Multifractality in the stochastic Burgers equation

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We investigate numerically the scaling properties of spatiotemporal correlation functions in the one-dimensional Burgers equation driven by noise with variance proportional to $|k|^{\beta}$. The long-distance behavior at $\beta<0$ is determined by shocks that lead to multifractality in the high-order structure functions and a dynamical exponent z close to unity. For $\beta>0$ earlier theoretical predictions for scaling exponents constrained by Galilean invariance obtain; these results are not expected to hold for $\beta<0$. Nevertheless, the continuation of the fixed point to $\beta<0$ correctly predicts some of the properties, an occurrence that we relate to the anomalous scaling of composite operators. [S1063-651X(96)05811-4]

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The Burgers equation [1] for a one-dimensional velocity field $u(x,t)$ has served as a simple model for investigating a variety of interesting issues that arise in fluid turbulence. Recently, there has been renewed interest in the Burgers equation with stochastic noise [2–5]

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}=\nu\frac{\partial^2 u}{\partial x^2}+\eta(x,t). \tag{1}$$

Here ν denotes the viscosity. The stochastic noise $\eta(x,t)$ is spatially correlated but has no temporal correlations. The spatial Fourier transform of the noise $\hat{\eta}(k,t)$ obeys

$$\langle\hat{\eta}(k,t)\hat{\eta}(k',t')\rangle=2D|k|^{\beta}(2\pi)\delta(k+k')\delta(t-t'). \tag{2}$$

In two stimulating papers Cheklov and Yakhot [2,3] have explored the special case of $\beta=-1$. We consider the system for positive and negative values of β and study how the spatiotemporal behavior in the inertial range varies. We discuss how the occurrence of shocks modifies long-distance, long-time properties for $\beta<0$ and leads to multifractality in contrast to the regime $\beta>0$. Nevertheless, as we will describe, results from a renormalization-group (RG) analysis of the model valid for $\beta>0$, where there are no shocks, continue to describe some of the properties for $\beta<0$. These observations can be understood in terms of the shocks themselves, which lead to a dynamical scaling exponent $z\approx1$ and to the anomalous scaling behavior of composite operators. Our specific numerical results include the long-wavelength behavior of (i) the energy spectrum $E(k)=\langle\hat{u}(k)\hat{u}(-k)\rangle$, where $\hat{u}(k)$ is the spatial Fourier transform of $u(x)$; (ii) the correlation function of the energy dissipation rate $\epsilon(x,t)=\nu(\partial u/\partial x)^2$; and (iii) the structure functions $S_q(r)$ moments of velocity differences

$$S_q(r)\equiv\langle|[u(x+r)-u(x)]|^q\rangle \tag{3}$$

for different values of $q\geqslant2$. One expects $S_q(r)\sim r^{\zeta_q}$ for r in the inertial range, delineated by the dissipation length scale set by the shock size and the distance between the shocks. Of particular interest is the dependence of ζ_q on q and its deviation from linearity $\zeta_q\neq cq$, which is referred to as intermittency and as displaying multifractal behavior. We study the dynamical behavior and find $z\approx1$ for $\beta<0$; we identify a (subdominant) exponent that we relate to the results of the RG analysis. Finally, we present results for the scaling of composite operators (local products of the fields) for $\beta<0$.

The one-dimensional noiseless Burgers equation displays shocks and scale-invariant behavior in the inertial range [1]: the energy spectrum $E(k)$ decays algebraically, i.e., $E(k)\sim k^{-2}$, and the structure functions $S_q(r)$ grow linearly with r for all $q\geqslant2$. In the presence of uncorrelated, conserving noise with $\beta=2$ shocks disappear; the energy spectrum tends to a constant for small k and $S_q(r)\sim r^{\zeta_q}$ with $\zeta_q=q$. These results correspond to those obtained for the Kardar-Parisi-Zhang (KPZ) equation [6,7] for interface growth, which is related to the noisy Burgers equation by a simple transformation [8]. The interface version of the problem was investigated by Medina *et al.* [9] using RG techniques for $0\leqslant\beta\leqslant2$. They studied fixed points to one-loop order and obtained the exponent χ characterizing the interface width and the dynamical exponent z exactly. We note that the case of $\beta=-1$ studied in Ref. [2] falls outside the scope of the analysis in Ref. [9], since for negative β a naive calculation reveals that higher-order nonlinear terms become relevant in the renormalization-group sense.

