\eta)}{4(I+1)}(\cos\theta - I)\right]\lambda^2 + \left[\eta\frac{1+r}{2}\cos\theta + \frac{r(1-\eta)(I\cos\theta-1)}{2(I+1)}\right]\lambda - \eta r = 0. \quad (9)$$

The long term behavior of the system is determined by the eigenmode of the eigenvalue with maximum modulus. A collapse attractor corresponds to the situation when this eigenvalue is real and within (0,1). The situation is very similar to the calculation in [11]. U_i^c, V_i^c, the time interval between collisions t_i, and the minimum distance between the spectator and particle 0 decrease exponentially with increasing collision number i. More specifically,

$$U_i^c \sim \lambda^i ,$$

$$V_i^c \sim \lambda^i ,$$

$$t_i \sim \left(\frac{r}{\lambda^2}\right)^i ,$$

$$d_i \sim \left(\frac{r}{\lambda}\right)^i .$$

Simulational results agree precisely with these ratios (Fig. 2).

The ''flat surface approximation'' is only valid when the time interval t_i decreases faster than the radial components of relative velocities U_i^c, so we need another condition, $\lambda^3 > r$.

We solve Eq. (9) numerically and give the result in graphical form. There are four parameters: r, η, I, and θ. We fix $I=1.5$, corresponding to homogeneous two-dimensional disks, and plot the collapse region in Fig. 3. If r is below the border shown, then collapse can occur. For comparison, the nonrotating case is shown by a dotted line at

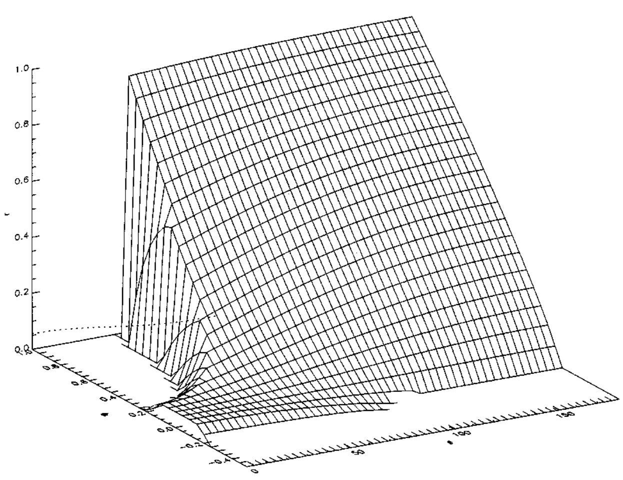

FIG. 3. Parameter region where collapse can occur, $I=1.5$. For comparison, the dotted line at $\eta=1$ shows the collapse regime for the nonrotating case.

$\eta=1$. Our result strongly contrasts with the nonrotating case. (i) Collapse can occur even for high values of r. In fact, r can be arbitrarily close to 1 (without rotation collapse requires $r<9-4\sqrt{5}\approx 0.056$). (ii) Collapse can occur for $\theta>90°$. (Figure 3 is extended beyond 120°, which is prohibited.) (iii) The rotating case does not go continously to the nonrotating case in the limit $\eta\to 1$. (iv) There is a prohibited parameter region at small angles and large η. Slight tangential friction prohibits collapse for small θ while allowing it for large θ. (However, the particles can collide many times in this region before they separate.)

The transition to the nonrotating case

When $\eta=1$, Eq. (9) reduces to the nonrotating case. Two eigenvalues are then ± 1. Their eigenmodes correspond to purely tangential velocities. Since these modes do not couple with the radial velocities, they do not interfere with the collapse process. The other two eigenvalues are

$$k_{\pm}=\frac{1+r}{4}\cos\theta \pm \sqrt{\left(\frac{1+r}{4}\cos\theta\right)^2-r}, \tag{10}$$

the relevant fixed points of the nonrotating problem.

