Traveling-wave solutions of the cubic-quintic nonlinear Schrödinger equation

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A subset of the exact analytical traveling-wave solutions of the equation $i\Psi_t+\Psi_{zz}=a_1\Psi|\Psi|^2+a_2\Psi|\Psi|^4$ is presented in compact form. The solutions are expressed in terms of Weierstrass's elliptic function $\wp$ and include periodic and solitary-wave-like solutions. The algebraic conditions for both cases are given. Examples are presented for simple cases. [S1063-651X(96)00810-0]

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I. INTRODUCTION

Apart from their applications in various fields of physics [1,2] the traveling-wave solutions of the nonlinear Schrödinger equation (NLSE)

$$i\Psi_t+\Psi_{zz}=a_1\Psi|\Psi|^2+a_2\Psi|\Psi|^4,\quad \Psi=\Psi(z,t)\in\mathbb{C},$$
$$a_i\in\mathbb{R},\quad \{a_1,a_2\}\neq\{0,0\} \tag{1}$$

are interesting in and of themselves. If $a_2\neq 0$ Eq. (1) is not integrable: it has no Lax pair and no analytical solution techniques are available. Thus methods that are not based on an inverse scattering transformation may be suitable for finding solutions of Eq. (1). In particular, the symmetry reduction method [3,4] and the Akhmediev approach [5,6] do not depend on the equation under study being integrable.

The technique presented here is based on the symmetry reduction method. The essence of this method is to rewrite Eq. (1) in terms of the invariants of a particular symmetry group of local point transformations in order to reduce the number of the independent variables in Eq. (1). The reduced equations are ordinary differential equations that are integrated by means of a Painlevé analysis [7,8]. In some cases (depending on the values of a_1,a_2) they can be solved in terms of elementary functions, Jacobi elliptic functions, or Painlavé transcendents.

The following analysis proceeds along the line just described but uses (only) translational invariance [cf. Eq. (3)] to reduce Eq. (1). The general exact solution to the reduced ordinary differential equation [cf. Eq. (4)] is given in terms of Weierstrass's elliptic function $\wp(z;g_2,g_3)$ [9]. The analytical properties of $\wp$ are well known [9]. $\wp$ is meromorphic with respect to z and holomorphic with respect to the invariants g_2,g_3. Thus a singularity analysis is straightforward and rather simple.

The results obtained in this way are partly contained in the profound analysis by Gagnon and Winternitz [3(c)] and in an article by Gagnon [3(a)]. Details will be discussed below.

Section II presents the exact (translational invariant) solutions of the NLSE in terms of Weierstrass's elliptic function $\wp$. The cubic and the cubic-quintic NLSE are investigated in some detail in Secs. III and IV, respectively. Section V summarizes the results in comparison with those of Gagnon and Winternitz [3(c)] and Gagnon [3(a)].

II. EXACT SOLUTIONS

A Galilean transformation

$$z'=z+vt,\quad t'=t,$$
$$\Psi'(z',t')=\Psi(z,t)e^{i(v/2)[z+(v/2)t]},\quad v=\text{const} \tag{2}$$

leaves Eq. (1) invariant, meaning that if $\Psi(z,t)$ is a solution so is $\Psi'(z',t')$. A stationary wave is transformed into a traveling one by Eq. (2). A large set of traveling-wave solutions of Eq. (1) can be found by using the local point transformation

$$\Psi(z,t)=\varphi(z)e^{-i\lambda t}, \tag{3}$$

where λ is a real constant and $\varphi(z)$ a complex function. Equation (3) reduces Eq. (1) to

$$\varphi_{zz}+\lambda\varphi=a_1\varphi|\varphi|^2+a_2\varphi|\varphi|^4. \tag{4}$$

The solution to Eq. (4) can in general be written as

$$\varphi(z)=f(z)e^{ig(z)}, \tag{5}$$

where $f(z)=|\varphi(z)|$ and $g(z)$ are real functions. The substitution of this expression into Eq. (4) and the separation of the real and imaginary parts lead to

$$f_{zz}-g_z^2f+\lambda f-a_1f^3-a_2f^5=0, \tag{6}$$
$$2f_zg_z+fg_{zz}=0. \tag{7}$$

Equation (7) can be integrated twice to give

$$g(z)=C\int dz'\,f^{-2}(z')+g_0, \tag{8}$$

where C and g_0 are real constants. By suitably adjusting the phase of Ψ, g_0 can be made equal to zero so that substitution of Eq. (8) into Eq. (6) yields

$$f_{zz}-\frac{C^2}{f^3}+\lambda f-a_1f^3-a_2f^5=0 \tag{9}$$

and, upon multiplying Eq. (9) by f_z and integrating,

$$(I_z)^2=4\left(\frac{a_2}{3}I^4+\frac{a_1}{2}I^3-\lambda I^2+kI-C^2\right)\equiv R(I), \quad (10)$$

where $I(z)\equiv f^2(z)$ and k is another real constant of integration.

The constant solutions $I=I_0$, given by the real positive roots I_0 of Eq. (10), lead to the solutions $\Psi(z,t)=\sqrt{I_0}\exp[i(Cz/I_0^2-\lambda t)]$ and will be disregarded in the following. In general, Eq. (10) can be solved by [10,11]

$$I(z)=I_0+\left.\frac{\sqrt{R(I)}\,\dfrac{d\wp(z;g_2,g_3)}{dz}+\frac{1}{2}R'(I)[\wp(z;g_2,g_3)-\frac{1}{24}R''(I)]+\frac{1}{24}R(I)R'''(I)}{2[\wp(z;g_2,g_3)-\frac{1}{24}R''(I)]^2-\frac{1}{48}R(I)R''''(I)}\right|_{I=I_0}, \quad (11)$$

where the primes denote differentiation with respect to I and I_0 is a real constant [not necessarily a real zero of $R(I)$]. The invariants g_2,g_3 of Weierstrass's function $\wp(z;g_2,g_3)$ are independent of I_0 and are given by

$$g_2=-\tfrac{16}{3}a_2C^2-2a_1k+\tfrac{4}{3}\lambda^2, \quad (12)$$

$$g_3=C^2(a_1^2+\tfrac{32}{9}a_2\lambda)-\tfrac{2}{3}k(2a_2k+a_1\lambda)+\tfrac{8}{27}\lambda^3. \quad (13)$$

If I_0 is a simple zero of $R(I)$ [12], Eq. (11) reads

$$I(z)=I_0+\left.\frac{R'(I)}{4[\wp(z;g_2,g_3)-\frac{1}{24}R''(I)]}\right|_{I=I_0}. \quad (14)$$

Inserting this solution into Eq. (8), integration [13] yields the phase function

$$g(z)=C\left(\frac{z}{I}+\frac{R'(I)\left(\ln\dfrac{\sigma(z+v)}{\sigma(z-v)}-2z\zeta(v)\right)}{4I^2\sqrt{4\wp^3(v;g_2,g_3)-g_2\wp(v;g_2,g_3)-g_3}}\right)_{I=I_0}+\text{const}, \quad (15)$$

with

$$v=\int_{u_0}^{\infty}\frac{du}{\sqrt{s(u)}}, \quad (16)$$

$$s(u)=4u^3-g_2u-g_3, \quad (17)$$

and $u_0=R''(I)/24-R'(I)/4I|_{I=I_0}$. The functions $\sigma(z)$ and $\zeta(z)$ denote Weierstrass's sigma and zeta functions, respectively [14].

