Fractal dimensions of chaotic saddles of dynamical systems

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A formula, applicable to invertible maps of arbitrary dimensionality, is derived for the information dimensions of the natural measures of a nonattracting chaotic set and of its stable and unstable manifolds. The result gives these dimensions in terms of the Lyapunov exponents and the decay time of the associated chaotic transient. As an example, the formula is applied to the physically interesting situation of filtering of data from chaotic systems. [S1063-651X(96)11811-0]

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I. INTRODUCTION

Kaplan and Yorke [1] conjectured a formula relating the fractal dimension of a chaotic attractor of a typical [2] N-dimensional dynamical system to the Lyapunov exponents of the attractor. More precisely, the formula gives the information dimension of the natural measure on the attractor (the natural measure is informally defined in Sec. II). Although the conjecture is still unproved in its most general form, some relevant rigorous results have been obtained. For the case of a two-dimensional invertible map ($N=2$), Young [3] proved that the information dimension of an ergodic invariant measure μ is

$$D(\mu)=(h_1^{-1}-h_2^{-1})H, \tag{1}$$

where H is the metric entropy of μ and $h_1 \geqslant 0 \geqslant h_2$ are the Lyapunov exponents for the measure μ. The invariant set in this case may or may not be an attractor. In the case where μ is the natural measure of an attractor, it is reasonable to assume that $H=h_1$ (this can be proved for hyperbolic attractors), and Eq. (1) then agrees with the $N=2$ version of the Kaplan-Yorke formula. In [4] it was shown that the Kaplan-Yorke formula provides a rigorous upper bound for the dimension of a chaotic attractor of an N-dimensional map (the question of whether the upper bound is actually attained remains open). Reference [5] uses a notion of ''partial dimensions'' to relate the Lyapunov exponents and entropy of an invariant measure μ (again not necessarily attracting) for general N to its dimension [as in (1) for $N=2$]. Reference [6] proves the Kaplan-Yorke conjecture for N-dimensional *iterated function systems*. (In contrast to a dynamical system, in an iterated function system, at each iterate, the map to be applied is drawn at random from a prespecified ensemble.)

In addition to attractors, nonattracting invariant sets are also of interest in a variety of situations. In particular, nonattracting invariant sets are responsible for such physical phenomena as fractal basin boundaries, chaotic transients, and chaotic scattering. Thus the dimensions of the natural measures of these sets and of their stable and unstable manifolds have attracted attention. In the case $N=2$ Refs. [7,8] use heuristic arguments to relate these dimensions to the Lyapunov exponents $h_1>0>h_2$ and the characteristic decay time τ of the associated chaotic transient. The results of Refs. [7,8] correspond to (1) with $H=h_1-(1/\tau)$. Motivated by the consideration of the problem of chaotic scattering, the case of general N was treated in [9], but was restricted to consideration of Hamiltonian systems.

The purpose of this paper is to present a heuristic derivation of a formula for the dimensions of the natural measures of a general invariant set and its stable and unstable manifolds. In the case where the set is an attractor, the Kaplan-Yorke formula is recovered. In the case where the set is nonattracting, the $N=2$ results of Refs. [7,8] and the general N-dimensional Hamiltonian results of [9] are recovered. The present formula also covers situations of typical nonattracting sets not covered by previous results, and these previously untreated cases have relevance to physical situations. We discuss one such physical example in Sec. II, acausal filtering of signals from chaotic systems [10]. Another example (not discussed here) is the convection through a ''scattering region'' of passive tracers by a three-dimensional time-dependent incompressible fluid flow [11].

II. DIMENSION FORMULAS

We consider a chaotic invariant ergodic set of an N-dimensional invertible smooth map [12]. Imagine that we enclose the invariant set by an N-dimensional cube and that we sprinkle a very large number $n(0)$ of initial conditions uniformly throughout the cube. We now iterate each initial condition forward in time. If an initial condition leaves the cube we regard it as ''lost'' and no longer follow it. Let $n(t)$, $t>0$, denote the number of initial conditions that have not yet been lost at time t. Then we define the forward exponential decay time τ as

$$1/\tau=\lim_{t\to+\infty}\lim_{n(0)\to\infty} t^{-1}\ln[n(t)/n(0)]. \tag{2}$$

If we examine the location of the $n(t)$ orbit points in the cube at large positive time t, they will be in the close vicinity of the unstable manifold of the invariant set. Thus we define a natural measure μ_+ for the unstable manifold such that the measure $\mu_+(C)$ of a small volume C within the cube is