An η different from 1 will introduce a coupling between the radial and tangential components of the velocities. We can treat the case $\eta\approx 1$ perturbatively. For the eigenvalues close to ± 1,

$$\pm\lambda_{\pm}=1-(1-\eta)\frac{1+\frac{3}{4}I+\frac{\cos\theta}{4}\mp\frac{1+\cos\theta}{2}}{(2\mp\cos\theta)(I+1)}+O(1-\eta)^2.$$

These values are independent of r. For collapse, the dominant eigenmode must correspond to a positive eigenvalue, i.e.,

$$\cos\theta>\frac{-3I+\sqrt{9I^2+16}}{2}. \tag{11}$$

In Fig. 3 this critical angle is $\theta=40.5°$.

The existence of additional eigenvalues, stemming from additional degrees of freedom (rotation), widens the parameter region for which collapse is possible.

IV. COLLAPSE PROBABILITIES

Figure 3 shows the region of parameters where collapse can occur. For three rotating particles, collapse is possible even for coefficients of restitution arbitrarily close to 1. This is a feature qualitatively different from the nonrotating case. However, from the requirement $\lambda^3>r$, we see that the decrease rate of the time intervals between collisions, r/λ^2, is rather slow when r is very close to unity. This means it takes a long time to reach the singularity. Consequently, the domain of appropriate initial conditions is small. We expect that collapse is very unlikely under such circumstances.

We investigate the collapse probabilities through numerical simulations. For each pair of r and η, 10^7 simulations are carried out with random initial conditions. A simulation ends either when the particles fly apart or when inelastic collapse occurs. The number of collapse events provides the probability information. Of course, the probability depends on the spatial area over which initial positions of particles are taken, or the density of the particles—it is roughly proportional to this density when the density is not very high. For fixed density, the probability is not sensitive to the distribution of initial data.

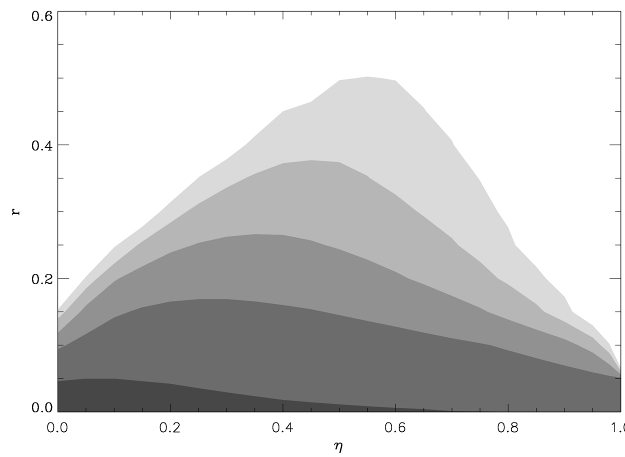

FIG. 4. Relative collapse probabilities as a function of r and η; $I=1.5$. Each gray scale corresponds to a collapse probability which is 10 times smaller than the previous one. In the white region we have detected no or very few events (out of 10^7 initial conditions per pair of parameter values).

Here we use the ''separation'' criterion for collapse [7]: if immediately after one collision the next collision involves two particles whose separation is less than 10^{-12}, the system is considered to have collapsed. This will inevitably include ''almost collapses'' in which the particles come very close but separate before the singularity is reached. One may not be able to or not wish to distinguish physically these events from collapses. Cases are observed where parameters are outside the collapse region, but simulations end due to the ''separation'' criterion. Further detailed investigation shows such cases will not lead to a singularity—there will not be an infinite number of collisions. Naturally, such situations also

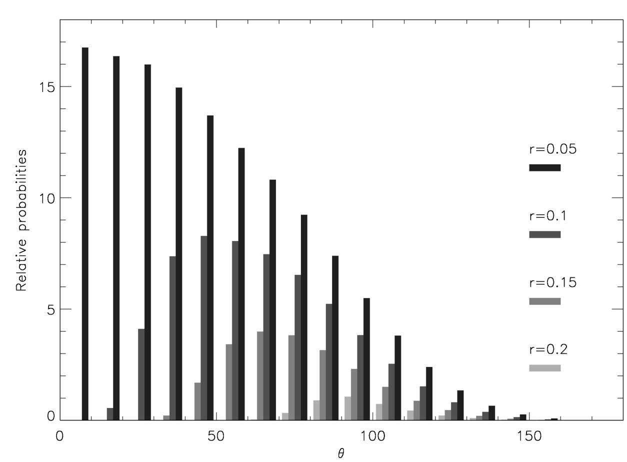

FIG. 5. Relative collapse probabilities as a function of θ for different values of r. $\eta=0.2$, $I=1.5$. θ is divided into bins of 10°.