The roots of e_1,e_2,e_3 of $s(u)=0$ are important for the behavior of $I(z)$ according to Eq. (11). If the discriminant Δ of $\wp$

$$\Delta=g_2^3-27g_3^2 \quad (18)$$

is negative, there are one real non-negative root and a pair of complex conjugate roots. If $\Delta\geqslant 0$, the roots are real (if $\Delta=0$, at least two roots are equal, if $\Delta>0$, the real roots are distinct) [9]. Since (if z is real) $\wp(z,g_2,g_3)$ is bounded from below by e_1 [15], real non-negative and bounded solutions according to Eq. (11) require that I_0 satisfies

$$\begin{aligned}
&I(0)=I_0\geqslant 0,\\
&R(I_0)\geqslant 0,\\
&2\left(e_1-\frac{R''(I)}{24}\right)^2-\left.\frac{R(I)R''''(I)}{48}\right|_{I=I_0}>0,\\
&I\left\{2\left(e_1-\frac{R''(I)}{24}\right)^2-\frac{R(I)R''''(I)}{48}\right\}+\frac{R'(I)}{2}\left(e_1-\frac{R''(I)}{24}\right)\\
&\quad+\left.\frac{R(I)R'''(I)}{48}\right|_{I=I_0}\geqslant 0,
\end{aligned} \quad (19)$$

where e_1 is the real (non-negative) root of $s(u)$ (if $\Delta>0$, e_1 is the largest root). It only depends on the coefficients of $R(I)$. Thus Eqs. (11), (14), (15), and (19) represent the complete solution to Eq. (4). Obviously the conditions (19) can be simplified if a simple root $I_0\geqslant 0$ of $R(I)$ exists. They show that it is not necessary to know the roots of $R(I)$ in order to formulate conditions for different non-negative and bounded solutions [cf. [3(a)], [3(c)] and Sec. V]. In particular, all non-negative and singular solutions are determined by conditions (19) if the third condition is replaced by

$$2\left(e_1-\frac{R''(I)}{24}\right)^2-\left.\frac{R(I)R''''(I)}{48}\right|_{I=I_0}\leqslant 0.$$

As is well known [1], the qualitative behavior of $I(z)$ can be determined by a phase diagram considering the graph of $R(I)$: real non-negative and bounded solutions $I(z)$ require that I_0 is in the closed interval between two positive zeros of R, with $R(I)\geqslant 0$ between the zeros (shaded in Fig. 1). For convenience this condition is referred to in the following as the phase diagram condition (PDC).

Obviously [see Fig. 1], the equation $R(I)=0$ has at least one simple real root if $C\neq 0$ and if the PDC is satisfied. Thus, in this case, Eq. (14) can be used instead of Eq. (11) resulting in Eq. (15). The lengthy integration in Eq. (8) is not necessary in this case. If $C=0$ it is possible that there is no simple root of $R(I)=0$, so that Eq. (11) must be used with I_0

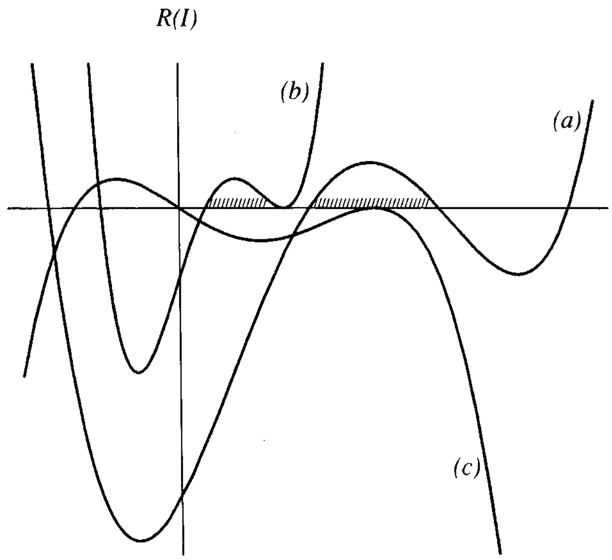

FIG. 1. Sketches of the graphs of $R(I)$. The phase diagram condition (PDC) for real positive and bounded solutions is satisfied only if I_0 is in the shaded regions. (a) represents a periodic and (b) a (gray) solitary-wave-like solution; (c) represents a solution for which the PDC is not fulfilled.

subject to the PDC. But in this case $g(z)\equiv 0$, which means that Eq. (15) is sufficiently general (cf. Ref. [12]).

The solutions $I(z)\geqslant 0$ [according to Eq. (11) or (14)] that fulfill the PDC are, in general, periodic or solitary-wave-like. In order to give conditions for both cases, it is useful, for obvious reasons, to consider the expressions for the real period of $\wp(z;g_2,g_3)$ [16]: if $\Delta\neq 0$, the real period ω of $\wp$ is finite (not zero). If $\Delta=0$, $g_2>0$, and $g_3>0$, the real period is also finite (not zero). If $\Delta=0$, $g_2\geqslant 0$, and $g_3\leqslant 0$, the real period is $\omega=\infty$. Thus the solutions $I(z)\geqslant 0$ that satisfy the PDC and $\Delta\neq 0$ or the PDC and $\Delta=0$, $g_2>0$, and $g_3>0$ are periodic. The period of $I(z)$ is determined by the elliptic integrals

$$2\omega=\begin{cases} 2\displaystyle\int_{e_1}^{\infty} du\ s(u)^{-1/2}, & \Delta>0 \\ 2\displaystyle\int_{e_2}^{\infty} du\ s(u)^{-1/2}, & \Delta<0. \end{cases} \tag{20}$$

The period is given by

$$2\omega=4\pi(6e_1)^{-1/2} \tag{21}$$

if $\Delta=0$, $g_2>0$, and $g_3>0$. In this case the zeros of $s(u)=0$ are $e_2=e_3=-e_1/2$ and $e_1>0$. The amplitude A of the periodic solution can be determined by means of Eq. (14):

$$A=\left|\frac{R'(I)}{4(e_1-R''(I)/24)}\right|_{I=I_0}, \tag{22}$$

where I_0 is a simple root of $R(I)=0$ (at $z=0$).

