Transforming signals with chaotic synchronization

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We introduce a general technique to alter the properties of chaotic signals used with coupled chaotic systems. The changes we introduce allow one to vary the synchronization properties of synchronized chaotic circuits, synchronize chaotic systems that do not otherwise synchronize, vary the spectral properties of chaotic signals, and produce a variety of chaotic signals from one chaotic circuit. The transformations we study could potentially aid designers of synchronous chaotic circuits, as it is far easier to design new transformations than to design new chaotic circuits. [S1063-651X(96)05411-6]

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I. INTRODUCTION

While there has been much speculation on the use of chaotic systems for communications or other applications [1–13], there are really only a few standard chaotic systems that have been used for examples. One may try to design other chaotic systems, but for now designing chaotic systems, especially systems that may be built as circuits, is a trial and error process. When one imposes some set of restrictions on the chaotic systems, such as ease of reproducibility, synchronization characteristics, or spectral properties, the problem of designing chaotic systems becomes even more difficult.

We will show in this paper that one may greatly modify the properties of existing coupled chaotic systems by transforming the original drive signal and one or more other chaotic signals from the existing drive system to create a new scalar drive signal. We then undo the transformation at the receiver using a procedure we call *synchronous substitution* to recover the original drive signal. We show below how synchronous substitution allows one to create a variety of chaotic signals from one source, tailor the Lyapunov exponents of response systems (one may even synchronize unstable subsystems), change the spectral properties of chaotic signals, and multiplex chaotic signals from different sources [13].

II. CHAOTIC SYNCHRONIZATION

We use the idea of chaotic synchronization [1–3,6,14,15] to reproduce the signals from a drive system at some response system. One may start with a *drive* (transmitter) system such as $\dot{x}=f(x,y,z)$, $\dot{y}=g(x,y,z)$, $\dot{z}=h(x,y,z)$ and divide it into component subsystems. The exact division may be done in many ways.

To build a receiver (or *response*) system, we reproduce one or both of the subsystems of the drive system and drive them with a signal from the drive system. There are many ways to apply the signal. For example, we could use the y variable to drive the response system $\dot{x}'=f(x',y',z')$, $\dot{y}'=g(x',y,z')$, $\dot{y}'=h(x',y',z')$, where the primed variables are response system variables only and we have applied the drive only in the ''y'' response subsystem. We could use different combinations of drive signals and subsystems; for example, we could replace the y' variable with the drive signal y everywhere that y' appears in the response system, or we could use a diffusive coupling [15,16]. It has been shown [1,2] that if all of the Lyapunov exponents in the response system are negative, then $y'-y\rightarrow 0$ as $t\rightarrow\infty$.

III. TRANSFORMATION AND SYNCHRONOUS SUBSTITUTION

Rather than simply sending a single signal such as y in the examples above, we may send a transformed version of y that may depend on other dynamical variables in the drive system. An example of such a transformation is $w=y+x$, where we call w the ''transmitted signal.'' If the response system is synchronized to the drive system, then $x'=x$, so we may construct the inverse transformation $\tilde{y}=w-x$ by using x' in place of x, a procedure we call *synchronous substitution*. In this case, $\tilde{y}=y$ (within some small error).

Superficially, the transformation procedure we use looks the same as the work of Kocarev and Parlitz or Peng *et al.* [17,18]. There are important differences in the physics between our work and previous work. In the Kocarev and Parlitz work, they do a change of variables on the drive and response systems by defining a new driving variable s which is a function of the old variables. This change of variables allows Kocarev and Parlitz to find new decompositions for an existing chaotic system. Some of these new decompositions will be stable. The driving variable will be different, depending on the particular decomposition used. We used a simple version of this idea in our original work on chaotic synchronization [1,2,14], in which we used a hysteretic circuit. We found that the circuit response system was not stable when driven with our original choice for driving variable. We had to define a new driving variable which was a function of the original circuit variables in order to synchronize the response to the drive.

In our approach, we do not use a change of variables. We do define a new variable w which is a function of the original driving variable and other variables. The new variable w is transmitted to the response system but is not used to drive the response system. We recover the original drive variable by inverting the transformation that generated w. We invert the transformation using only variables from the synchronized response system, a process that we call synchronous substitution. In inverting the transformation, we create a feedback loop in the response system, so that the new response system is no longer identical to the drive system. One may decide which variables one would like to feed back into the response system, for example, to improve stability, and then design the transformations appropriately. In the Kocarev and Parlitz technique, the response system is still identical to the drive system; there is no feedback involving the response variables.

