Renormalization group and singular perturbations: Multiple scales, boundary layers, and reductive perturbation theory

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Perturbative renormalization group theory is developed as a unified tool for global asymptotic analysis. With numerous examples, we illustrate its application to ordinary differential equation problems involving multiple scales, boundary layers with technically difficult asymptotic matching, and WKB analysis. In contrast to conventional methods, the renormalization group approach requires neither *ad hoc* assumptions about the structure of perturbation series nor the use of asymptotic matching. Our renormalization group approach provides approximate solutions which are practically superior to those obtained conventionally, although the latter can be reproduced, if desired, by appropriate expansion of the renormalization group approximant. We show that the renormalization group equation may be interpreted as an amplitude equation, and from this point of view develop reductive perturbation theory for partial differential equations describing spatially extended systems near bifurcation points, deriving both amplitude equations and the center manifold. [S1063-651X(96)00506-5]

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I. INTRODUCTION

Asymptotic and perturbative analysis has played a significant role in applied mathematics and theoretical physics. In many cases, regular perturbation methods are not applicable, and various singular perturbation techniques must be used [1–6]. Examples of widely used techniques for ordinary differential equations (ODEs) include [1,2] the methods of multiple scales, boundary layers, or asymptotic matching, WKB, stretched coordinates, averaging, the method of reconstitution [4], and center manifold theory [6]. Although these methods are well known, each has its own drawbacks, preventing mechanical (or algorithmic) application. Indeed, it is probably fair to say that the practice of asymptotic analysis is something of an art.

Multiple-scales analysis has proven to be a particularly useful tool for constructing uniform or global approximate solutions for both small and large values of independent variables. In this method a set of scaled variables, which are regarded as independent variables (although they are ultimately related to one another), is introduced to remove all secular terms. The choice of the set is, in some cases, nontrivial, and may only be justified *post hoc*. Nevertheless, this method is usually considered the most general, subsuming the others mentioned below.

Differential equations whose highest-order derivatives are multiplied by a small parameter ϵ often yield solutions with narrow regions of rapid variation, known as boundary layers. Boundary-layer techniques can be applied if the thickness of these regions tends to zero as $\epsilon \to 0$; otherwise, WKB must be used. The limitation of WKB is that it applies to linear equations only. Although boundary-layer methods apply to nonlinear as well as to linear problems, the determination of the expansion parameter can be subtle. Furthermore, matching of outer and inner expansions via intermediate expansions is required, sometimes involving delicate arguments that are difficult to perform mechanically.

Another class of related problems concerns partial differential equations (PDEs) describing nonequilibrium, spatially extended systems near bifurcation points. Such systems often exhibit spatial-temporal patterns modulated by an envelope function (or amplitude) which varies slowly compared with the pattern itself. Extracting the long wavelength, slow time scale behavior of such systems is the task of reductive perturbation methods [7], which are themselves related to multiple-scale analysis.

The purpose of this paper is to present a unified, and physically motivated approach to these classes of problems, based upon the renormalization group (RG). The essence of the renormalization group method is to extract structurally stable features of a system which are insensitive to details [8–11]. For example, field theories, critical phenomena, polymers, and other statistical mechanical systems exhibit universal scaling functions and critical exponents in the limit $\Lambda/\xi \to 0$, where Λ is some ultraviolet cutoff and ξ is the (temperature-dependent) correlation length. The renormalization group is the principal tool with which to elucidate this universal behavior and is properly regarded as a means of asymptotic analysis.

The usefulness of this point of view has been amply demonstrated [12–15] by the relationship between the renormalization group and intermediate asymptotics [16]. In particular, the large-time asymptotic behavior of certain initial-value problems is given by a similarity solution of the governing PDE, where the similarity variable contains anomalous exponents which may not be determined *a priori* by elementary dimensional considerations. Nevertheless, the renormalized perturbation theory combined with the renormalization group, gives an expansion for the anomalous exponents and the solution [17].