$$\mu_+(C)=\lim_{t\to+\infty}\lim_{n(0)\to\infty} n_+(t,C)/n(t), \tag{3}$$

where $n_+(t,C)$ is the number of the $n(t)$ orbit points in the cube at time t that are also in C. We can also define a natural measure for the stable manifold by

$$\mu_-(C)=\lim_{t\to+\infty}\lim_{n(0)\to\infty} n_-(t,C)/n(t), \tag{4}$$

where $n_-(t,C)$ is the number of initial conditions in C that do not leave the cube before time t. To define the natural measure on the invariant set itself, we consider the $n(t)$ orbits that do not leave the cube before time t and ask where they were located at some intermediate time ξt, where $0<\xi<1$ (e.g., we might take $\xi=1/2$). Letting $n_0(\xi,t,C)$ denote the number of these orbits that are in C at time ξt, we define the natural measure on the invariant set as

$$\mu_0(C)=\lim_{t\to+\infty}\lim_{n(0)\to\infty} n_0(\xi,t,C)/n(t), \quad 0<\xi<1. \tag{5}$$

[Note that with this notation $n_+(t,C)=n_0(1,t,C)$ and $n_-(t,C)=n_0(0,t,C)$.] The natural measure of the invariant set will have N associated Lyapunov exponents that characterize the stretching or compression of differential volumes following orbits generated by those initial conditions sprinkled in the cube that do not leave for a large number of forward iterates. Let U (for unstable) denote the number of positive Lyapunov exponents and let S (for stable) denote the number of nonpositive exponents; then $U+S=N$. We label the exponents as

$$h_U^{(+)}\geq h_{U-1}^{(+)}\geq\cdots\geq h_1^{(+)}>0\geq -h_1^{(-)}\geq -h_2^{(-)} \geq\cdots\geq -h_S^{(-)}, \tag{6}$$

where we have arranged the exponents in decreasing order, starting with the largest positive exponent on the left and ending with the most negative exponent on the right. Note that in this notation $h_j^{(+,-)}$ are all non-negative and that smaller values of the subscripts j correspond to values of $h_j^{(+,-)}$ closer to zero.

Making the simplifying assumption that the invariant set is hyperbolic, it is appropriate to conceptualize the edges of the enclosing cube as parallel to directions of stretching and compression by the Lyapunov exponents $h_j^{(+)}$ and $-h_i^{(-)}$. We also suppose that, by suitable normalizations, we can take the cube edges to be of unit length. Iteration of the sprinkled points forward in time then results in a distribution restricted to slabs within the cube, where these slabs have dimensions

$$1\times 1\times\cdots\times 1\times e^{-h_1^{(-)}t}\times e^{-h_2^{(-)}t}\times\cdots\times e^{-h_S^{(-)}t} \tag{7}$$

and there are U slab edge dimensions of unit length. Let $N(t)$ denote the number of these slabs. For large t we write

$$N(t)\sim e^{Ht},$$

where we call H the forward entropy. Mapping these $N(t)$ slabs backward t iterates, we obtain $N(t)$ slabs of initial conditions each of dimension,

$$e^{-h_U^{(+)}t}\times e^{-h_{U-1}^{(+)}t}\times\cdots\times e^{-h_1^{(+)}t}\times 1\times\cdots\times 1, \tag{8}$$

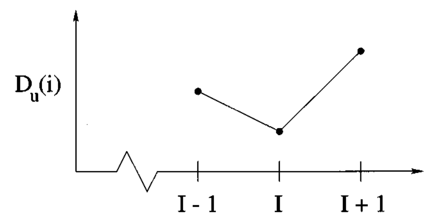

FIG. 1. $D_u(i)$ versus i around $i=I$.

where there are S slab edge dimensions of unit length. Since these are the locations of the initial conditions that have not left the cube in t iterates, we have

$$N(t)\exp\left(-\sum_{j=1}^{U} h_j^{(+)}t\right)\sim\exp(-t/\tau).$$

Thus

$$H=\left(\sum_{j=1}^{U} h_j^{(+)}\right)-1/\tau. \tag{9}$$

To obtain the dimension D_u of the natural measure of the unstable manifold, we wish to cover the $N(t)$ slabs of dimensions given by (7) by small N-dimensional cubes. Let the edge length of one of these cubes be

$$\epsilon_i=\exp(-h_{(i+1)}^{(-)}t). \tag{10}$$

The required number of cubes is

$$\left(\frac{1}{\epsilon_i}\right)^U\left(\frac{e^{-h_1^{(-)}t}}{\epsilon_i}\right)\left(\frac{e^{-h_2^{(-)}t}}{\epsilon_i}\right)\cdots\left(\frac{e^{-h_i^{(-)}t}}{\epsilon_i}\right)N(t).$$