One major difference between our synchronous substitution and the change of variables technique of Kocarev and Parlitz is that our transformations are not limited to recombinations of the existing drive system variables. In [19], the transformation included a filter. Any transformation will work, including transformations that introduce new variables, as long as the transformation is invertible and the response system (including the inverse transformation) is stable. If we consider transformations such as filtering [19], we see the synchronous substitution technique may be used to alter the spectrum of the transmitted chaotic signal w.

Below we first show a numerical example involving the Lorenz equations before giving a more general description of synchronous substitution. Afterwards we show circuit examples of synchronous substitution.

IV. NUMERICAL EXAMPLE

Our first simple example illustrates the technique using the Lorenz equations. The drive system is

$$\frac{dx}{dt}=10(y-x), \tag{1}$$

$$\frac{dy}{dt}=-xz+60x-y, \tag{2}$$

$$\frac{dz}{dt}=xy-2.667z, \tag{3}$$

$$w=y+x. \tag{4}$$

The response system is

$$\tilde{y}=w-x', \tag{5}$$

$$\frac{dx'}{dt}=10(\tilde{y}-x'), \tag{6}$$

$$\frac{dz'}{dt}=x'\tilde{y}-2.667z'. \tag{7}$$

The stability of the synchronous state is determined from the conditional Lyapunov exponents of the response system (5)–(7). They are found from the Jacobian of the response system evaluated on the synchronous state,

$$\begin{bmatrix} \frac{\partial \dot{x}'}{\partial x'} & \frac{\partial \dot{x}'}{\partial z'} \\ \frac{\partial \dot{z}'}{\partial x'} & \frac{\partial \dot{z}'}{\partial z'} \end{bmatrix}_{\text{sync state}} = \begin{bmatrix} -20 & 0 \\ y-x & -2.667 \end{bmatrix}. \tag{8}$$

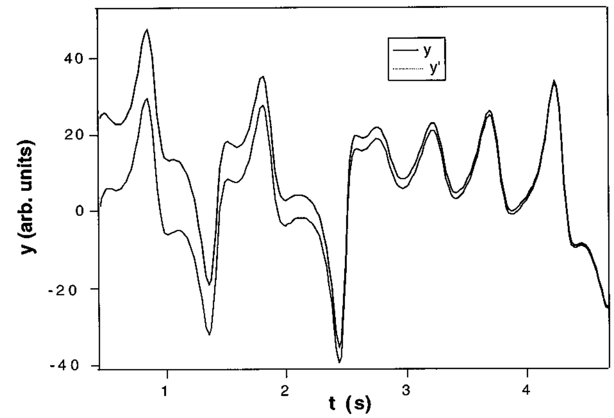

FIG. 1. Signal y from the drive system (solid line) and response signal y' from the Lorenz system of Eqs. (3)–(10) showing that the response system converges to the drive system when the driving signal is $w=y+x$.

Since the Jacobian is lower triangular, the conditional Lyapunov exponents are simply the diagonal elements, -20 and -2.667, indicating that the response system is stable. Figure 1 shows the convergence of y' to y when the drive and response systems are started with different initial conditions.

V. GENERAL FORMULATION OF SYNCHRONOUS SUBSTITUTION

We can generalize this combined use of transformation and synchronous substitutions as follows. Let T be a transformation from $\mathbb{R}^n \rightarrow \mathbb{R}$: $w=T(x,y,z,\ldots)$, where $(x,y,z,\ldots)\in\mathbb{R}^n$. Suppose the response system is near synchronization. We send the transmitted signal w which may be a combination of several drive system signals, including the original drive signal y. In order to synchronize the response system, we need an estimate for the value of y given only the signal w. By the implicit function theorem, if $D_yT\neq 0$, then there exists an inverse transformation which we denote by T_y^{-1} such that $y=T_y^{-1}(w,x,z,\ldots)$. At the response we only know w. But we can get a good estimate of y by using the response variables x', z', etc. This use of response variables in place of drive variables is what we term synchronous substitution. We write $\tilde{y}=T_y^{-1}(w,x',z',\ldots)$. We can now put $\tilde{y}$ into the response where we would like to apply the drive variable y.

The question that remains is that of stability. Using the above formulation we can write the general form of the variational problem for the response stability. If the vector field of the response is $F(x',y',z',\ldots,\tilde{y})$, then the variational equations become

$$\frac{d\,\delta\mathbf{r}}{dt}=[D_{(x',y',z',\ldots)}F|_{\text{sync state}} + D_{\tilde{y}}FD_{(x',y',z',\ldots)}T_y^{-1}|_{\text{sync state}}]\delta\mathbf{r}, \tag{9}$$

where $\delta\mathbf{r}=(x'-x,y'-y,z'-z,\ldots)$. The first term in brackets is the usual Jacobian that results in the standard variational problem. The second term depends on the transformation and the synchronous substitution. The latter can cause changes in stability and allow more interesting and varied synchronization schemes to be developed. Note that the second term will have a column of zeroes in the y' position.