The similarities between the renormalization group and

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singular perturbation methods extend also to technical details: both perturbative renormalization group and conventional singular perturbation methods remove secular or divergent terms from the perturbation series. These formal similarities invite a natural question: what is the relation, if any, between conventional asymptotic methods and the renormalization group?

In this paper, which is an extended version of our preliminary report [18], we demonstrate that singular perturbation methods may be naturally understood as renormalized perturbation theory, and that amplitude equations obtainable by reductive perturbation methods may be derived as renormalization group equations. The basic approach of our method uses the interpretation of renormalization found in the physics literature on quantum field theory or statistical mechanics [10]. There, a quantity that is not directly observable, such as the bare charge on an electron, is renormalized by the interactions in the theory to yield an effective value, which can be directly observed under stated conditions, for example, at some energy or momentum scale of interest. In our approach, the Cauchy data are the analogues of the bare quantities in quantum field theory, and are renormalized by the perturbation [10,19]. Typically, this is most conveniently performed by a multiplicative renormalization. In the examples that we study here, however, some which involve periodic motions are formulated in such a way that the amplitude of the variable of interest becomes multiplicatively renormalized, whereas the phase is additively renormalized. In such problems, use of a complex variable technique will always lead to multiplicative renormalization.

Our studies indicate that the renormalization group method may have several practical advantages compared with conventional methods. Although we recognize that our analysis is at the formal, heuristic level, we suggest that a more careful mathematical analysis would be worthwhile, given the potential usefulness of our central claim.

One advantage of the renormalization group method is that the starting point is a straightforward naive perturbation expansion, for which very little *a priori* knowledge is required. That is, one does not need to guess or otherwise introduce unexpected fractional power laws or logarithmic functions of ϵ in an *ad hoc* manner. It seems that these ϵ-dependent space-time scales arise naturally during the analysis.

We will show that the renormalization group approach sometimes seems to be more efficient and accurate in practice than standard methods in extracting global information from the perturbation expansion. Standard methods often attempt to represent an asymptotic solution in terms of asymptotic sequences of a few simple functions of the expansion parameter, such as exp, log, powers, and so on. The renormalization group can generate its own problem-adapted asymptotic sequence without matching: in the examples given in Sec. IV, these turn out to be complicated functions conveniently defined by an integral representation. For small ϵ, this asymptotic sequence can be expanded to reproduce the solutions conventionally obtained by asymptotic matching, although in the examples that we have studied so far, the conventional approximant is practically inferior to the one obtained by the RG. Because we only utilize the inner expansion, the RG perturbation series may need to be carried out to a higher rather than a lowest order, then expanded in ϵ, in order to reproduce the (inferior) conventional result. A related feature of the renormalization group seems to be the lack of necessity to perform asymptotic matching. To illustrate this assertion, in Sec. III we solve several ODEs with boundary layers, and in Sec. IV we address the difficult technical problem of switchback terms.

The renormalization group methods for partial differential equations such as the Barenblatt equation [12,15,16], and front propagation problems in reaction-diffusion equations [11] are, in retrospect, examples of the general approach discussed in this paper. We emphasize that our renormalization group method has no connection with the so-called method of renormalization or uniformization [1] in the conventional perturbation literature; the latter is a variant of the method of stretched coordinates, and of narrow limited use.

Lastly, we wish to point out that recently, a method utilizing an invariance condition in the solution of multiple-scale singular perturbation problems was proposed independently by Woodruff [20], based on ideas related to the renormalization group. In addition, Kunihiro [21] has demonstrated the general relation between the renormalization group equation and the envelope equation in the classical theory of envelopes.

The outline of this paper is as follows. In Sec. II, we discuss the general relation between multiple-scale analysis and the renormalization group. In Sec III, we show how the boundary layer and WKB problems can be solved using the renormalization group. In particular, we comment on the renormalization of Cauchy data in the context of boundary-layer theory, using Wilson's RG procedure rather than the Gell-Mann—Low procedure used elsewhere in this paper. In Sec. IV, we demonstrate with several examples that the renormalization group approach has technical advantages to conventional asymptotic methods. In Sec. V, the renormalization group is applied as a reductive perturbation tool to the derivation of global slow motion equations for partial differential equations. Center manifold theory is also briefly considered from the same point of view. We conclude in Sec. VI.