Since $\epsilon_i\to 0$ as $t\to+\infty$, the box-counting definition of dimension, $\lim_{\epsilon\to 0}\ln[\#(\epsilon)]/\ln(1/\epsilon)$, where $\#(\epsilon)$ is the number of ϵ cubes in the covering, yields an estimate for the dimension [13],

$$D_u(i)=U+i+[H-(h_1^{(-)}+h_2^{(-)}+\cdots+h_i^{(-)})]/h_{i+1}^{(-)}. \tag{11}$$

Since the covering by ϵ_i cubes may not be optimal, Eq. (11) is an upper bound on the dimension: $D_u\leq D_u(i)$. To obtain the best upper bound, we minimize $D_u(i)$ over the index i (i.e., over the possible edge sizes ϵ_i). Since the choices of edge length given by (10) appear to be the most natural choices, it is reasonable to conjecture that the minimum over i gives the true value of D_u (in the attractor case this assumption yields the Kaplan-Yorke conjecture). To find the minimum of (11), consider the quantity $D_u(i+1)-D_u(i)$,

$$D_u(i+1)-D_u(i)=\left(\frac{1}{h_{i+1}^{(-)}}-\frac{1}{h_{i+2}^{(-)}}\right)\times[(h_1^{(-)}+h_2^{(-)}+\cdots+h_{i+1}^{(-)})-H].$$

Since $h_{i+1}^{(-)}\leq h_{i+2}^{(-)}$ [see (6)], the dimension increases (decreases) if the term in the square brackets is positive (negative). Thus the minimum occurs at that value of i (denoted I) such that (see Fig. 1)

$$h_1^{(-)}+\cdots+h_I^{(-)}+h_{I+1}^{(-)}\geq H\geq h_1^{(-)}+\cdots+h_I^{(-)} \quad (12)$$

and the conjecture is

$$D_u=D_u(I). \quad (13)$$

Proceeding in the same way, we obtain the following result for the information dimension of the stable manifold:

$$D_s(j)=S+j+[H-(h_1^{(+)}+h_2^{(+)}+\cdots+h_j^{(+)})]/h_{j+1}^{(+)}, \quad (14)$$

$$D_s=D_s(J), \quad (15)$$

where J is defined by

$$h_1^{(+)}+\cdots+h_J^{(+)}+h_{J+1}^{(+)}\geq H\geq h_1^{(+)}+\cdots+h_J^{(+)}. \quad (16)$$

Since the invariant set lies in both its stable manifold and its unstable manifold, the invariant set is the intersection of these two manifolds. Assuming this intersection to be generic, we have that the dimension of the invariant set is [14]

$$D=D_u+D_s-N=(I+J)+\left(H-\sum_{i=1}^{I} h_i^{(-)}\right)(h_{I+1}^{(-)})^{-1}+\left(H-\sum_{j=1}^{J} h_j^{(+)}\right)(h_{J+1}^{(+)})^{-1}. \quad (17)$$

As a check, consider the case of the attractor. In this case $\tau=\infty$ and (9) and (16) yield $J=U-1$ and $D_s=S+U=N$. (Formally we could also take $J=U$ and get the same value for D_s, though $h_{U+1}^{(+)}$ is undefined.) Equation (17) then gives $D=D_u$, which with (9) is just the Kaplan-Yorke formula.

Finally, we note that our definitions of the natural measures, Eqs. (3)–(5), are not the only possible definitions [15], but (3)–(5) appear to be the most natural choices and, perhaps more importantly, they are the relevant measures for our considerations in Sec. III and for the fluid tracer problem mentioned in Sec. I.

III. FILTERING OF DATA FROM CHAOTIC SYSTEMS

Badii *et al.* [16] consider the effect of filtering on the dimension measured from a time series generated by a dynamical system and show how to compute, for an ideal low-pass or high-pass filter, the amount by which filtering increases the dimension of the attractor reconstructed from the time series. More recently [10,17] there has been interest as well in the effect of acausal filters. In this section we present an acausal filter for which we can show that the ''attractor'' reconstructed from the filtered signal is actually a chaotic saddle for an associated dynamical system. We then apply the formula from the preceding section to compute the dimension increase due to filtering.