Obviously this approach is not limited to using the variable y' and can be applied in situations where more than one signal, say w_1 and w_2, are transmitted.

VI. CIRCUIT EXAMPLES

We demonstrated synchronous substitution in a piecewise linear Rossler (PLR) circuit [20]. The circuit is described by

$$\frac{dx}{dt} = -\alpha(rx + \beta y + z), \tag{10}$$

$$\frac{dy}{dt} = -\alpha(-x - ay + \gamma y), \tag{11}$$

$$\frac{dz}{dt} = -\alpha[z - g(x)], \tag{12}$$

$$g(x) = \begin{cases} 0 & \text{if } x < 3 \\ 15(x-3) & \text{if } x \geq 3 \end{cases}, \tag{13}$$

$$w = T(x,y,z), \tag{14}$$

where $\alpha = 10^4$, $a = 0.12$, $b = 1.0$, $\beta = 0.5$, $\gamma = 0.02$, and $r = 0.05$. The y term in Eq. (11) is divided into two parts to make the correspondence with the response system of Eq. (17) more obvious. The synchronized response circuit is described by

$$\tilde{y} = T_y^{-1}(w, x', y, z'), \tag{15}$$

$$\frac{dx'}{dt} = -\alpha(rx' + \beta\tilde{y} + z'), \tag{16}$$

$$\frac{dy'}{dt} = -\alpha(-x' - a\tilde{y} + \gamma y'), \tag{17}$$

$$\frac{dz'}{dt} = -\alpha[z' - g(x')]. \tag{18}$$

The term $\gamma y'$ in Eq. (17) is necessary to stabilize the operational amplifier integrator used in the above circuit. We used the response circuit of Eqs. (14)–(17) with two different versions of T. For our first circuit, we used $w = y - x$ and $\tilde{y} = w + x'$. The plot of Fig. 2 shows y' vs y from the circuit for the preceding transformation. The largest Lyapunov exponent for the response circuit is $-196\ \mathrm{s}^{-1}$ (calculated numerically from the equations of motion by the method of Eckmann and Ruelle [21]).

Nonlinear transformations are also possible with synchronous substitution. The transformation $w = -y/(x+4.2)$ and $\tilde{y} = -w(x'+4.2)$ also resulted in synchronization in the circuit. The largest Lyapunov exponent of the response circuit for this second transformation was calculated to be $-651\ \mathrm{s}^{-1}$.

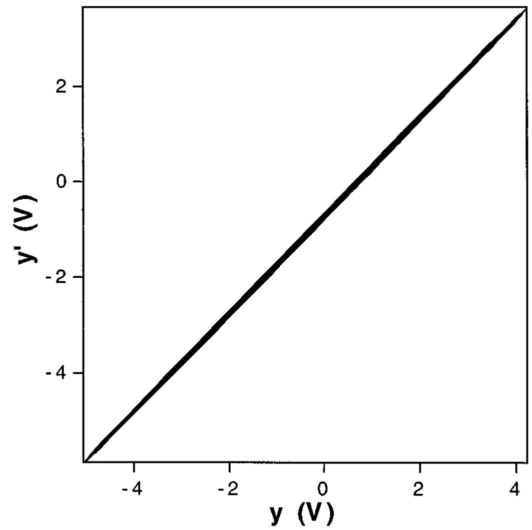

FIG. 2. y' signal from the response system vs y signal from the drive signal for two piecewise linear Rossler (PLR) circuits when the driving signal is $w = y - x$ and the reconstructed driving signal $\tilde{y} = s_t + x'$.

VII. DRIVING UNSTABLE SUBSYSTEMS

Synchronous substitution may also be used in control the stability of the response system and even synchronize response systems that normally do not synchronize. We demonstrate this control of stability with the drive circuit of Eqs. (10)–(14) and the response circuit described by

$$w = z - kx, \tag{19}$$

$$\tilde{z} = w + kx', \tag{20}$$

$$\frac{dx'}{dt} = (rx' + \beta y' + \tilde{z}), \tag{21}$$

$$\frac{dy'}{dt} = -\alpha(-x' - \rho y'), \tag{22}$$

where $\rho = a - \gamma = 0.12$ and the other symbols are defined with Eqs. (10)–(14).