II. MULTIPLE-SCALE THEORY AND RG

In this section, we show that multiple-scale analysis is equivalent to the RG, and that the solvability condition used in multiple scales to remove the secular divergences is equivalent to the physical assumption of renormalizability in RG theory.

A. Rayleigh equation

The example we consider below is the Rayleigh equation [22], closely related to the van der Pol oscillator

$$\frac{d^2y}{dt^2}+y=\epsilon\left\{\frac{dy}{dt}-\frac{1}{3}\left(\frac{dy}{dt}\right)^3\right\}. \tag{2.1}$$

It is known that the method of uniformization or renormalization [1] fails here, and this example is a textbook illustration of multiple-scales analysis. We show here that from only the simple-minded straightforward expansion, not only is the RG capable of identifying automatically all different

multiple scales required by multiple-scales analysis, but also produces a uniformly valid asymptotic solution without encountering the ambiguity which often plagues higher-order calculations in multiple-scales analysis.

A naive expansion $y=y_0+\epsilon y_1+\epsilon^2 y_2+\cdots$ gives

$$y(t)=R_0\sin(t+\Theta_0)+\epsilon\left\{-\frac{R_0^3}{96}\cos(t+\Theta_0)+\frac{R_0}{2}\left(1-\frac{R_0^2}{4}\right)\times(t-t_0)\sin(t+\Theta_0)+\frac{R_0^3}{96}\cos 3(t+\Theta_0)\right\}+O(\epsilon^2). \tag{2.2}$$

where R_0, Θ_0 are constants determined by the initial conditions at arbitrary $t=t_0$. This naive perturbation theory breaks down when $\epsilon(t-t_0)>1$ because of the secular terms. The arbitrary time t_0 may be interpreted as the (logarithm of the) ultraviolet cutoff in the usual field theory [11]. To regularize the perturbation series, we introduce an arbitrary time τ, split $t-t_0$ as $t-\tau+\tau-t_0$, and absorb the terms containing $\tau-t_0$ into the renormalized counterparts R and Θ of R_0 and Θ_0, respectively. This is allowed because R_0 and Θ_0 are no longer constants of motion in the presence of the nonlinear perturbation.

We introduce a multiplicative renormalization constant $Z_1=1+\sum_1^\infty a_n\epsilon^n$ and an additive one $Z_2=\sum_1^\infty b_n\epsilon^n$ such that $R_0(t_0)=Z_1(t_0,\tau)R(\tau)$ and $\Theta_0(t_0)=\Theta(\tau)+Z_2(t_0,\tau)$. The coefficients a_n and b_n ($n\geqslant 1$) are chosen order by order in ϵ to eliminate the terms containing $\tau-t_0$ as in the standard RG [23–28]. The choice $a_1=-(1/2)(1-R^2/4)(\tau-t_0)$, $b_1=0$ removes the secular terms to order ϵ, and we obtain the following renormalized perturbation result [29]

$$y(t)=\left\{R+\epsilon\frac{R}{2}\left(1-\frac{R^2}{4}\right)(t-\tau)\right\}\sin(t+\Theta)-\epsilon\frac{1}{96}R^3\cos(t+\Theta)+\epsilon\frac{R^3}{96}\cos 3(t+\Theta)+O(\epsilon^2), \tag{2.3}$$

where R,Θ are now functions of τ. Since τ does not appear in the original problem, the solution should not depend on τ. Therefore $(\partial y/\partial\tau)_t=0$ for any t. This is the RG equation, which in this case consists of two independent equations

$$\frac{dR}{d\tau}=\epsilon\frac{1}{2}R\left(1-\frac{1}{4}R^2\right)+O(\epsilon^2),\quad \frac{d\Theta}{d\tau}=O(\epsilon^2). \tag{2.4}$$