Here we study the stochastic Burgers equation numerically for $-1\leqslant\beta\leqslant2$ using a pseudospectral method [10], typically with 4096 points in a system of size $L=1024$ and occasionally with larger sizes. We chose the noise from a uniform distribution with the appropriate variance and the initial state to be either a single sine wave or a random superposition of sinusoids. We used the parameters $\nu=0.1$ or smaller (down to $\nu=0.03$) and a noise intensity D, in the discrete version of Eq. (1), of $D=10^{-6}$.

For positive β we find good agreement with the theoretical results of Medina *et al.* in Ref. [9]. Namely, for β between 1.5 and 2, the system flows to the standard KPZ fixed point [11] with interface exponents $z=\frac{3}{2}$ and $\chi=\frac{1}{2}$ [6,7]. We define the exponent σ that characterizes the behavior of the energy spectrum:

$$E(k)\propto|k|^{-\sigma}. \tag{4}$$

At $\beta=1.6$ we find that [12] $E(k)$ tends to a constant consistent with the results for the KPZ fixed point $\sigma=2\chi-1=0$ crossing over from the bare free-field behavior of

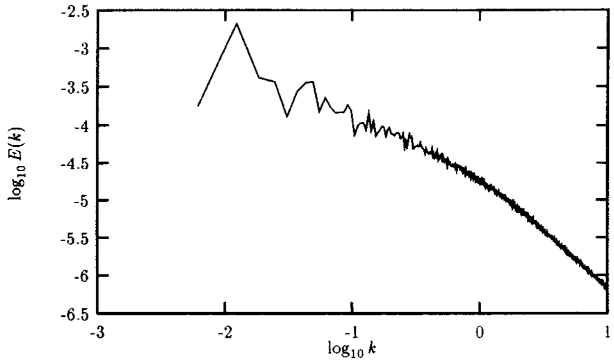

FIG. 1. Energy spectrum $E(k)$ vs k on a log-log plot for $\beta=0.5$. Here $\nu=0.05$ and $D=10^{-6}$. The system size is $L=1024$. At large k the behavior is that of the free field and at low k is in agreement with the result derived from the non-KPZ fixed point. See Eq. (5).

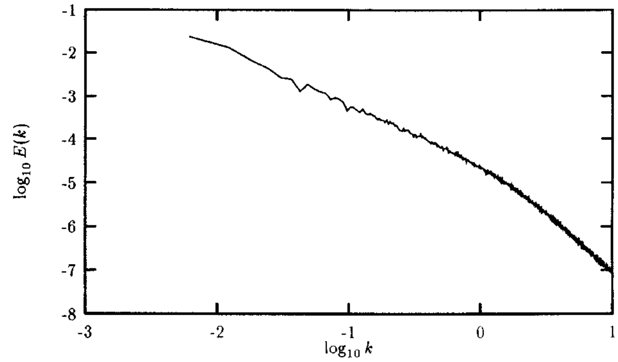

FIG. 2. Energy spectrum $E(k)$ vs k on a log-log plot for a system of size $L=1024$ for $\beta=-0.5$. Here $\nu=0.04$ and $D=10^{-6}$. The behavior at small k is in agreement with the result extrapolated from the non-KPZ fixed point: $E(k)\propto|k|^{-\sigma}$ with $\sigma=1-\frac{2}{3}\beta=\frac{4}{3}$.

$E(k)\sim k^{-0.4}$. The measured dynamical exponent is in good agreement with the KPZ value and the velocity structure functions approach a constant value for large r as expected [12].

For $0<\beta<1.5$, Ref. [9] finds a new fixed point with exponents given by [13]

$$\sigma=1-\tfrac{2}{3}\beta, \quad z=1+\tfrac{1}{3}\beta. \tag{5}$$

Both this new (strong-coupling) fixed point and the KPZ fixed point are Galilean invariant and this leads to the exponent relation $\chi+z=2$ [9]. There exist several numerical simulations [7] in the interface representation for the case $\beta\geqslant1$; the numerical results for $\beta>1$ do not all agree [14]; for $\beta<1$ a ballistic deposition model studied by Meakin and Jullien [15] does not in fact yield $\chi+z=2$, but the authors point out possible difficulties with crossover in the determination of z.