The stability of the response system described by Eqs. (19)–(22) is determined by the conditional Lyapunov exponents of the $x' - y'$ response subsystem. Since this subsystem is linear, the exponents can be computed analytically; they are found from the eigenvalues of the sub-Jacobian (setting $\alpha = 1$ for this calculation)

$$\begin{bmatrix} -r-k & -\beta \\ 1 & \rho \end{bmatrix}. \tag{22'}$$

For $k = 0$, $\rho = 0.12$, $\beta = 0.5$, and $r = 0.05$ the eigenvalues are $0.035 \pm 0.702i$, and therefore the subsystem is unstable—chaotic synchronization is not possible (which is a known result for the Rossler $x - y$ subsystem [1,2].

The stability of the response system varies for $k \neq 0$. Figure 3 shows the maximum real part of the eigenvalues of this matrix (μ_{max}) as a function of k (with other parameters given above). The eigenvalues cross into the left half plane when

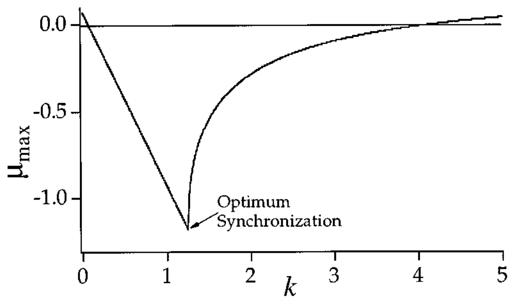

FIG. 3. Plot of $\mu_{max}=\max\{\mathrm{Re}(\lambda_1),\mathrm{Re}(\lambda_2)\}$ versus k for the piecewise linear Rossler response system of Eqs. (19)–(21).

$k=\rho-r=0.07$. ''Optimum'' synchronization (minimum μ_{max}) occurs when the eigenvalues are degenerate at $k=-(\rho+r)+2\sqrt{\beta}\approx 1.244$.

A circuit was built to simulate Eqs. (19)–(22) (with $\alpha=10^4$). For $k=0$ (corresponding to z driving only), no synchronization of the drive and response circuits was seen (as discussed above, the response system is unstable for $k=0$). Synchronization was seen for $k=1$ (the system is stable for $k=1$ in the theoretical example above). The response system was also seen to synchronize for $k=0.15$ and 2.20 (well within the region of stability shown in Fig. 3), while no synchronization was seen for $k=0.075$, just on the boundary of stability shown in Fig. 3. The drive and response circuits did not synchronize for $k=2.67$, but, as can be seen in Fig. 3, μ_{max} for $k=2.67$ is just below 0, so the response circuit may be especially sensitive to noise and parameter mismatch.

One may stabilize other normally unstable subsystems; we drove a y-z subsystem of the PLR circuit of Eqs. (8)–(14) with $w=x+ky$ and $\tilde{x}=w-ky'$. The response system is unstable for x driving [1,2]—the numerically determined largest Lyapunov exponent for the response system is 1100 s^{-1}. The circuits did synchronize when $k=1$, for which the largest Lyapunov exponent was $-8899\ s^{-1}$. The response circuit could also be set to have neutral stability; the largest Lyapunov exponent for the response equations was 0 for $k=0.11$.

VIII. SIGNALS FROM MULTIPLE SYSTEMS

Note that nothing in the definition of the transformation T requires that all signals come from the same dynamical system. For example, we could use a transformation T which combines signals from different dynamical systems as a way to multiplex different chaotic signals or to use one chaotic signal to change the spectral properties of another through a nonlinear transformation. Tsimring and Sushchick [13] have numerically demonstrated a simple version of chaotic multiplexing by adding two chaotic signals. We have demonstrated a similar process both numerically and in circuits.

We have combined signals from two PLR circuits. The pair of driving circuits is described by Eqs. (10)–(14) above. The response circuits were driven by a diffusive coupling [15,16] to allow more control over the stability of the response system. The response circuits were described by