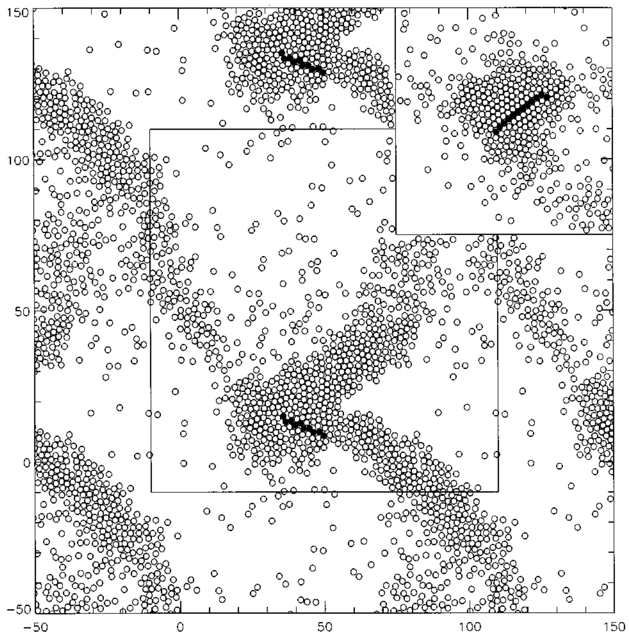

FIG. 6. A configuration of 1024 inelastic disks with $r=0.7$, $\eta=0.3$, $I=1.5$, and the volume fraction $\nu=0.22$ when collapse happens, or more accurately, when the ''separation'' criterion is satisfied. The solid disks are those involved in the last 200 collisions. The linear collapse structure for the nonrotating case is shown in the upper right box, which is from a simulation of 1024 nonrotating disks with $r=0.6$, $\nu=0.22$.

occur when parameters are inside the collapse region. The collapse probabilities calculated using the ''separation'' criterion are thus generally larger than the real probabilities.

Figure 4 shows the collapse probabilities for different r and η accumulated over θ. As expected, collapses for large r are very unlikely. One sees a strong dependence on r and a weaker dependence on η. Figure 5 shows the probabilities as a function of θ for different values of r. Collapses for large θ are unlikely. Small θ can be unlikely too. We clearly see that the most probable θ can be rather different from 0°. It can even be larger than 90°.

V. MANY-PARTICLE SIMULATIONS

From previous sections, we see that collapse between rotating particles is quite different from that of the nonrotating particles. What does rotation change in the many-particle collapse process?

We performed simulations of a two-dimensional system containing 1024 inelastic disks with periodic boundary conditions. The volume fraction occupied by the disks is kept at 0.22. Simulations for various values of r and η are carried out. For the nonrotating case, the critical value of the coefficient of restitution for collapse to happen is about $r=0.62$ [7]. Collapses for rotating particles are observed for r as high as 0.75. The participants of a collapse generally form a linear structure, as in [7]. However, an interesting feature is observed: instead of forming a straight line, as nonrotating particles do, rotating particles tend to form a wiggled line when collapse happens. This corresponds qualitatively to the above result that collapse between three particles can have a nonzero most likely θ. A typical collapse configuration is shown in Fig. 6.

VI. CONCLUSIONS

We demonstrated analytically the existence of inelastic collapse for three rotating particles and found that rotation changes the collapse conditions dramatically. Collapse is now possible for large coefficients of restitution r and large angles. The likelihood of collapse, however, decreases rapidly with increasing r. The most likely angle between collapsing particles is often not 0°. In many-particle simulations wiggled collapse structures are observed.

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