Origin of the persistent oscillations of solitary waves in nonlinear quadratic media

C. Etrich, U. Peschel, and F. Lederer
Institut für Festkörpertheorie und Theoretische Optik, Friedrich-Schiller-Universität Jena, Max-Wien-Platz 1, 07743 Jena, Germany

B. A. Malomed
School of Mathematical Sciences, University of Tel Aviv, Tel Aviv 69978, Israel

Y. S. Kivshar
Australian Photonics Cooperative Research Centre, Australia National University, Australian Capital Territory, 0200, Canberra, Australia

(Received 30 April 1996)

We reveal the origin of the persistent oscillations of solitary waves in a quadratically nonlinear medium. It is found that the oscillations are closely correlated to a nontrivial discrete eigenmode of the corresponding linear eigenvalue problem. In addition to this discrete eigenmode a quasibound mode, weakly coupled to the continuous spectrum, is identified. This gives rise to a long-living beating of the solitary wave amplitude. [S1063-651X(96)01410-9]

PACS number(s): 42.65.Tg, 42.65.Sf, 42.65.Ky

Recently, solitary waves characterized by a mutual locking of the fundamental and second harmonics in quadratically nonlinear media attracted a lot of attention [1–10]. The study of these localized waves is strongly complicated by the fact that the evolution equations of the system are not integrable and even stationary solitary waves have to be determined numerically [5]. The exception is a single element of the one-parameter family, which is known analytically [1,3,4]. An important issue of these solutions is their stability. By means of both numerical and analytical methods a narrow instability region was identified [8].

In the case of spatial solitary waves in quadratically nonlinear media the Galilean invariance of the governing equations allows us to generate moving solitary waves, necessary for the analysis of collisions between them [9]. Such numerical collision experiments with appropriate velocities, as well as the excitation of a single solitary wave [7] or the evolution of an unstable one [8], reveal another prominent feature of the solutions under consideration. It turns out that persistent oscillations appear in many cases. They are practically undamped and may exhibit fairly large amplitudes. The extreme stability of the excited solitary waves is really astonishing for nonintegrable systems. This also is of great importance for future applications. Given a realistic experimental situation the second harmonic is excited via the fundamental wave only and strong oscillations cannot be avoided. Thus a stationary solitary wave with no oscillations seems to be more or less an exception.

So far the nature of the oscillations has not been clearly identified. However, we would like to mention that the asymptotic analysis near the instability threshold [8] suggests that these oscillations should be connected to a localized eigenstate of the corresponding linearized problem, being somehow an analytical continuation of the unstable mode into the stability region. Similar oscillating states were also observed for some types of nonintegrable nonlinear Schrödinger equations [11–13].

The main objective of the present paper is the first systematic analysis of the oscillations of solitary waves in a quadratically nonlinear medium and a discussion of their origin. Here we concentrate on the spatial case which can be encountered in planar waveguides (see, e.g., [10]). First we study the mode structure of the linearized evolution equations corresponding to a solitary wave solution. We find nontrivial eigenstates which are related to the oscillations of the solitary waves. The linear analysis is followed by systematic simulations aimed to examine the nonlinear properties of the oscillations. To find the origin of their extreme stability we investigate the response of the solitary waves to different amplitudes of excitation.

For the propagation of the fundamental and second harmonics in a planar waveguide geometry with a quadratic nonlinearity, in the presence of diffraction the properly scaled evolution equations for the slowly varying amplitudes a_1 and a_2 of the two fields are [5,7,9]

$$\begin{aligned} i\frac{\partial a_1}{\partial z}+\frac{1}{2}\frac{\partial^2 a_1}{\partial y^2}-a_1+a_1^* a_2=0,\\ i\frac{\partial a_2}{\partial z}+\frac{\sigma}{2}\frac{\partial^2 a_2}{\partial y^2}-\alpha a_2+a_1^2=0, \end{aligned} \tag{1}$$

where α describes the phase mismatch between the fundamental and second harmonics and σ is the ratio of their wave numbers. The asterisk denotes the complex conjugate. Throughout this work we consider the case $\sigma=1/2$, appropriate for the spatial case. In this notation the stationary solitary wave solutions are obtained equating the z derivatives in Eqs. (1) to zero and applying the appropriate boundary conditions. Thus these solutions depend on the single parameter α. As mentioned above, they have to be found numerically with the exception of the analytical solution at $\alpha=\sigma=1/2$ [1,3,4].