Solving (2.4), and equating τ and t eliminates the secular term; we get

$$R(t)=R(0)/[e^{-\epsilon t}+\tfrac{1}{4}R(0)^2(1-e^{-\epsilon t})]^{1/2}+O(\epsilon^2 t),$$
$$\Theta(t)=\Theta(0)+O(\epsilon^2 t) \tag{2.5}$$

where $R(0),\Theta(0)$ are constants to be determined by the initial condition. Assuming the initial condition $y(0)=0$, $y'(0)=2a$, we find $R(0)=2a$, $\Theta(0)=0$, and the final uniformly valid result reads

$$y(t)=R(t)\sin(t)+\frac{\epsilon}{96}R(t)^3\{\cos(3t)-\cos(t)\}+O(\epsilon^2), \tag{2.6}$$

which approaches a limit circle of radius 2 as $t\to\infty$.

The second-order RG calculation shows the assumption of perturbative renormalizability is consistent and no ambiguity arises at all. The corresponding amplitude and phase equation to order $O(\epsilon^3)$ are

$$\frac{dR}{dt}=\epsilon\frac{1}{2}R\left(1-\frac{1}{4}R^2\right)+O(\epsilon^3),$$
$$\frac{d\Theta}{dt}=-\frac{\epsilon^2}{8}\left(1-\frac{R^4}{32}\right)+O(\epsilon^3), \tag{2.7}$$

from which the multiple time scales $T_1=\epsilon t$, $T_2=\epsilon^2 t,\ldots$ used in multiple-scale analysis appear naturally (although the RG does not require such identifications). When $R=2$, (2.7) reduces to

$$\frac{dR}{dt}=0+O(\epsilon^3),\quad \frac{d\Theta}{dt}=-\frac{1}{16}\epsilon^2+O(\epsilon^3). \tag{2.8}$$

In this simple example, it was straightforward to determine the multiple time scales. However, it is well known that in many cases, within multiple-scale analysis hidden intermediate scales must be included in the perturbation expansion so as to obtain the correct result. In the next example, will show that the RG method is a more straightforward but secure way to determine multiple slow time scales than the multiple-scale method.

B. Mathieu equation

The second illustrative example we examine using RG is the Mathieu equation [30]

$$\frac{d^2y}{dt^2}+(a+2\epsilon\cos t)y=0, \tag{2.9}$$

where a and ϵ are parameters.

The Floquet theory of linear periodic differential equations [1] predicts that in the (a,ϵ) plane there are some regions where the solutions to (2.9) remain bounded for all t and stable, and others where the solutions are unstable. Perturbative investigation shows that for sufficiently small ϵ, all solutions $y(t)$ are stable for $a>0$, $a\neq n^2/4$, $n=0,1,2,\ldots$. Without loss of generality, we investigate the stability of solutions near $a=1/4$ and $\epsilon=0$ to find the stability boundary in the (a,ϵ) plane. We treat the boundary curve a as a function of ϵ and expand a in powers of ϵ: $a(\epsilon)=1/4+a_1+a_2\epsilon^2+\cdots$. It is our goal to determine values of $a_1,a_2,\ldots$ perturbatively. Multiple-scale analysis can be applied to this problem, and the coefficients $a_1=1$, $a_2=-1/2$ are determined. However it turns out that the introduction of multiple time scales $\tau_1=\epsilon t$, $\tau_2=\epsilon^2 t,\ldots$ is not sufficient to determine the second-order coefficient a_2 even after the first-order coefficient a_1 is set to 1. Through careful analysis, it is found that a new hidden time scale $\sigma=\epsilon^{3/2}t$ must be introduced into the problem, and the perturbative expansion must be done in powers of $\epsilon^{1/2}$, rather than the original expansion

in powers of ϵ. It is necessary to go to the fourth order in powers of $\epsilon^{1/2}$ to determine a_2. Thus the procedure required to determine all necessary time scales is not mechanical: if any hidden scales are omitted or cannot be determined, correct results will not be guaranteed. This represents a typical shortcoming of multiple scales analysis.