Solutions $I(z)\geqslant 0$ that satisfy the PDC and $\Delta=0$, $g_2\geqslant 0$, and $g_3\leqslant 0$ are solitary-wave-like. If $g_2>0$ and $g_3<0$ ($e_1=e_2>0$ and $e_3=-2e_1$), the solutions $I(z)$ can be expressed by hyperbolic functions (as degenerate cases of $\wp$ [17]). If $g_2=g_3=0$ ($e_1=e_2=e_3=0$), $\wp$ degenerates to a z^{-2} dependence [17]. Accordingly, there are nonalgebraic and algebraic solitary-wave solutions [18]. Examples are given in Secs. III and IV.

If there is a simple root I_0 (as is the case for bright and dark solitons), Eq. (22) gives the amplitude of the solitary wave. If there is no simple root I_0, it is necessary to use Eq. (11). In this case $C^2=k=0$ must hold, so that the PDC yields $a_2>0$, $a_1<0$, and $\lambda=-3a_1^2/16a_2$. Thus the amplitude of the kink solitary wave is

$$A=\frac{-3a_1}{4a_2}. \tag{23}$$

Equations (20)–(22) can be used to investigate the dependence of the half period ω and of the amplitude A on the parameters a_2,a_1,λ,k,C^2.

Summarizing this section, the periodic and solitary-wave-like solutions of Eq. (1) that obey Eq. (3) all have the form

$$\Psi(z,t)=\sqrt{I(z)}e^{i[g(z)-\lambda t]}, \tag{24}$$

with $I(z),g(z)$ determined by Eq. (11) [or Eq. (14)] and Eq. (15), respectively. The constants I_0 and a_1,a_2,λ,k,C^2 must satisfy the PDC or Eq. (19).

The period of the periodic solutions can be evaluated as a complete elliptic integral of the first kind [16] according to Eq. (20) or (21). Solitary-wave-like solutions are determined by the PDC and $\Delta=0$, $g_2\geqslant 0$, and $g_3\leqslant 0$, where the upper and the lower signs correspond to nonalgebraic and algebraic solutions, respectively. The amplitude A of $I(z)$ is determined by Eqs. (22) and (23), respectively. The transformation of $\Psi(z,t)$ according to Eq. (2) yields the traveling-wave solutions to Eq. (1)

$$\Psi'(z',t')=\sqrt{I(z'-vt')}e^{i[g(z'-vt')+(v/2)z'-(\lambda+v^2/4)t']}. \tag{25}$$

III. SOLUTIONS OF THE CUBIC NONLINEAR SCHRÖDINGER EQUATION

We now consider Eq. (1) and the corresponding reduced equation (10) with $a_2=0$, $a_1\neq 0$, and $C^2\neq 0$. For a_1,λ,k,C^2,I_0 to be consistent with the PDC the zeros of $R(I)$ must fall under one of the three categories depicted in Fig. 2.

Obviously, the PDC cannot be fulfilled if $R(I)=0$ has a triple root. Thus no algebraic solitary waves exist if $a_2=0$. There are one real and two complex roots of R if $\Delta<0$. Hence the PDC is not satisfied if $\Delta<0$.

If $a_1<0$, $\Delta>0$, and there are two changes of sign in the sequence $a_1,-\lambda,k,-C^2$, two positive zeros I_1,I_2 [Fig. 2(a)] of $R(I)$ exist (according to the Cartesian sign rule [19]), so that periodic solutions $I(z)$ according to Eq. (14) with $I_0\in[I_1,I_2]$ are possible. Since a double root cannot occur in this case (see Fig. 2), there are no solitary-wave-like solutions for $a_1<0$ and $C^2\geqslant 0$.

To find the periodic solutions in this case it is necessary to determine the ranges of the parameters a_1,λ,k,C^2 consistent with $\Delta>0$. This is done by solving $\Delta=0$ for those parameters that are appropriate for the problem in question. An example is shown in Fig. 3. If $a_1<0$ and C^2 are given, the solution

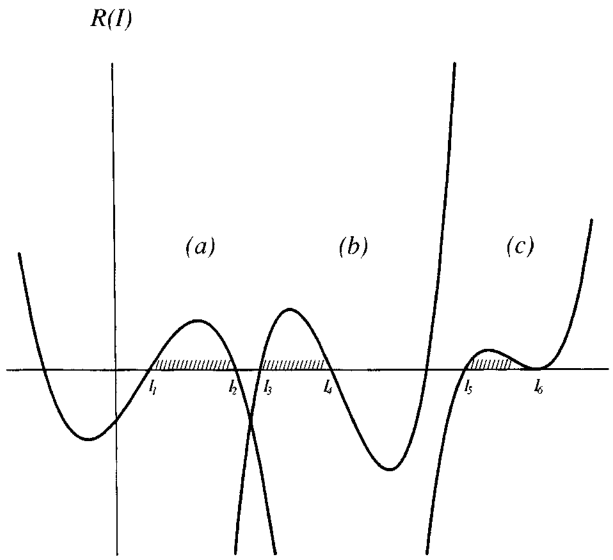

FIG. 2. Graphs of $R(I)$ if $a_2=0$ and $C^2\neq 0$: (a) periodic solution, if $a_1<0$; (b) periodic solution, if $a_1>0$; (c) solitary-wave-like solution, if $a_1>0$. Shading is as in Fig. 1.