To launch the oscillations, a solitary wave is perturbed and propagated over a certain distance. A particular shape of

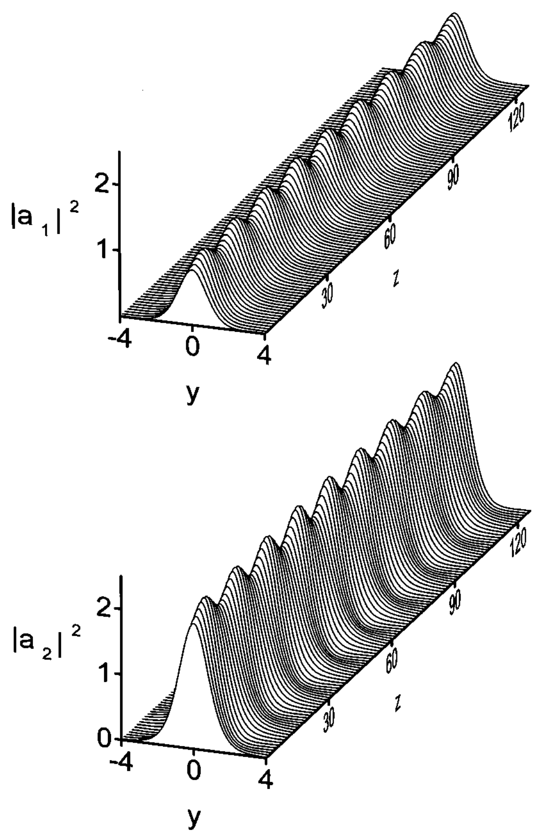

FIG. 1. Typical example of persistent internal oscillations of a solitary wave solution excited with ξ=0.4 for α=0.5. Displayed are the intensities of the fundamental and second harmonics.

the initial perturbation will be chosen, such that the energy of the initial wave is not changed

$$a_n(z=0)=\left[a_{n0}^2(y)+\xi\,\frac{a_{n0}^2(0)}{\left.\frac{\partial^2}{\partial y^2}\,a_{n0}^2(y)\right|_{y=0}}\,\frac{\partial^2}{\partial y^2}\,a_{n0}^2(y)\right]^{1/2}. \tag{2}$$

Here a_{n0}, $n=1,2$, denotes a stationary solitary wave solution of Eqs. (1) and ξ stands for the perturbation amplitude. A similar type of energy-preserving perturbation was used in [13] recently. A typical example of the persistent oscillations of a perturbed solitary wave is displayed in Fig. 1. The oscillations seem to be quite regular. This is compared with the solutions of the linearized Eqs. (1). Linearizing Eqs. (1) around a stationary solution, $a_n=a_{n0}+\delta a_n\exp(i\lambda z)$, $a_n^*=a_{n0}+\overline{\delta a_n}\exp(i\lambda z)$, we arrive at the following eigenvalue problem for the propagation constant of the perturbation λ:

$$\begin{aligned}
&\frac{1}{2}\,\frac{\partial^2\delta a_1}{\partial y^2}-\delta a_1+a_{10}\delta a_2+a_{20}\overline{\delta a_1}=\lambda\,\delta a_1,\\
&\frac{\sigma}{2}\,\frac{\partial^2\delta a_2}{\partial y^2}-\alpha\,\delta a_2+2a_{10}\delta a_1=\lambda\,\delta a_2,\\
&\frac{1}{2}\,\frac{\partial^2\,\overline{\delta a_1}}{\partial y^2}-\overline{\delta a_1}+a_{10}\overline{\delta a_2}+a_{20}\delta a_1=-\lambda\,\overline{\delta a_1},\\
&\frac{\sigma}{2}\,\frac{\partial^2\,\overline{\delta a_2}}{\partial y^2}-\alpha\,\overline{\delta a_2}+2a_{10}\overline{\delta a_1}=-\lambda\,\overline{\delta a_2}.
\end{aligned} \tag{3}$$

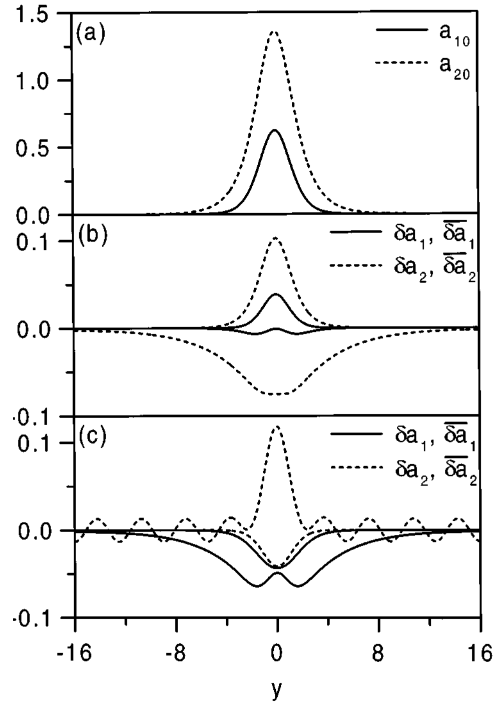

FIG. 2. (a) Solitary wave solution, (b) discrete eigenstate, and (c) quasibound eigenstate of the corresponding linearized problem for α=0.15. Solid and dashed lines refer to the fundamental and second harmonics, respectively.

Note that a_{n0} is real and that a_n and a_n^* have to be varied independently. Introducing the variables $a=\delta a_1+\overline{\delta a_1}$, $b=\delta a_2+\overline{\delta a_2}$, $c=\delta a_1-\overline{\delta a_1}$, and $d=\delta a_2-\overline{\delta a_2}$, Eqs. (3) can be reduced to an eigenvalue problem for λ^2

$$L_-L_+\begin{pmatrix}a\\b\end{pmatrix}=\lambda^2\begin{pmatrix}a\\b\end{pmatrix},\quad L_+L_-\begin{pmatrix}c\\d\end{pmatrix}=\lambda^2\begin{pmatrix}c\\d\end{pmatrix}, \tag{4}$$

with

$$L_\pm=\begin{pmatrix}\frac{1}{2}\,\frac{\partial^2}{\partial y^2}-1\pm a_{20} & a_{10}\\ 2a_{10} & \frac{\sigma}{2}\,\frac{\partial^2}{\partial y^2}-\alpha\end{pmatrix}. \tag{5}$$

The linear problem always has two zero eigenvalues, corresponding to translational invariance and the invariance due to an arbitrary phase of the two fields. The corresponding eigenvectors are

$$\begin{aligned}
&\begin{pmatrix}a^{(0)}\\b^{(0)}\end{pmatrix}=\begin{pmatrix}\frac{\partial a_{10}}{\partial y}\\ \frac{\partial a_{20}}{\partial y}\end{pmatrix},\quad \begin{pmatrix}c^{(0)}\\d^{(0)}\end{pmatrix}=0\\
&\begin{pmatrix}a^{(0)}\\b^{(0)}\end{pmatrix}=0,\quad \begin{pmatrix}c^{(0)}\\d^{(0)}\end{pmatrix}=\begin{pmatrix}a_{10}\\2a_{20}\end{pmatrix}.
\end{aligned} \tag{6}$$