Now we demonstrate how the unexpected time scales such as $\sigma=\epsilon^{3/2}t$ appear automatically from the RG equation, starting only with a straightforward perturbative expansion. Substituting $a=1/4+a_1\epsilon+a_2\epsilon^2+\cdots$ in (2.9) and expanding in powers of ϵ (*not* $\epsilon^{1/2}$) as $y=y_0+\epsilon y_1+\epsilon^2 y_2+\cdots$, we get

$$\frac{d^2y_0}{dt^2}+\frac{1}{4}y_0=0, \tag{2.10}$$

$$\frac{d^2y_1}{dt^2}+\frac{1}{4}y_1=-(a_1+2\cos t)y_0, \tag{2.11}$$

$$\frac{d^2y_2}{dt^2}+\frac{1}{4}y^2=-a_2y_0-(a_1+2\cos t)y_1, \tag{2.12}$$

and so on. First, let us determine the first-order coefficient a_1. The straightforward perturbation result, to $O(\epsilon)$, is given by

$$\begin{aligned}y(t)=&R_0\cos(t/2+\Theta_0)+\epsilon R_0\{-\tfrac{1}{2}\cos(t/2+\Theta_0)\\&+\tfrac{1}{2}\cos(3t/2+\Theta_0)-a_1(t-t_0)\sin(t/2+\Theta_0)\\&-(t-t_0)\sin(t/2-\Theta_0)\}+O(\epsilon^2),\end{aligned} \tag{2.13}$$

where R_0,Θ_0 are constants dependent on initial conditions given at some arbitrary time t_0. Similarly, the secular divergences can be removed by regarding t_0 as a regularization parameter and renormalizing the bare amplitude A_0 and bare phase Θ_0: $R_0(t_0)=Z_1(t_0,\mu)R(\mu)$, $\Theta_0(t_0)=Z_2(t_0,\mu)+\Theta(\mu)$, where μ is some arbitrary time scale, as was done in previous problems. The renormalized perturbation result is

$$\begin{aligned}y(t)=&\{R(\mu)+\epsilon R[-1/2+(t-\mu)\sin2\Theta(\mu)]\}\cos(t/2+\Theta)\\&-\epsilon R(a_1+\cos2\Theta)(t-\mu)\sin(t/2+\Theta)\\&+\epsilon\frac{R}{2}\cos(3t/2+\Theta)+O(\epsilon^2).\end{aligned} \tag{2.14}$$

The RG equation $\partial y/\partial\mu=0$ for any t gives

$$\frac{dR}{d\mu}=\epsilon R\sin2\Theta+O(\epsilon^2),\quad \frac{d\Theta}{d\mu}=\epsilon(a_1+\cos2\Theta)+O(\epsilon^2). \tag{2.15}$$

For convenience, we introduce the complex amplitude $A=Re^{i\Theta}$ as $A=B+iC$, with its real and imaginary parts $B=R\cos\Theta$, $C=R\sin\Theta$. The equations for $B(\mu)$ and $C(\mu)$ are

$$B'(\mu)=\epsilon(1-a_1)C(\mu),\quad C'(\mu)=\epsilon(1+a_1)B(\mu). \tag{2.16}$$

Thus we have

$$B''(\mu)=\epsilon^2(1-a_1^2)B(\mu). \tag{2.17}$$

Solving this and setting $\mu=t$, we get

$$B(t)=K_1e^{\pm(1-a_1^2)^{1/2}\epsilon t}, \tag{2.18}$$

where K_1 is a constant, and the first slow time scale $\tau_1=\epsilon t$ has appeared automatically. Obviously, for $|a_1|<1$, instability sets in, where the solution grows exponentially with time t, while for $|a_1|>1$, the solutions are bounded and stable. Therefore, near $\epsilon=0$, the stability boundary is $a=1/4\pm\epsilon+O(\epsilon^2)$.