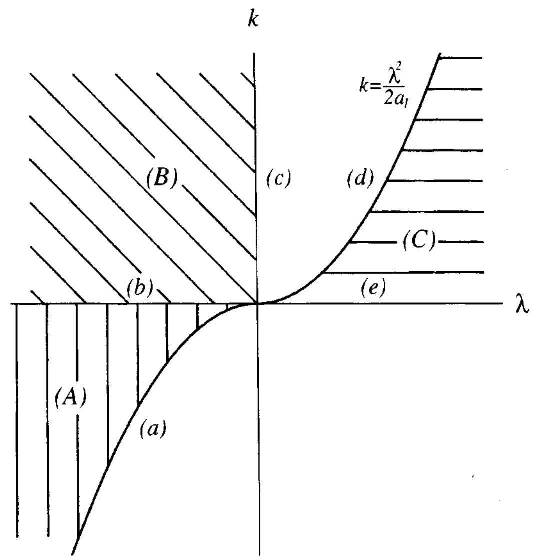

FIG. 4. Pairs $\{\lambda,k\}$ associated with different solutions $I(z)$ for $a_2=0$ and $C^2=0$: region (A), solution according to Eq. (27) ($a_1<0$); region (B), solution according to Eq. (26) ($a_1<0$); region (C), solution according to Eq. (26) ($a_1>0$). Solutions on the boundaries: (a) singular ($a_1<0$, $\Delta=0$), (b) solitary-wave-like ($a_1<0$, $\Delta=0$), (c) periodic ($a_1<0$, $\Delta=-8a_1^3k^3$), (d) solitary-wave-like ($a_1>0$, $\Delta=0$), and (e) singular ($a_1>0$, $\Delta=0$).

$k(\lambda)$ of $\Delta=0$ can be evaluated [curve (a) in Fig. 3]. Pairs $\{\lambda,k\}$ in the shaded region (A) (excluding the boundary) are associated with periodic solutions of Eq. (10) given by Eq. (14), where I_0 can be taken as one of the simple roots of $R(I)=0$.

If $a_1>0$, $\Delta\geqslant 0$, and (according to the Cartesian sign rule) there are three changes of sign in the sequence $a_1,-\lambda,k,-C^2$, leading to $\lambda>0$ and $k>0$, periodic solutions [$\Delta>0$; Fig. 2, curve (b)] or gray solitary-wave-like solutions [$\Delta=0$; Fig. 2, curve (c)] according to Eq. (14) are possible, where $I_0\in[I_3,I_4]$ must be chosen in the first case and $I_0\in[I_5,I_6]$ in the second case. Assuming, as in the former example, that $a_1>0$ and C^2 are fixed, evaluation of $\Delta=0$ yields results shown in Fig. 3. Pairs $\{\lambda,k\}$ in region (B) are associated with periodic solutions ($\Delta>0$) and pairs $\{\lambda,k\}$ of curve (b) represent all (gray) solitary-wave-like solutions [points on curve (c) have to be excluded since the associated solutions are singular].

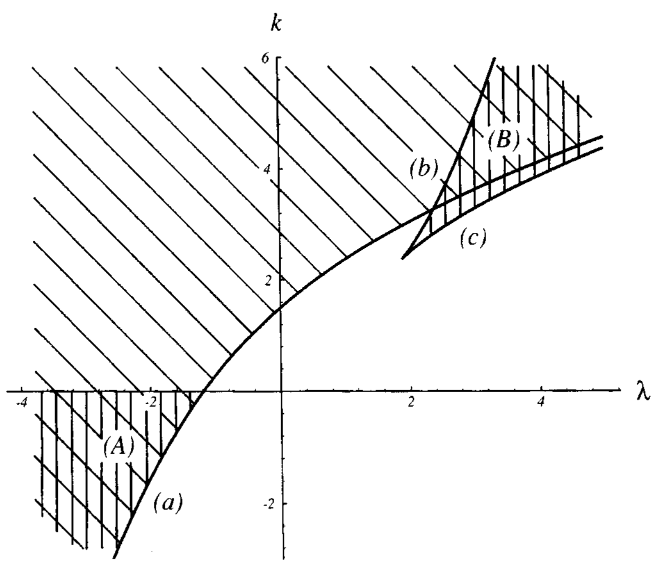

FIG. 3. Pairs $\{\lambda,k\}$ associated with solutions $I(z)$ for $a_2=0$ and given a_1 and C^2: region (A) (boundary excluded), periodic solutions for $a_1=-1$ and $C^2=1$; region (B), periodic solutions for $a_1=1$ and $C^2=1$; curve (b), solitary-wave-like solutions; curve (c), singular solutions.

It is instructive to illustrate the previous analysis by assuming $C^2=0$ [20]. An evaluation of $\Delta\geqslant 0$ determines the permitted pairs $\{\lambda,k\}$ shown in Fig. 4. The roots of $R=0$ are $\{0,(\lambda\pm\sqrt{\lambda^2-2a_1k})/a_1\}$ and the roots of $s=0$ are $\{-\lambda/3,\lambda/6\pm\frac{1}{2}\sqrt{\lambda^2-2a_1k}\}$. The invariants of $\wp$ are given by $g_2=\frac{4}{3}\lambda^2-2a_1k$ and $g_3=\frac{8}{27}\lambda^3-\frac{2}{3}a_1k\lambda$ in this case.

If $\Delta>0$, $a_1>0$, $\lambda>0$, and $k>0$ (see Fig. 4) or $\Delta>0$, $a_1<0$, $\lambda<0$, and $k>0$, $I_0=0$ is a simple root of $R=0$. Insertion in Eq. (14) yields the solutions

$$I(z)=\frac{k}{\frac{\lambda}{3}+\wp(z;g_2,g_3)}. \tag{26}$$

If $\Delta>0$, $a_1<0$, $\lambda<0$, and $k<0$ (see Fig. 4), $I_0=(\lambda+\sqrt{\lambda^2-2a_1k})/a_1$ is a simple root of $R=0$. Hence Eq. (14) yields the solutions

$$I(z)=\frac{\lambda+\sqrt{\lambda^2-2a_1k}}{a_1}\times\left(1+\frac{\sqrt{\lambda^2-2a_1k}}{\wp(z;g_2,g_3)-\frac{\lambda}{6}-\frac{1}{2}\sqrt{\lambda^2-2a_1k}}\right). \tag{27}$$

The period of $I(z)$ is determined by

$$2\omega=\frac{2K(m)}{\sqrt{e_{\max}-e_{\min}}},\tag{28}$$

with $m=(e_m-e_{\min})/(e_{\max}-e_{\min})$ and K denoting the complete elliptic integral of the first kind [9]. The roots of $s(u)=0$ are ordered according to $e_{\min}<e_m\leqslant e_{\max}$ (for m and $e_{\min},e_m,e_{\max}$ associated with the various parameters a,λ,k see the Appendix). The amplitude of $I(z)$ is determined by the zeros of $(dI/dz)^2=R(I)$:

$$A=\begin{cases}\dfrac{1}{a_1}(\lambda-\sqrt{\lambda^2-2a_1k}), & \Delta>0,\quad a_1>0,\quad \lambda>0,\quad k>0\\ \dfrac{1}{a_1}(\lambda-\sqrt{\lambda^2-2a_1k}), & \Delta>0,\quad a_1<0,\quad \lambda<0,\quad k>0\\ \dfrac{-2}{a_1}\sqrt{\lambda^2-2a_1k}, & \Delta>0,\quad a_1<0,\quad \lambda<0,\quad k<0,\end{cases}\tag{29}$$

in agreement with Eq. (22).