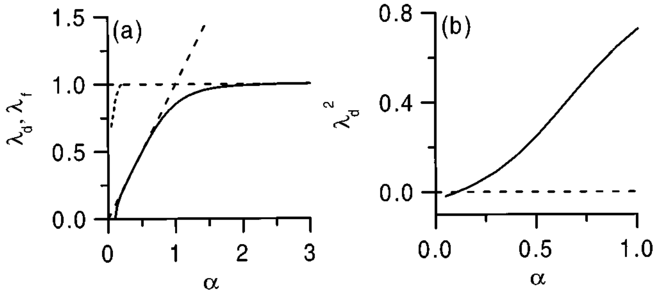

FIG. 3. (a) Eigenvalues λ_d of the discrete eigenstate (solid) and λ_f of the quasibound eigenstate (dashed) and (b) squared eigenvalue λ_d^2 of the discrete eigenstate versus control parameter α. The straight dashed lines in (a) mark the limit of the continuous spectrum.

These solutions are bound or discrete eigenstates of the linear system. Solving Eqs. (3) or Eqs. (4) numerically, we were always able to find one additional discrete eigenstate with real $\lambda_d^2 \neq 0$. The existence of this single nontrivial discrete eigenstate is a first essential result of the present work. An example is displayed in Fig. 2(b), together with the underlying solitary wave solution in Fig. 2(a). As can be seen, e.g., from Eq. (5), this discrete eigenvalue must obey the inequality $\lambda_d^2 < \min\{1,\alpha^2\}$, which marks the boundary of the continuous spectrum. The behavior of λ_d as a function of α is displayed in Fig. 3(a). It is noteworthy that the eigenvalue almost touches the continuum at $\alpha \simeq 0.4$. For $\alpha \rightarrow \infty$ it is approaching the continuum. In this limit Eqs. (1) reduce to an effective nonlinear Schrödinger equation where this kind of localized mode is absent. We checked that exactly at the instability border, $\alpha = \alpha_c \simeq 0.106$ as found in [8], λ_d^2 changes its sign, which causes the onset of the instability [Fig. 3(b)].

Comparing the numerically evaluated frequencies of the internal oscillations of the perturbed solitary wave with the above-mentioned discrete eigenvalue we find a good agreement even for stronger perturbations [$\xi \simeq 0.15$ in Eq. (2)]. Thus the assumption that the oscillations are adequately described by the linearized system seems to be correct.

However, an important open question remains: How large can the amplitude of the stable oscillations be, and why are the oscillations so persistent. If the amplitude of the initial perturbations is increased, we observed always a saturation of the established amplitude [Fig. 4(a)]. The actual frequency of the oscillations at the saturation level may be conspicuously smaller than the one predicted by the linear analysis [Fig. 4(b)]. Obviously this implies a strong softening anharmonism of the nonlinear oscillations, which pushes the frequency farther from the (lower) limit of the continuous spectrum. The limitation of the achievable amplitude of the persistent internal oscillations is due to emission of radiation in the transient period of evolution, i.e., coupling of higher harmonics of the oscillations to the continuous spectrum of the linear problem. Thus one should expect that the saturation mechanism is extremely nonlinear and the saturation amplitude is strongly dependend on the separation of the discrete state from the continuum. This is indeed the case (Fig. 5). If the discrete eigenvalue is very close to the continuum ($\alpha \simeq 0.4$ and $\alpha > 2$) the response of the solitary wave to an initial perturbation of fixed size is rather weak (and