We now set $a_1=1$ and go to the second order to determine a_2. For order ϵ^2, a special solution to (2.12) is obtained

$$\begin{aligned}y_2(t)=&-R_0(a_2-\tfrac{1}{2}\cos2\Theta_0)(t-t_0)\sin(t/2+\Theta_0)\\&-\tfrac{1}{2}R_0\sin2\Theta_0(t-t_0)\cos(t/2+\Theta_0)\\&-\tfrac{1}{2}R_0(1+\cos2\Theta_0)(t-t_0)\sin(3t/2+\Theta_0)\\&+\tfrac{1}{2}R_0\sin2\Theta_0(t-t_0)\cos(3t/2+\Theta_0)\\&-\tfrac{3}{4}R_0(1+\cos2\Theta_0)\cos(3t/2+\Theta_0)\\&-\tfrac{3}{4}R_0\sin2\Theta_0\ \sin(3t/2+\Theta_0)+\tfrac{1}{12}R_0\cos(5t/2+\Theta_0).\end{aligned} \tag{2.19}$$

Extending the renormalization procedure to the second order, we find all the secular divergences to this order can be removed completely, a sign of the consistency of perturbative renormalizability. Keeping only the two lowest harmonics with prime frequency and omitting other higher frequency terms which are not important for determining the stability boundary, we obtain the renormalized perturbation result, to order ϵ^2,

$$\begin{aligned}y(t)=&\left\{R(\mu)+\epsilon R[-1/2+(t-\mu)\sin2\Theta(\mu)]\right.\\&\left.-\epsilon^2\frac{R}{2}(t-\mu)\sin2\Theta\right\}\cos(t/2+\Theta)\\&+\{-\epsilon R(1+\cos2\Theta)(t-\mu)+\epsilon^2R(a_2-\tfrac{1}{2}\cos2\Theta)\\&\times(t-\mu)\}\sin(t/2+\Theta)+\mathscr{T},\end{aligned} \tag{2.20}$$

where $\mathscr{T}$ represents all higher frequency terms. The RG equation to order ϵ^2 now reads

$$\frac{dR}{d\mu}=\epsilon R\sin2\Theta+O(\epsilon^3),$$

$$\frac{d\Theta}{d\mu}=\epsilon(1+\cos2\Theta)+\epsilon^2(a_2+1/2)+O(\epsilon^3). \tag{2.21}$$

Accordingly, the equations for $B(\mu)$ and $C(\mu)$ become

$$B'(\mu)=-\epsilon^2(a_2+1/2)C(\mu),$$

$$C'(\mu)=[2\epsilon+\epsilon2(a2+1/2)]B(\mu). \tag{2.22}$$

Thus we get

$$B''(\mu)=-2\epsilon^3(a_2+1/2)B(\mu), \tag{2.23}$$

which gives

$$B''(\mu)\approx-\epsilon^3(2a_2+1)B(\mu)+O(\epsilon^4), \tag{2.24}$$

which has the solution (setting $\mu=t$)

$$B(t)=K_2e^{\pm\sqrt{2a_2+1}\epsilon^{3/2}t}, \tag{2.25}$$

where K_2 is a constant, and the second and the third slow time scales $\sigma=\epsilon^{3/2}t$, $\tau_2=\epsilon^2t$ appear naturally. We apparently have stable solutions for $a_2>-1/2$ and unstable solutions for $a_2<-1/2$. Therefore, to order ϵ^2, the instability boundary is given by

$$a(\epsilon)=\tfrac{1}{4}+\epsilon-\tfrac{1}{2}\epsilon^2+O(\epsilon^2),\quad \epsilon\to0. \tag{2.26}$$

C. Oscillator with time-dependent spring constant

The third illustrative example is an oscillator governed by the equation [31]

$$\frac{d^2y}{dt_2}+y-\epsilon ty=0. \tag{2.27}$$

The initial conditions are $y(0)=1$ and $y'(0)=0$. The regular perturbation theory breaks down for $t\to\infty$, and multiple-scale analysis can be applied to eliminate the secular behavior. However it turns out that multiple time scales must be chosen as $\tau_0=t$, $\tau_1=\epsilon^{1/2}t$, $\tau_2=\epsilon t,\ldots$. Since the frequency of the oscillator is found to be time dependent, the method of stretched coordinates or the so-called method of uniformization or renormalization (in the conventional applied mathematics sense) does not work here.