Evaluation of the conditions $\Delta=0$, $g_2>0$, and $g_3<0$ for solitary waves yields $a_1>0$, $\lambda>0$, and $k=\lambda^2/2a_1$ or $a_1<0$, $\lambda<0$, and $k=0$ (see Fig. 4). In the first case the simple root of $R=0$ is $I_0=0$. In the second case the simple root of $R=0$ is $I_0=2\lambda/a_1$. Using $\wp(z)=e_1[1+3/\sinh^2(\sqrt{3e_1}z)]$ [17] (e_1 is equal to $\frac{1}{6}\lambda$ in the first case and equal to $-\frac{1}{3}\lambda$ in the second case) and inserting I_0 and $\wp$ into Eq. (14) yields

$$I(z)=\begin{cases}\dfrac{\lambda}{a_1}\tanh^2\left(\sqrt{\dfrac{\lambda}{2}}\,z\right), & a_1>0,\ \lambda>0,\ k=\dfrac{\lambda^2}{2a_1} \quad (30)\\ \dfrac{2\lambda}{a_1}\,\mathrm{sech}^2(\sqrt{-\lambda}z), & a_1<0,\ \lambda<0,\ k=0. \quad (31)\end{cases}$$

Equations (30) and (31) describe dark and bright solitary waves, respectively. The amplitudes are consistent with Eq. (22).

IV. SOLUTIONS OF THE CUBIC-QUINTIC NONLINEAR SCHRÖDINGER EQUATION

If $a_2\neq 0$ and $C^2\neq 0$, the solution of Eq. (10) proceeds along the lines described in Sec. III. It is useful to consider the cubic resolvent [21] of R

$$\mathrm{Res}=z^3+2pz^2+(p^2-4r)z-q^2,\tag{32}$$

where

$$p=-\frac{3}{a_2}\left(\frac{9a_1^2}{32a_2}+\lambda\right),$$

$$q=-\frac{3}{a_2}\left(\frac{81a_1^4}{4096a_2^3}+C^2+\frac{3a_1k}{8a_2}+\frac{9a_1^2\lambda}{64a_2^2}\right),$$

$$r=\frac{3}{a_2}\left(k+\frac{9a_1^3}{64a_2^2}+\frac{3a_1\lambda}{4a_2}\right).$$

The discriminant of R [22] is equal to Δ. Thus the roots of $R=0$ can be discriminated by Δ.

If $\Delta>0$, all roots of $R=0$ are real provided that $p<0$ and $p^2-4r>0$; otherwise there are two pairs of complex conjugate roots, for which the PDC cannot be satisfied. If $p<0$ and $p^2-4r>0$, the number of positive roots of $R=0$ determines whether or not the PDC can be fulfilled. According to the Cartesian sign rule the number of positive roots is equal to the number of changes of sign in the sequence $a_2,a_1,-\lambda,k,-C^2$ since all roots are real in this case.

If $\Delta<0$, there is a pair of real roots and a pair of complex conjugate roots [23]. If $a_2>0$, one root must be negative according to the Viète relations, with the result that the PDC cannot be satisfied in this case. If $a_2<0$, two positive (and simple) roots that satisfy the PDC can exist. In this case there must be at least two changes of sign in the sequence $a_2,a_1,-\lambda,k,-C^2$.

If $\Delta=0$, there are multiple real roots leading to solitary-wave-like or periodic solutions, if the PDC is fulfilled. Examples of phase diagrams are shown in Fig. 5. Since $\Delta=0$, all zeros of the cubic resolvent Res are real. If these zeros are all positive (if $p<0$ and $p^2-4r>0$), there are four real roots of $R=0$; otherwise (one zero of Res positive, two zeros negative) there are two pairs of complex conjugate roots of $R=0$. Hence $p<0$ and $p^2-4r>0$ are necessary for physical solutions of Eq. (10). In particular, if $a_2>0$ [see Fig. 5(a)] there must be (only) one negative root (otherwise the PDC

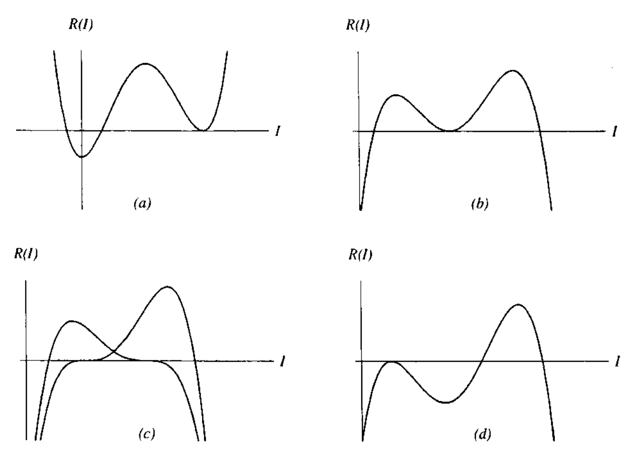

FIG. 5. Phase diagrams of the cubic-quintic case if $\Delta=0$ and $C^2\neq 0$. Associated solutions: (a) gray solitary-wave-like, (b) two (gray and bright) solitary-wave-like, (c) gray and bright solitary-wave-like, and (d) periodic.

cannot be fulfilled) and three positive roots. Thus three changes of sign in the coefficients of R are necessary, and the PDC is satisfied, if the simple (positive) root is smaller than the double root [Fig. 5(a)]. The associated solution is (gray) solitary-wave-like. Further possibilities of fulfilling the PDC do not exist if $\Delta=0$ and $a_2>0$.

If $a_2<0$, two negative roots and one positive root of $R=0$ are possible; however, the PDC is not satisfied in this case. Possibilities consistent with the PDC are illustrated in Figs. 5(b)–5(d). A double root between the simple roots leads to a dark and a bright solitary wave [Fig. 5(b)]. A bright (gray) solitary wave is possible if the triple root is to the left (right) of the simple one [Fig. 5(c)] and a periodic solution exists if the double root is to the left of the simple ones [Fig. 5(d)]. In all cases [Fig. 5(c) and 5(d)] four changes of sign in the sequence $a_2, a_1, -\lambda, k, C^2$ are necessary.