In Fig. 1 we show the behavior of $E(k)$ versus k for $\beta=0.5$. At high k, i.e., small distances, one finds the bare free-field behavior $E(k)\sim k^{-1.5}$. This behavior crosses over at smaller k to $E(k)\sim k^{-2/3}$, in agreement with the value of σ given in Eq. (5). We find similar agreement for $\beta=1$. Thus there is clear indication that one is at a new fixed point for $0<\beta<1.5$. We find that the velocity structure functions $S_q(r)$ approach a constant value A_q at large r and their ratios are consistent with the results of a Gaussian distribution. The conclusion is that the long-wavelength behavior is determined by the strong-coupling fixed point for positive β and there is no remnant of shocklike behavior; the velocity profile shows no shocks and this is confirmed by the lack of any indication of intermittency in $\{S_q\}$. We emphasize that the noise variance behaves as $|k|^\beta$ for all k; no cutoff has been introduced [4,12]. From a direct calculation of the velocity correlation function $C(t)\langle u(x,t)u(x,0)\rangle\sim t^{-(1-\sigma)/z}$ we find $z\approx1.2$, which means that the relation $\chi+z=2$ is obeyed within numerical errors. However, in the interface representation, from the temporal correlations in the steady state, we find a value for z much larger than the theoretical value, which leads to a violation of the identity $\chi+z=2$ by as much as 20%, in rough agreement with the results of Ref. [15].

We next discuss the regime $\beta<0$. A visual inspection of the profile reveals a few large, well-defined shocks. As mentioned earlier, negative values of β lie outside the realm of a RG analysis of the interface model because of higher-order nonlinear terms. Nevertheless, we find that at $\beta=-0.5$ and -1, $E(k)\sim k^{-1.34\pm0.04}$ (see Fig. 2) and $E(k)\sim k^{-1.65\pm0.05}$, respectively, in agreement with (4) and (5). The result at $\beta=-1$ is the one reported in Ref. [2] with a hyperviscosity term with a 12th derivative in Eq. (1). Reference [2] found clear evidence for $\sigma=\frac{5}{3}$ and provided somewhat less persuasive evidence that $z=\frac{2}{3}$.

The agreement of the exponent σ with that predicted from a naive extrapolation of the results in Ref. [9] is surprising; it appears that the continuation of the non-KPZ fixed point to negative values of β determines the behavior of $\langle u(k)u(-k)\rangle$ and higher-order nonlinearities are not relevant (see the later discussion). Indeed, we have explicitly checked that at $\beta=-1$ the addition of a small nonlinearity of the form $u^3(\partial u/\partial x)$ does not change the value of the exponent σ. As to the velocity structure functions, it was already pointed out in Ref. [2] that at $\beta=-1$ they grow almost linearly, $\zeta_q\approx0.9$, with distance for $q=4,6,8$. At $\beta=-0.5$ also, the profile clearly indicates the presence of the shocks. With the noise of the form assumed and within our numerical limitations, $\zeta_q\approx0.87$ for $q=6$ and 8 and somewhat lower for $q=4$ (see Fig. 3). In order to clarify this strongly intermittent behavior we studied a cutoff noise that further suppresses the stochastic driving at short length scales. We employed a noise that has correlations of the form $|k|^\beta$ for small k and assumes a small constant value for larger k (equal to the smallest value obtained in the original model) with smooth interpolation between the two limits. The shocks are better defined and for both $\beta=-1$ and -0.5 we find a value of $\zeta_q=0.98\pm0.04$ for $q=4$, 6, and 8. The value of σ remains unaltered. This clearly establishes the role of shocks in causing strong intermittency.

Since we expect most of the dissipation to occur in the shocks a useful probe of the system is the spatial correlation of the rate of energy dissipation $\epsilon(x)$ defined by $\epsilon=\nu(\partial u/\partial x)^2$. We compute its correlation

$$G_d(r)=\langle\epsilon(x)\epsilon(x+r)\rangle \tag{6}$$

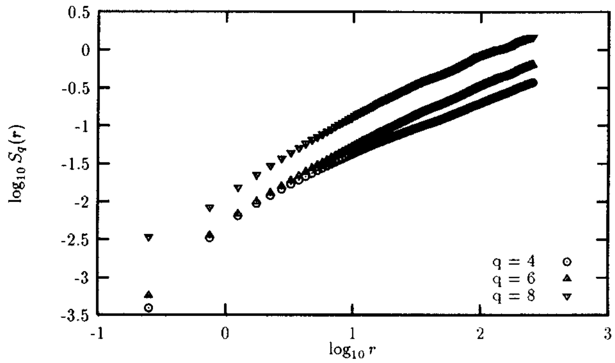

FIG. 3. Structure factors $S_q(r)$ vs r plotted on a log-log plot for $q=4$, 6, and 8 when $\beta=-0.5$. The system size is $L=1024$ and distances are measured in lattice units. For distances in the inertial range, the growth of $S_q(r)$ is close to linear. Here $\nu=0.03$ and $D=10^{-6}$.