We will see that the RG is able to provide a uniformly valid solution for times less than, say, $O(\epsilon^{-1/2})$, but can never give results which are reliable for times of order ϵ^{-1}, due to a singularity at $\epsilon t=1$. Renormalized perturbation theory techniques naively applied are no more able to solve this sort of problem than are multiple-scale techniques (see the discussion in Ref. [1]). The problem must be treated as a WKB problem.

To attempt to solve (2.27), we assume a straightforward expansion in powers of ϵ (*not* $\epsilon^{1/2}$), $y=y_0+\epsilon y_1+\epsilon^2y_2+\cdots$. The bare perturbation result, to order ϵ, is given by

$$y(t)=R_0\cos(t+\Theta_0)+\epsilon R_0\{\tfrac{1}{4}(t^2-t_0^2)+\tfrac{1}{4}(t-t_0)\}\sin(t+\Theta_0)+O(\epsilon^2). \tag{2.28}$$

As in the preceding examples, renormalizing the bare amplitude R_0 and phase Θ_0 removes the secular divergences. The renormalized perturbation result is

$$y(t)=\{R+\tfrac{1}{4}\epsilon R(t-\mu+a_1)\}\cos(t+\Theta)+\tfrac{1}{4}\epsilon R(t^2-\mu^2+b_1)\sin(t+\Theta)+O(\epsilon^2), \tag{2.29}$$

where R, Θ are functions of an arbitrary time scale μ, and a_1,b_1 are arbitrary constants. The RG equation reads

$$\frac{dR}{d\mu}=\frac{1}{4}\epsilon R+O(\epsilon^2),\quad \frac{d\Theta}{d\mu}=-\frac{1}{2}\epsilon\mu+O(\epsilon^2). \tag{2.30}$$

Solving (2.30) and setting $\mu=t$ in (2.29) gives

$$R(t)=R(0)e^{(1/4)\epsilon t}+O(\epsilon^2t),$$
$$\Theta(t)=-\tfrac{1}{4}\epsilon t^2+\Theta(0)+O(\epsilon^2t). \tag{2.31}$$

Thus we obtain the uniformly valid result

$$y(t)=R(t)\cos[t+\Theta(t)]+\tfrac{1}{4}\epsilon R(t)[a_1\cos(t+\Theta)+b_1\sin(t+\Theta)]+O(\epsilon^2). \tag{2.32}$$

Imposing the boundary conditions $y(0)=1$, $y'(0)=0$ gives $R(0)=1$, $\Theta(0)=0$, $a_1=0$, $b_1=-1$. Therefore the final result is

$$y(t)=e^{(1/4)\epsilon t}\cos(t-\tfrac{1}{4}\epsilon t^2)-\tfrac{1}{4}\epsilon e^{(1/4)\epsilon t}\sin(t-\tfrac{1}{4}\epsilon t^2)+O(\epsilon^2), \tag{2.33}$$

where the frequency defined as $\omega=d\Theta/dt$ becomes time dependent: $\omega=1-(1/2)\epsilon t+O(\epsilon^2)$. Rewriting ϵt^2 as $(\epsilon^{1/2}t)^2$, two slow time scales $T_1=\epsilon^{1/2}t$, $T_2=\epsilon t$ are easily identified from the RG result (but these identifications are unnecessary in our approach).

Note that our solution to this problem is valid up to times of $O(\epsilon^{-1/2})$. Incorporating higher-order terms into the naive perturbation series will not generate a solution uniformly valid for longer times, in this particular case. To see why this is so, make the change of variables $x=\epsilon t$: (2.27) becomes

$$\epsilon^2\frac{d^2y}{dx^2}+Q(x)y=0 \tag{2.34}$$

with $Q(x)=1-x$. The transformed equation is in the canonical form for WKB problems, with a turning point at $x=1$. Such problems are treated in Sec. III G; note, in particular, that the singularity arising from the large argument behavior of the Airy function cannot be captured by a finite number of terms in the perturbative expansion given above.