To elucidate the preceding procedure the case illustrated in Fig. 5(c) will be considered in some detail. Since there is a triple root of $R=0$,

$$g_2=g_3=0 \tag{33}$$

must hold. Solving Eq. (33) for C^2 and k,

$$C^2_{b,d}=\frac{27a_1^4+144a_1^2a_2\lambda+128a_2^2\lambda^2\mp a_1\sqrt{(9a_1^2+32a_2\lambda)^3}}{512a_2^3}, \tag{34}$$

$$k_{b,d}=\frac{-27a_1^3-144a_1a_2\lambda\pm\sqrt{(9a_1^2+32a_2\lambda)^3}}{192a_2^2}, \tag{35}$$

inserting Eqs. (34) and (35) into Eq. (10), and solving for I yields the simple roots

$$I_{0_{b,d}}=\frac{-3}{8a_2}\,(a_1\pm\sqrt{9a_1^2+32a_2\lambda}). \tag{36}$$

Since $p<0$ holds, $9a_1^2+32a_2\lambda>0$ is necessary for $C^2_{b,d}, k_{b,d}, I_{0_{b,d}}$ to be real. The sign rule (for four positive roots) yields $a_2<0$, $a_1>0$, $\lambda>0$, $k>0$, and $C^2>0$ as necessary conditions for physical solutions. $p^2-4r>0$ is always satisfied by C^2 and k according to Eqs. (34) and (35). Selecting $a_2<0$, $a_1>0$, and $\lambda>0$, the upper or lower sign or both signs in Eqs. (34) and (35) can result in $C^2>0$ and $k>0$. Accordingly, the upper sign in Eqs. (34)–(36) represents a bright solitary wave while the lower sign is related to a dark (gray) solitary wave (if $k>0$ and $C^2>0$). Since $e_1=e_2=0$, Weierstrass's function $\wp$ has to be replaced by z^{-2} so that Eq. (14) reads

$$I(z)=I+\left.\frac{R'(I)z^2}{4\left(1-\frac{R''(I)}{24}z^2\right)}\right|_{I=I_{0_{b,d}}}. \tag{37}$$

This equation describes the intensity of the algebraic solitary waves with C^2 and k determined by Eqs. (34) and (35) and subject to the conditions listed above. The numerical evaluation of Eq. (37) is straightforward. The phase $g(z)$ can be obtained by inserting Eqs. (37), (34), and (35) into Eq. (8):

$$g_{b,d}(z)=\sqrt{C^2_{b,d}}\left.\left(\frac{zR''(I)}{IR''(I)-6R'(I)}+\frac{12\sqrt{6}R'(I)\arctan\left(\frac{z\sqrt{6R'(I)-IR''(I)}}{2\sqrt{6I}}\right)}{\sqrt{I[6R''(I)-IR''(I)]^3}}\right)\right|_{I=I_{0_{b,d}}}. \tag{38}$$

It is remarkable that a bright and a dark (gray) solitary wave can exist for the same values of a_2, a_1, λ if these are chosen according to Eqs. (34) and (35) with $C^2_{b,d}>0$ and $k_{b,d}>0$ (e.g., $a_2=-1$, $a_1=1.9$, and $\lambda=1$).

It is intriguing to illustrate some further features of the procedure for the case $C^2=0$. The cases considered are represented by the phase diagrams illustrated in Fig. 6. If the PDC is satisfied and $\Delta>0$, $p<0$, and $p^2-4r>0$, and if there are three changes of sign in the sequence $a_2, a_1, -\lambda, k$, Figure 6(a) is the associated phase diagram. If $a_2<0$, two periodic solutions, determined by Eq. (14), are possible, where I_0 must be chosen according to $I_0\in[0,I_3]$ or $I_0\in[I_5,I_6]$, respectively. If $a_2>0$, only one periodic solution is possible with $I_0\in[I_1,I_2]$ and the above conditions being fulfilled. The PDC, $\Delta=0$, $p<0$, $p^2-4r>0$, $g_2>0$, and $g_3<0$, and three changes of sign in the sequence a_2 $(<0), a_1, -\lambda, k$ yield a phase diagram as shown in Fig. 6(b). The sign rule yields $a_1>0$ and $\lambda>0$. Since $p<0$ and $a_1^2+4a_2\lambda>0$, there are two possible solutions k to $\Delta=0$,

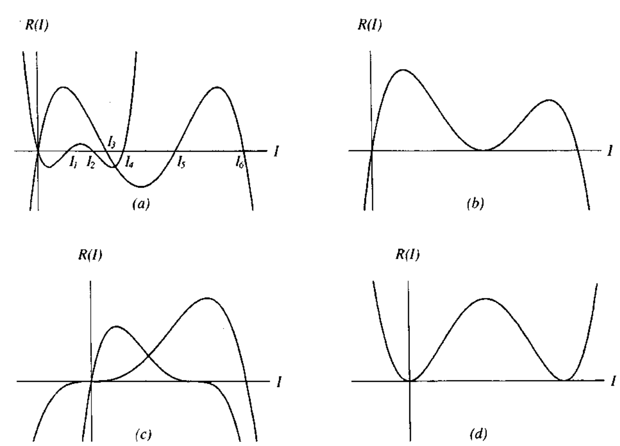

FIG. 6. Phase diagrams of the cubic-quintic case if $C^2=0$. Associated solutions: (a) periodic $a_2<0$, (b) dark and bright solitary-wave-like, (c) algebraic bright and dark solitary-wave-like, and (d) kink solitary-wave-like.

$$k_{1,2}=\frac{-a_1^3-6a_1a_2\lambda\mp\sqrt{(a_1^2+4a_2\lambda)^3}}{12a_2^2}. \tag{39}$$

The simple nonvanishing roots I_s ($\neq 0$) and the double roots I_d of $R=0$ are, accordingly,

$$I_{s_{1,2}}=-\frac{1}{a_2}\left(\frac{a_1}{2}\mp\sqrt{a_1^2+4a_2\lambda}\right), \tag{40}$$

$$I_{d_{1,2}}=-\frac{1}{2a_2}\,(a_1\pm\sqrt{a_1^2+4a_2\lambda}). \tag{41}$$

For a double solitary-wave solution according to Fig. 6(b) $I_d<I_s$ must hold. Hence $k=k_1$ must be excluded ($I_s<I_d$ and $k=k_1$ leads to a periodic solution). For $k=k_2$ Eq. (14) can be evaluated with $I_s=0$ and I_{s_2}. Thus [24]

$$I(z)=\frac{-a_1^3-6a_1a_2\lambda+\sqrt{a_1^2+4a_2\lambda}}{4a_2^2(3\wp(z;g_2,g_3)+\lambda)} \tag{42}$$