Consider a time series $\{x_n\}$ and an associated filtered time series

$$z_n=x_n+\sum_{k=1}^{\infty} \lambda_c^k x_{n-k}+\sum_{k=1}^{\infty} \lambda_a^k x_{n+k}, \quad (18)$$

where $0\leq\lambda_a,\lambda_c<1$. In general this filter is acausal; however, if $\lambda_a=0$ then we have a discrete version of the causal, low-pass filter studied by Badii *et al.* [16]. In that paper, it is observed that the dynamics reconstructed from a delay-coordinate embedding of the filtered signal $\{z_n\}$ is that of the dynamical system that generated $\{x_n\}$ coupled to linear contracting dynamics for the filtered coordinate z, given by the recursion

$$z_{n+1}=\lambda_c z_n+x_{n+1}. \quad (19)$$

Thus an additional Lyapunov exponent $-h_c=\ln\lambda_c$ is introduced to the dynamics by the causal filter and the information dimension D of the filtered attractor is given by the Kaplan-Yorke formula, using the Lyapunov exponents of the original system together with the new exponent $-h_c$. Thus, in general, $d\leq D\leq d+1$, where d is the information dimension of the unfiltered attractor.

For the acausal filter (18) with λ_a and λ_c nonzero, there is no analog to (19); that is, there is no equation relating z_{n+1}, z_n and any finite number of the $\{x_k\}$. We can, however, decompose $\{z_n\}$ into causal and acausal components $\{u_n\}$ and $\{v_n\}$, each of which satisfies a recursion such as (19). Specifically, we write $z_n=u_n+v_n$, where

$$u_n=\sum_{k=1}^{\infty} \lambda_c^k x_{n-k}, \quad (20)$$

$$v_n=\sum_{k=0}^{\infty} \lambda_a^k x_{n+k}. \quad (21)$$

Then

$$u_{n+1}=\lambda_c(u_n+x_n), \quad (22)$$

$$v_{n+1}=\lambda_a^{-1}(v_n-x_n). \quad (23)$$

Thus the dynamics reconstructed from the filtered signal $\{z_n\}$ (e.g., by use of delay coordinates) are those of the original system generating $\{x_n\}$ coupled with (22) and (23). The u and v dynamics are linear and result in Lyapunov exponents $-h_c=\ln\lambda_c$ and $h_a=\ln\lambda_a^{-1}$. Though the v dynamics are expanding, we know from (21) that $\{v_n\}$ is bounded and thus the dynamics of the filtered system confines itself to the chaotic saddle, which is repelling in the v direction and attracting in all other directions.

To apply the dimension formula (17) from Sec. II we must also determine the entropy H of the filtered dynamics. We show now that H is the same as for the unfiltered dynamics; that is, H is equal to the sum of the positive Lyapunov exponents of the unfiltered system. The filtered system has an additional positive Lyapunov exponent $h_a=\ln\lambda_a^{-1}$, but since the v dynamics (23) are linear with expanding eigenvalue λ_a^{-1}, trajectories are repelled from a neighborhood of the chaotic saddle with exponential decay time $\tau=1/h_a$; thus $h_a-1/\tau=0$ and the v dynamics have no net effect on the entropy of the system.

We thus make the following conclusions about the dimension increase due to the filter (18). Because H is unchanged by the filter, the effects of the additional Lyapunov exponents $-h_c$ and h_a on the dimension formula (17) can be

TABLE I. Dimension increase d_c as a function of the causal Lyapunov exponent h_c.

h_c	d_c
$h_c \geq h_{I+1}^{(-)}$	0
$h_{I+1}^{(-)} \geq h_c \geq \max(h_I^{(-)},\delta)$	$\frac{\delta}{h_c} - \frac{\delta}{h_{I+1}^{(-)}}$
$h_I^{(-)} \geq h_c \geq \delta$	$1 - \frac{h_c - \delta}{h_I^{(-)}} - \frac{\delta}{h_{I+1}^{(-)}}$
$h_c \leq \delta$	$1 - \frac{h_c}{h_{I+1}^{(-)}}$

treated separately. That is, we can write the dimension D of the chaotic saddle reconstructed from the filtered time series as

$$D = d + d_c + d_a, \tag{24}$$

where d is the dimension of the unfiltered attractor, d_c is the increase in the right side of (17) due to $-h_c$, and d_a is the increase due to h_a.