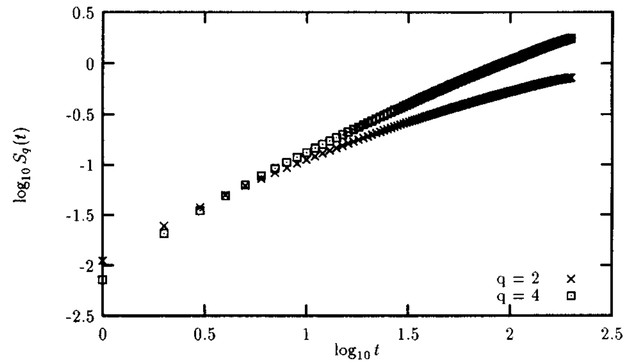

FIG. 5. Autostructure function $S_q(r=0,t)=\langle[u(x,t)-u(x,0)]^q\rangle$ vs t plotted on a log-log plot for $q=2$ (lower curve) and $q=4$ (upper curve) when $\beta=-1$. From the long-time behavior the value of the dynamical scaling exponent z can be estimated to be $z\approx 1$.

and determine the exponent μ defined by its large r behavior: $G_d(r)\sim|r|^{-\mu}$. We computed the spatial Fourier transform $\hat{G}_d(k)=\langle\hat{\epsilon}(k)\hat{\epsilon}(-k)\rangle$ directly; the data obtained from our calculations are displayed in Fig. 4. For $\beta=-1$ we find $\mu\approx 0.38\pm 0.08$ compared to the value of 0.25 obtained in the presence of hyperviscosity [2]. At $\beta=-0.5$, the value of μ changes somewhat to 0.45 ± 0.06 [16]. Note that for $\beta>0$, $G_d(r)$ is short ranged and does not show scaling behavior.

We next consider dynamical correlations and determine the dynamical exponent z for $\beta>0$. We evaluate the autostructure functions defined by $S_q(r=0,t)=\langle[u(x,t)-u(x,0)]^q\rangle$, which are expected to scale as $S_q(0,t)\sim t^{\zeta_q/z}$. We see from Fig. 5 that there are two scaling regimes and z can be deduced from the values of ζ_q determined earlier. Our findings may be summarized by the statement that at short times the behavior of $S_2(0,t)$ is consistent with $z=1+\beta/3$, while $z=1$ at longer times. The value $z=1$ is due to the presence and ballistic motion of shocks; the latter leads to linear behavior in time similar to that in space. This is reminiscent of Taylor's frozen hypothesis in that the time correlations at a given point are similar to equal time spatial correlations with a spatial separation determined by the mean velocity of shocks. The value of $z=1$ would imply that the exponent identity $z+\chi=2$ imposed by Galilean invariance at the strong-coupling fixed point is violated; however, the Galilean invariant fixed point does describe the behavior of the system: it correctly predicts the exponent σ that characterizes $E(k)$; it also predicts the dynamical exponent except when the effective exponent of $z=1$ due to the motion of shocks simply dominates the lower fixed-point value.

Finally, we draw attention to the anomalous scaling behavior of the composite operators $\langle u^2(x)u^2(y)\rangle$ and $\langle u^4(x)u^4(y)\rangle$. We calculate $\langle\widehat{u^n}(k)\widehat{u^n}(-k)\rangle$, where $\widehat{u^n}(k)$ is the Fourier transform of $u^n(x)$ for $n=2$ and 4. The results are plotted in Fig. 6. We find numerically that $\langle\widehat{u^n}(k)\widehat{u^n}(-k)\rangle\propto k^{-\sigma_n}$ at $\beta=-1$ with $\sigma_n\approx 1.6$ for both $n=2$ and 4 [16]. This behavior does not obey, within the limitations of our numerical calculations, expectations based on so-called gap scaling and is consistent with the occurrence of multifractality. The ultraviolet behavior determined by the existence of shocks alters the scaling of composite operators and the precise behavior can be related quantitatively to the value $z=1$. Our results strongly suggest that the higher-order operators are not increasingly relevant. This behavior underlies the success of the prediction for the exponent $\sigma=1-\frac{2}{3}\beta$ based on a simple balancing argument ignor-