$$w=y_1+y_2, \tag{23}$$

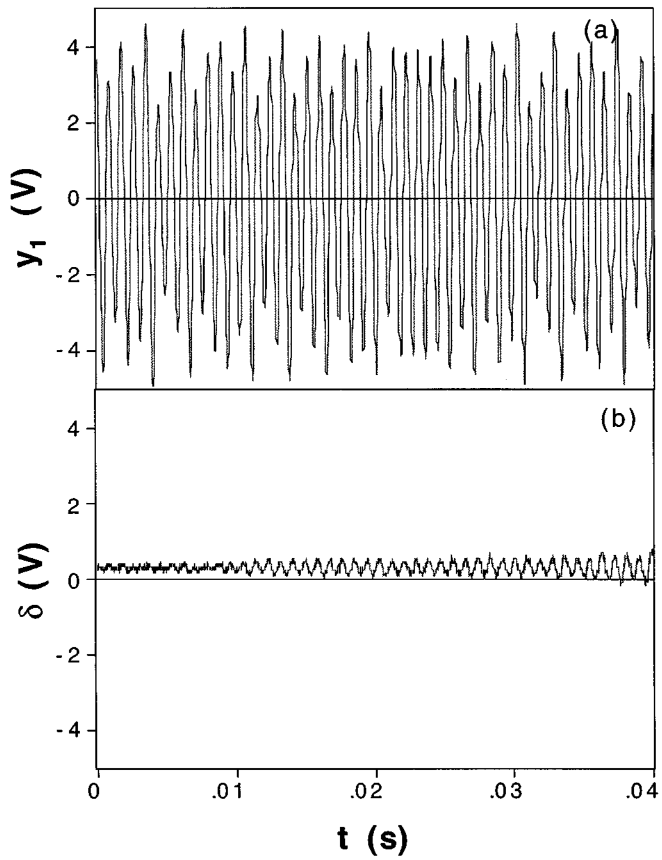

FIG. 4. (a) y_1 signal from a PLR circuit described by Eqs. (10)–(13). (b) Difference between y signals in drive and response circuits ($\delta=y_1-y_1'$) when signals from two PLR circuits are added to together, transmitted, and separated by synchronous substitution before driving synchronized response systems as in Eqs. (23)–(29).

$$\tilde{y}_1=w-y_2', \quad \tilde{y}_2=w-y_1', \tag{24}$$

$$i=1,2, \tag{25}$$

$$\frac{dx_i'}{dt}=-\alpha(rx_i'+\beta y_i'+z_i'), \tag{26}$$

$$\frac{dy_i'}{dt}=-\alpha[-x_i'-\rho y_i'-c(\tilde{y}_i-y_i')], \tag{27}$$

$$\frac{dz_i'}{dt}=-\alpha[z_i'-g(x_i')], \tag{28}$$

$$g(x_i')=\begin{cases}0 & \text{if } x_i'<3\\ 15(x_i'-3) & \text{if } x_i'\geq 3,\end{cases} \tag{29}$$

where $\alpha=10^4$, $\rho=0.12$, $b=1.0$, $\beta=0.5$, $r=0.05$, and $c=0.5$. When the response system of Eqs. (23)–(29) is integrated numerically, y_1' is seen to synchronize with y_1, and y_2' synchronizes with y_2. The largest conditional Lyapunov exponent [2] for the six-dimensional response system is -140 s^{-1}, compared with a largest conditional Lyapunov exponent of $-2252\ s^{-1}$ for a single driven PLR response system. For identical systems, the initial conditions determine whether response system 1 synchronizes with drive system 1 or 2; in building actual circuits, response system 1 is most closely matched to drive system 1, and the same for systems 2.

Noise free numerical simulations do not reveal an accurate picture of the system of Eqs. (23)–(29), however. While the global conditional Lyapunov exponents are less than zero, there are regions on the response system attractor where a conditional Lyapunov exponent is greater than zero. When circuits corresponding to Eqs. (23)–(29) were driven, bursting was seen instead of perfect synchronization. Figure 4(a) shows a time series of the signal y_1 from the drive circuit, while Fig. 4(b) shows $\delta = y_1 - y_1'$. The drive and response systems are close to synchronization, but local regions where a conditional Lyapunov exponent is greater than zero cause bursting away from synchronization. Tsimring and Sushchick [13] see the same local instabilities that we have seen here. The local instabilities are related to the fact that the two response systems are coupled to each other. We were also able to observe synchronization in numerical experiments when one of the drive circuits was a PLR circuit and the other circuit was a four-dimensional circuit described in [5], but local instabilities still caused bursting in circuit experiments.

IX. CONCLUSIONS

The use of transformations that may be undone by synchronous substitution will be a useful tool in the application of chaos in fields such as communication. If one desires to send many different chaotic signals to many different users, one could use signal transformation by synchronous substitution. It is easier to design new synchronous transformations than it is to design new chaotic circuits, so one may engineer whole sets of chaotic signals with some desired properties. The combination of signals from different chaotic systems, which may be undone by synchronous substitution, has been proposed as a method to multiplex many chaotic signals together [13], and also offers a way to alter the spectral properties of chaotic signals; unfortunately, the presence of local instabilities in the response system currently makes this method impractical.

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