To describe the dimension increase d_c due to $-h_c$, let I be defined as in (12) and let $\delta = H - (h_1^{(-)} + \cdots + h_I^{(-)})$; then $0 \leq \delta \leq h_{I+1}^{(-)}$ and the portion of the dimension formula (17) for the unfiltered attractor due to the stable Lyapunov exponents is

$$I + \frac{\delta}{h_{I+1}^{(-)}}. \tag{25}$$

If $h_c \geq h_{I+1}^{(-)}$, then h_c has no effect on the dimension formula and $d_c = 0$. If $h_{I+1}^{(-)} \geq h_c \geq \max(h_I^{(-)},\delta)$, then h_c simply replaces $h_{I+1}^{(-)}$ in the dimension formula and thus $d_c = \delta/h_c - \delta/h_{I+1}^{(-)}$. If $h_I^{(-)} \geq h_c \geq \delta$ (this case is empty if $\delta > h_I^{(-)}$), then

$$h_1^{(-)} + \cdots + h_{I-1}^{(-)} + h_c \leq H \leq h_1^{(-)} + \cdots + h_{I-1}^{(-)} + h_c + h_I^{(-)}. \tag{26}$$

It does not matter whether h_c is greater or smaller than $h_{I-1}^{(-)}$; either way (26) bounds H between the sum of the magnitudes of the I smallest stable Lyapunov exponents of the filtered system and the sum of the $I+1$ smallest of these magnitudes. (Similar considerations apply to the cases discussed below.) Thus

$$d_c = \frac{H - (h_1^{(-)} + \cdots + h_{I-1}^{(-)} + h_c)}{h_I^{(-)}} - \frac{\delta}{h_{I+1}^{(-)}} = \frac{\delta + h_I^{(-)} - h_c}{h_I^{(-)}} - \frac{\delta}{h_{I+1}^{(-)}} = 1 - \frac{h_c - \delta}{h_I^{(-)}} - \frac{\delta}{h_{I+1}^{(-)}}. \tag{27}$$

Finally, if $h_c \leq \delta$, then

$$h_1^{(-)} + \cdots + h_I^{(-)} + h_c \leq H \leq h_1^{(-)} + \cdots + h_I^{(-)} + h_c + h_{I+1}^{(-)} \tag{28}$$

TABLE II. Dimension increase d_a as a function of the acausal Lyapunov exponent h_a.

h_a	d_a
$h_a \geq h_U^{(+)}$	0
$h_a \leq h_U^{(+)}$	$1 - \frac{h_a}{h_U^{(+)}}$

and thus

$$d_c = 1 + \frac{H - (h_1^{(-)} + \cdots + h_I^{(-)} + h_c)}{h_{I+1}^{(-)}} - \frac{\delta}{h_{I+1}^{(-)}} = 1 - \frac{h_c}{h_{I+1}^{(-)}}. \tag{29}$$

We summarize these cases in Table I.

The dimension increase d_a due to h_a is much simpler to describe because $H = h_1^{(+)} + \cdots + h_U^{(+)}$ and the portion of the dimension formula (17) due to the unstable Lyapunov exponents is just $J = U$. If $h_a \geq h_U^{(+)}$ then it has no effect on the dimension formula and $d_a = 0$. If $h_a \leq h_U^{(+)}$ then

$$h_1^{(+)} + \cdots + h_{U-1}^{(+)} + h_a \leq H \leq h_1^{(+)} + \cdots + h_{U-1}^{(+)} + h_a + h_U^{(+)} \tag{30}$$

and thus

$$d_a = 1 + \frac{H - (h_1^{(+)} + \cdots + h_{U-1}^{(+)} + h_a)}{h_U^{(+)}} = \frac{h_U^{(+)} - h_a}{h_U^{(+)}} = 1 - \frac{h_a}{h_U^{(+)}}. \tag{31}$$

Again, we summarize the cases in a table (Table II). This result is illustrated in Fig. 2, which shows the information dimension D as a function of h_a, assuming that h_c and the system generating the x dynamics are held fixed.

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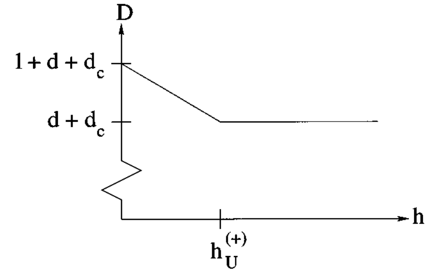

FIG. 2. D versus h_a.

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[12] The arguments in this section generalize those of Refs. [8,9]. See also E. Ott, *Chaos in Dynamical Systems* (Cambridge University Press, Cambridge, 1993), pp. 176–179.

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