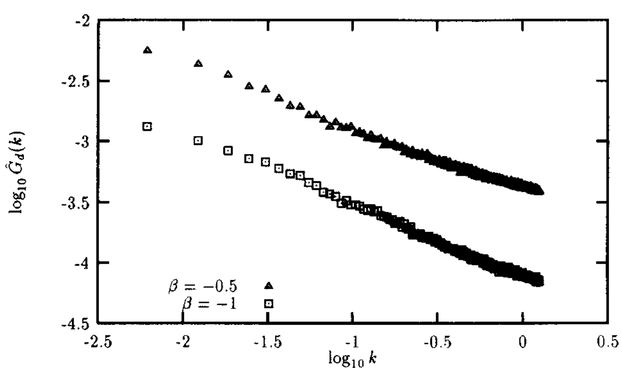

FIG. 4. Fourier transform of the correlations of the energy dissipation $\epsilon=\nu(\partial u/\partial x)^2$ [see Eq. (6)], $G_d(k)$ vs k on a log-log plot for $\beta=-0.5$ and $\nu=0.04$ (upper curve) and $\beta=-1$ and $\nu=0.075$ (lower curve). Here $D=10^{-6}$ in both cases.

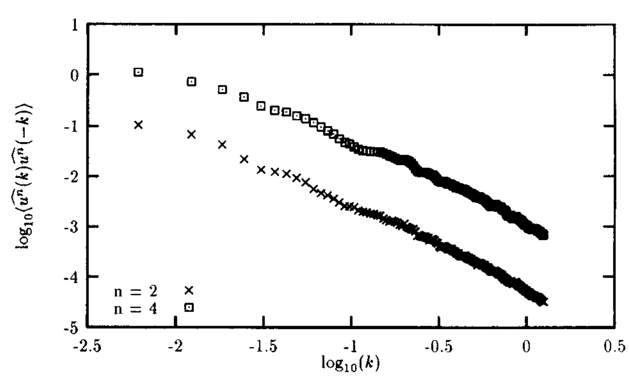

FIG. 6. Correlations of composite operators $\langle\widehat{u^n}(k)\widehat{u^n}(-k)\rangle$ plotted on a log-log plot for $n=2$ (lower curve) and $n=4$ (upper curve) for $\beta=-1$. Here $\nu=0.075$ and $D=10^{-6}$.

ing higher-order terms [13], which is also the value obtained by a continuation to negative β of the exponent computed by Medina *et al.* for $\beta>0$. Note that the naive arguments apply for correlations involving two u's at widely separated points but fail in the (singular) limit when separations are taken to be zero. Recently, Polyakov [5] has analyzed the stochastic Burgers equation using point-splitting methods and the operator product expansion; however, his results do not appear to be directly applicable to the $\beta<0$ case studied here. In conclusion, we have described a variety of intriguing behavior that occurs in the Burgers equation with strongly (spatially) correlated stochastic noise: the existence of shocks and the consequent occurrence of multifractality and the scaling of composite operators and dynamical correlations.

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[8] Recall that the height $h(x,t)$ in the KPZ equation is related to the velocity in the Burgers equation by $u(x,t)=\partial h(x,t)/\partial x$.

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[11] Note that the corresponding noise in the interface problem has a spectrum proportional to $|k|^{\beta-2}$.

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[13] In the Burgers equation rescaling $x=bx'$, $t=b^z t'$, and $u=b^{\chi-1}u'$ casts the equation in the form

$$b^{\chi-1-z}\frac{\partial u'}{\partial t'}=b^{\chi-3}\boldsymbol{\nu}\frac{\partial^2 u'}{\partial x'^2}-b^{2\chi-3}u'(\partial u'/\partial x')+b^{-(\beta+1+z)/2}\eta.$$

Now balancing the time derivative term against the nonlinear term yields $\chi+z=2$ and balancing it against the noise term yields $2\chi-z=1-\beta$. Together these two yield the exponents derived in Ref. [9] for $0<\beta\leqslant 3/2$; defining $\sigma=2\chi-1$ yields Eq. (5).

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[16] We point out that the evaluation of the correlations of composite operators and especially the energy dissipation are prone to greater statistical and numerical uncertainties and limited by our computational resources.