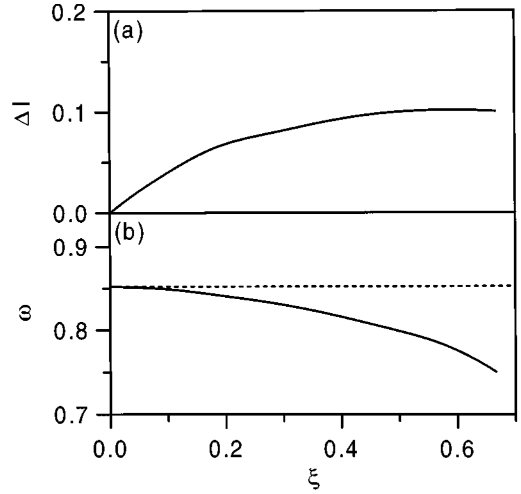

FIG. 4. (a) Oscillation amplitude of the intensity (fundamental) ΔI and (b) frequency ω of the internal oscillations vs initial excitation ξ for $\alpha=1$. The dashed line marks the frequency of the corresponding discrete eigenstate of the linear problem.

practically linear). On the other side, a natural feature of the response is that it strongly diverges near the instability threshold.

Another interesting fact revealed by the simulations is a stable beating at $\alpha < 0.2$ (Fig. 6). In this case most energy is in the second harmonic. The existence of the beating state is noteworthy, since to produce such an effect there are two frequencies necessary and there is only one discrete eigenfrequency. The frequency, which gives rise to the beatings, belongs to the continuous spectrum [$\alpha^2 < \lambda_f^2 < 1$, cf. Fig. 3(a)]. Such resonances (quasibound modes) in the continuum are well known in quantum mechanics (so-called Fano resonances [14]). They can be produced by the coupling of a bound state of one subsystem (in our case the fundamental) to the continuum of another subsystem (here the second harmonic). This leads to the appearance of a quasibound mode in the continuum with a finite (but sufficiently large) life-

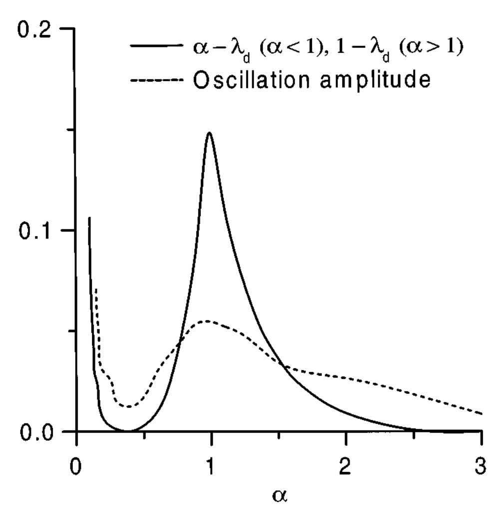

FIG. 5. Gap between the continuous spectrum and the discrete eigenvalue λ_d (solid) and final oscillation amplitude of the intensity (fundamental, dashed) of the internal oscillations versus control parameter α for $\xi=0.15$.

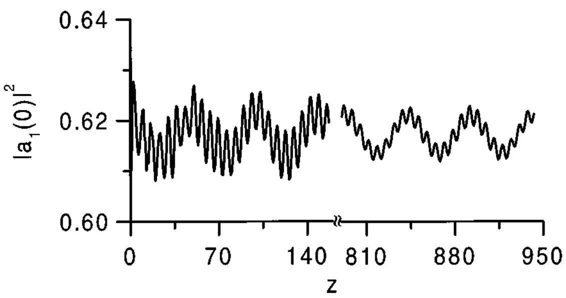

FIG. 6. Intensity of the fundamental at $y=0$ vs the propagation distance z for $\alpha=0.15$ showing the beating of the oscillations.

time. In our system, the strong second harmonic at $\alpha<0.2$ acts like an effective potential in the linearized equation corresponding to the fundamental [cf. Eqs. (3)]. The second harmonic is weakly coupled to the continuum through the weak fundamental. As a result this quasibound mode can really be observed [Fig. 2(c)]. The corresponding eigenfrequency is indeed in agreement with the frequency of the beatings displayed in Fig. 5. The larger the separation of the eigenvalue corresponding to the quasibound mode from the continuum of the fundamental ($\lambda=1$), the more easily it can be excited (as we have observed in additional simulations not shown here). There is a fairly weak radiative damping of the quasibound mode through the continuum of the second harmonic ($\lambda=\alpha$), which was found numerically to be $\sim\exp(-z/500)$.