represents a dark solitary wave, while

$$I(z)=-\frac{1}{a_2}\left(\frac{a_1}{2}+\sqrt{a_1^2+4a_2\lambda}\right)+\frac{18\sqrt{(a_1^2+4a_1\lambda)^3}-9a_1^2-36a_1a_2\lambda}{4a_2[12a_2\wp(z;g_2,g_3)-7a_1^2-28a_2\lambda-2a_1\sqrt{a_1^2+4a_1\lambda}]} \tag{43}$$

represents a bright one. The invariants of $\wp$ are

$$g_2=\frac{4}{3}\lambda^2+\frac{a_1[a_1^3+6a_1a_2\lambda-\sqrt{(a_1^2+4a_2\lambda)^3}]}{6a_2^2}, \tag{44}$$

$$g_3=\frac{-a_1^6-9a_1^4a_2\lambda-24a_1^2a_2^2\lambda^2-16a_2^3\lambda^3+(a_1^3+3a_1a_2\lambda)\sqrt{a_1^2+4a_2\lambda}}{54a_2^3}. \tag{45}$$

Figure 6(c) is the phase diagram of two solitary waves. The conditions to be solved are $g_2=g_3=0$ subject to $a_2<0$, $a_1>0$, $p<0$, $p^2-4r>0$, and the PDC. Obviously there are various possibilities to fulfill $g_2=g_3=0$. Solving for λ and k yields $\lambda=k=0$ with a simple root $I_0=-3a_1/2a_2$. This leads to a bright solitary wave [18]

$$I(z)=\frac{2a_1}{a_1z^2-\frac{4}{3}a_2}. \tag{46}$$

Another solution of $g_2=g_3=0$ is $\lambda=-a_1^2/4a_2$ and $k=a_1^3/24a_2^2$, with a simple root $I_0=0$, and is thus associated with a dark solitary wave

$$I(z)=\frac{a_1^3z^2}{2a_1^2a_2z^2+24a_2^2}. \tag{47}$$

Further solutions to $g_2=g_3=0$ are

$$I(z)=\frac{kz^2}{1+\sqrt{\frac{a_1k}{6}z^2}},\quad a_2=-\frac{1}{2}\left(\frac{a_1^3}{6k}\right)^{1/2},\quad \lambda=\left(\frac{3a_1k}{2}\right)^{1/2}, \tag{48}$$

$$I(z)=\frac{kz^2}{1+\frac{\lambda}{3}z^2},\quad a_2=-\frac{\lambda^3}{9k^2},\quad a_1=\frac{2\lambda^2}{3k}, \tag{49}$$

$$I(z)=\frac{2\lambda^2z^2}{3a_1\left(1+\frac{\lambda}{3}z^2\right)},\quad a_2=-\frac{a_1^2}{4\lambda},\quad k=\frac{2\lambda^2}{3a_1}, \tag{50}$$

$$I(z)=\frac{k\sqrt{3}z^2}{\sqrt{3}-(a_2k^2)^{1/3}},\quad a_1=-2(-3a_2^2k)^{1/3},$$

$$\lambda=-(9a_2k^2)^{1/3}, \tag{51}$$

which are consistent with $a_2<0$, $a_1>0$, $\lambda>0$, and $k>0$. All solutions are algebraic [18]. Solution (46) is Lorentzian shaped and solutions (47)–(51) exhibit a non-Lorentzian shape.

Considering Fig. 6(d) and solving $\Delta=0$ for k,λ, subject to $a_2>0$, $a_1<0$, and the PDC, yields $k=0$ and $\lambda=-3a_1^2/16a_2$. There is no simple root of $R(I)=0$ in this case, so that Eq. (11) must be used instead of Eq. (14). Noting that $e_1=a_1^2/16a_2$ and choosing $I_0=-3a_1/8a_2$, evaluation [17] of Eq. (11) gives

$$I_\pm(z)=\frac{-3a_1}{4a_2\left\{1+\exp\left[\pm\left(\frac{3}{a_2}\right)^{1/2}\frac{a_1}{2}z\right]\right\}}. \tag{52}$$

Equation (52) represents a kink solitary wave.

V. CONCLUSION

There are some connections between the foregoing analysis and the literature [1,3]. In particular, the articles by Gagnon and Winternitz [3(c)] and Gagnon [3(a)] are of interest here. Thus it is appropriate to compare the above results with those of Refs. [3(c)] and [3(a)].

Obviously, Eq. (1.1) of Ref. [3(c)] is more general than

Eq. (1). In Sec. 2 of Ref. [3(c)] Eq. (1.1) is reduced to 14 different partial differential equations, one of which [Eq. (2.13)] is equivalent to Eq. (1). Section 3 of Ref. [3(c)] presents several reductions to ordinary differential equations. There are 15 inequivalent ones, three of which [Eqs. (3.12), (3.13), and (3.22)] provide solutions of Eq. (1).

In Sec. 4 of Ref. [3(c)] two of these solutions are investigated further: Eq. (4.4) is equivalent to Eq. (9). Sections 4.3 and 4.4 of Ref. [3(c)] show that Eq. (4.4) passes the Painlevé test for all values of the parameters. Integration of Eq. (4.4) [3(c)] and of Eq. (9) yields equivalent results [Eqs. (4.62a) and (10), respectively]. Sections 4.3. and 4.4. of Ref. [3(c)] provide various solutions of Eq. (4.4) depending on the roots of the polynomial $P(W)$ in Ref. [3(c)]. In the present article the solution of Eq. (10) [and thus of Eq. (4)] is given by Eqs. (11) and (15) depending compactly on the coefficients of the polynomial $R(I)$. The various solutions to Eq. (4.4) are discussed with the roots of $P(W)$ yielding a list of solutions. The above treatment (Secs. III and IV) shows that some statements about the solutions are possible without explicitly knowing the roots of $R(I)$.

In Ref. [3(a)] the translationally invariant solutions of Sec. 4.4 in Ref. [3(c)] are treated in detail. In this respect Refs. [3(a)] and [3(c)] and the present article deal with the same problem [cf. Eq. (3)]. In Ref. [3(a)] the reduced equation (17) [which is equivalent to Eq. (10)] is solved for different ordered quadruples $\{W_i\}$ (for the cubic-quintic case), which are obtained by a phase diagram analysis [cf. remark (d), p. 1479 in Ref. [3(a)]]. The result is a list [Table 1 in Ref. [3(a)]] of solutions and of conditions on the parameter values for each solution with a specification of its general behavior (column 2 of Table 1).

Within the frame of the symmetry reduction method the mathematical approach in the present paper is different from the approach in Ref. [3(a)]. Since the general solution of Eq. (4) is given by Eqs. (11) and (15), there is, in principle, no need for a list. All functions W and N in Ref. [3(a)] [W and χ in Ref. [3(c)]] can be expressed by Eqs. (11) and (15), respectively.

As shown in Secs. III and IV, a combination of the general solution Eq. (11) with a phase diagram analysis based on Eq. (10) leads to a different and simpler classification of the solutions since the roots (and their degeneracy) of $R=0$ can be described by the discriminant Δ of Weierstrass's function and thus, according to Eq. (11), leading to general conditions for periodic and solitary-wave-like solutions (Sec. II). This kind of classification yields several simple results: for instance, different families of solutions [Eqs. (26), (27), (37), (42), (43), and (48)–(51)], a necessary condition for solitary-wave-like solutions ($\Delta=0$, $g_2\geqslant 0$, and $g_3\leqslant 0$), the nonexistence of algebraic solitary-wave-like solutions if $a_2=0$ or if $a_1<0$ and $C^2\neq 0$, and the nonexistence of physical solutions if $a_2>0$ and $\Delta>0$.

A further result of the above approach is an algebraic version of the phase diagram conditions (19). As outlined in Sec. II, all singular solutions [not listed in Ref. [3(a)]] can be excluded by the third condition (19), so that a singularity analysis based on Eq. (11) is simplified considerably. Furthermore, Eq. (11) yields general expressions for periods and amplitudes [Eqs. (20), (22), and (23)].

Solutions to Eq. (16) in Ref. [3(a)] and to Eq. (4.62a) in Ref. [3(c)] are given in terms of Jacobi elliptic functions. Certainly, these are completely equivalent to Weierstrass's elliptic function $\wp$ used above. But the use of $\wp$ is not only a matter of taste. As is obvious from Table 1 in Ref. [3(a)], the solutions strongly depend on the degeneracy of the roots W_i. Thus it is rather inconvenient to regard W_i as the ultimate input parameters of the problem [cf. Ref. [3(a)]]. It may happen in practice [20] that $I(z)$ [or W in Ref. [3(a)]] has to be evaluated numerically for different values of the original parameters $\lambda, a_1, a_2, k, C^2$ (e.g., in a parametric plot). Conveniently, in this case Eq. (11) subject to the constraints (19) can be used instead of the various solutions of Table 1 in Ref. [3(a)].

Finally, it should be noted that solutions 1, 2, and (18) in Ref. [3(a)] are special cases of Eqs. (30), (52), and (31), respectively. Thus it seems that the solutions of Table 1 in Ref. [3(a)] are consistent with Eqs. (11) and (15).

To sum up, the analysis of Ref. [3(c)] has more content than the above analysis. This is natural because a much more general problem is treated in Ref. [3(c)]. As pointed out, some results of Refs. [3(c)] and [3(a)] are equivalent or consistent with results of the present article; some of the previous results are not contained in Refs. [3(c)] and [3(a)]. As indicated in Secs. III and IV, it seems that the closed-form solution [Eqs. (11), (14), and (15)] in connection with the constraints (19) simplifies the phase diagram analysis as well as the classification of solutions. Furthermore, Eqs. (11), (14), (15), and (19) suitably can be used in a stability analysis based on the standard Floquet theory [25].

APPENDIX: m AND ROOTS OF $s=0$ IF $a_2=0$ AND $\Delta>0$

The period of the solutions (26) and (27) is given by Eq. (28), where $e_{\min}<e_m<e_{\max}$ and m are related to a_1, λ, k (see Fig. 4) according to

a_1 λ k	$e_{\max}$	e_m	$e_{\min}$	m
$+$ $+$ $+$	$\frac{\lambda}{6}+\frac{1}{2}\sqrt{\lambda^2-2a_1k}$	$\frac{\lambda}{6}-\frac{1}{2}\sqrt{\lambda^2-2a_1k}$	$-\frac{\lambda}{3}$	$\frac{\lambda-\sqrt{\lambda^2-2a_1k}}{\lambda+\sqrt{\lambda^2-2a_1k}}$
$-$ $-$ $+$	$\frac{\lambda}{6}+\frac{1}{2}\sqrt{\lambda^2-2a_1k}$	$-\frac{\lambda}{3}$	$\frac{\lambda}{6}-\frac{1}{2}\sqrt{\lambda^2-2a_1k}$	$\frac{-\lambda+\sqrt{\lambda^2-2a_1k}}{2\sqrt{\lambda^2-2a_1k}}$
$-$ $-$ $-$	$-\frac{\lambda}{3}$	$\frac{\lambda}{6}+\frac{1}{2}\sqrt{\lambda^2-2a_1k}$	$\frac{\lambda}{6}-\frac{1}{2}\sqrt{\lambda^2-2a_1k}$	$\frac{2\sqrt{\lambda^2-2a_1k}}{-\lambda+\sqrt{\lambda^2-2a_1k}}$

[1] See, for instance, L. Gagnon and P. Winternitz, Phys. Rev. A **39**, 296 (1989), where some references are given in the Introduction.

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[12] As will be seen below, this assumption is not a restriction if $C^2>0$.

[13] W. Jahnke-Emde, *Tafeln Höherer Funktionen* (Teubner, Leipzig, 1952), p. 101.

[14] See Ref. [9], p. 629.

[15] Weierstrass's function $\wp(z,g_2,g_3)$ is real for real z and real g_2,g_3. It has a double pole at $z=0$ (see Ref. [9], p. 635), so that $I(z)$ is holomorphic at $z=0$. As z increases from 0 to ω (the real half period of $\wp$), $\wp(z,g_2,g_3)$ decreases monotonically from $+\infty$ to e_1. Thus e_1 is the minimum of $\wp(z,g_2,g_3)$ if $z\in(0,\omega]$.

[16] See Ref. [9], pp. 649 and 652.

[17] See Ref. [9], pp. 651 and 652.

[18] See, for instance, K. Hayata and M. Koshiba, Phys. Rev. E **51**, 1499 (1995).

[19] I. N. Bronstein and K. A. Semendjajew, *Taschenbuch der Mathematik* (Deutsch, Frankfurt, 1989), p. 137.

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[23] A general, necessary, and sufficient condition for two positive roots to exist can be given by using the roots of the cubic resolvent, but this condition is rather complicated.

[24] It is useful for the numerical evaluation of Eq. (42) to replace $\wp(z;g_2,g_3)$ by $e_1+(3e_1)/\sinh^2(\sqrt{3e_1}z)$ [17]; since e_1 is a rather lengthy expression, $I(z)$ has been written in terms of $\wp$.

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