In conclusion, in this work we have presented a systematic analysis of internal oscillations of spatial solitary waves supported by the coupling between the fundamental and second harmonics in a quadratically nonlinear medium. These oscillations seem to be undamped even if the propagation distance exceeds $z=1000$. Referring to realistic experimental situations ($LiNbO_3$ at $\lambda=1.32$ μm [10]) this corresponds to a propagation length of about half a meter. Thus from the experimental point of view these oscillations are completely stable and have to be regarded as a fundamental feature of the solitary waves. Linearizing the evolution equations around the stationary solitary wave solutions, we obtained exactly one nontrivial discrete eigenstate. The onset of the instability of the solitary waves can be naturally explained as the change of sign of the corresponding squared eigenvalue. We simulated also strongly nonlinear oscillations of the solitary wave. It was found that there is a certain maximum (saturation) amplitude of the oscillations. When the amplitude is close to the maximum, the oscillations become strongly anharmonic. Then their frequency is significantly smaller than predicted by the linear analysis. We identified also an additional discrete quasibound mode, which overlaps weakly with the continuum, but nevertheless is remarkably persistent. The mutual excitation of both the discrete and the quasibound mode leads to long-living beatings of the oscillations of the initially perturbed solitary wave.

The authors are indebted to L. Leine for valuable help with the numerics. They acknowledge a grant from Deutsche Forschungsgemeinschaft in the framework of Sonderforschungsbereich 196. Y.K. acknowledges fruitful discussions with A. V. Buryak.

[1] Y. N. Karamzin and A. P. Sukhorukov, Pis'ma Zh. Eksp. Teor. Fiz. **20**, 734 (1974) [JETP Lett. **20**, 339 (1974)].
[2] R. Schiek, J. Opt. Soc. Am. B **10**, 1848 (1993).
[3] M. J. Werner and P. D. Drummond, J. Opt. Soc. Am. B **10**, 2390 (1993).
[4] K. Hayata and M. Koshiba, Phys. Rev. Lett. **71**, 3275 (1993).
[5] A. V. Buryak and Yu. S. Kivshar, Opt. Lett. **19**, 1612 (1994); Phys. Lett. A **197**, 407 (1995).
[6] V. Steblina, Yu. S. Kivshar, M. Lisak, and B. A. Malomed, Opt. Commun. **118**, 345 (1995).
[7] L. Torner, C. R. Menyuk, and G. I. Stegeman, Opt. Lett. **19**, 1615 (1994); J. Opt. Soc. Am. B **12**, 889 (1995).
[8] D. E. Pelinovsky, A. V. Buryak, and Y. S. Kivshar, Phys. Rev. Lett. **75**, 591 (1995).
[9] C. Etrich, U. Peschel, F. Lederer, and B. Malomed, Phys. Rev. A **52**, B3444 (1995).
[10] R. Schiek, Y. Baek, and G. I. Stegeman, Phys. Rev. E **53**, 1138 (1996).
[11] V. E. Zakharov, V. V. Sobolev, and V. S. Synakh, Zh. Eksp. Teor. Fiz. **60**, 136 (1971) [Sov. Phys. JETP **33**, 77 (1971)].
[12] A. W. Snyder, S. Hewlett, and D. J. Mitchell, Phys. Rev. E **51**, 6297 (1995).
[13] D. E. Pelinovsky, V. V. Afanasjev, and Yu. S. Kivshar, Phys. Rev. E **53**, 1940 (1996).
[14] U. Fano, Phys. Rev. **124**, 1866 (1961).