The RG scheme given above is also applicable to quantum systems with discrete or continuous energy spectrums, especially those which involve resonance phenomena, e.g., the Rabi flopping, the Stark shift, the Bloch-Siegert shift [32]. The multiple-time scale perturbation analysis has successfully given a unified framework for all quantum resonance [33]. In a similar way, the RG method simply recovers all resonance equations which turn out to be simply RG equations. The application of RG to the time-dependent Schrödinger equation also reproduces the Fermi's golden rule [34]. Here we will not give detailed calculations of these problems. In the next section, we will show that WKB problems can be easily solved using the RG method. Therefore many quantum problems which are usually solved using WKB and/or multiple-scale analysis can also be studied using the RG approach.

To summarize, it seems that the RG method is more efficient and mechanical than the multiple-scale method in determining the multiple slow time scales. In the RG approach, the starting point is simply a straightforward naive perturbation series, and all necessary multiple scales arise naturally from the RG equations. The above examples reveal two important points, demonstrated more generally below: (1) the results of multiple-scale, analysis can be obtained from the

renormalized perturbation theory, and (2) the RG equation describes the long-time scale motion of the amplitude and the phase.

III. BOUNDARY-LAYER THEORY, WKB, AND RG

Another important class of singular problems is that for which the highest-order derivative of the equation is multiplied by a small parameter ϵ, e.g., WKB and boundary-layer problems.

Boundary-layer theory and asymptotic matching are a collection of singular perturbation methods for constructing a uniformly and globally valid solution by calculating the separated outer and inner solutions and then matching them across intermediate scale solutions. Quite often, the intermediate matching is very lengthy and only some particular matching method will work. WKB theory is well known to be a powerful tool for obtaining a global approximation to solutions of a linear differential equation whose highest derivative is multiplied by a small parameter ϵ. Many linear problems often solved by the WKB theory can be solved by the boundary-layer theory; indeed, in these cases, the boundary-layer theory (thickness of the boundary layer goes to zero as $\epsilon\to 0$) is a special case of WKB (thickness of the boundary layer remains finite even as $\epsilon\to 0$). The limitation of the conventional WKB method is that it applies only to linear problems, while boundary-layer theory works for linear as well as nonlinear problems.

In this section we will demonstrate explicitly that many boundary-layer problems, linear or nonlinear, can be solved by the RG. The uniformly valid asymptotics of boundary-layer problems can actually be constructed from the inner expansion alone, with the aid of the RG, without the need for intermediate matching.

A. Simple linear example

Consider the following simple example, which describes the motion of an overdamped linear oscillator

$$\epsilon\frac{d^2y}{dt^2}+\frac{dy}{dt}+y=0,\quad \epsilon\ll 1 \tag{3.1}$$

where ϵ is a small parameter. A standard dominant-balance argument shows that there exists a boundary layer of thickness $\delta=O(\epsilon)$ at $t=0$. Thus we set $t=\epsilon\tau$, and rewrite Eq. (3.1) as

$$\frac{d^2y}{d\tau^2}+\frac{dy}{d\tau}+\epsilon y=0. \tag{3.2}$$

Naive expansion gives

$$y(\tau)=A_0+B_0e^{-\tau}+\epsilon[-A_0(\tau-\tau_0)+B_0(\tau-\tau_0)e^{-\tau}]+O(\epsilon), \tag{3.3}$$

where the coefficients A_0, B_0 are constants of integration and $O(\epsilon)$ refers to all the regular terms of order ϵ and higher, which are finite even in the limit $\tau-\tau_0\to\infty$. This naive perturbation theory breaks down due to the divergence of secular terms for large $\tau-\tau_0$. However this divergence can be removed by regarding τ_0 as a regularization parameter and renormalizing A_0,B_0 as $A_0(\tau_0)=Z_1A(\mu)$, and $B_0(\tau_0)=Z_2B(\mu)$. Here μ is an arbitrary time, and A,B are the renormalized counterparts of A_0,B_0. The renormalization constants $Z_1=\sum_0^\infty a_n(\tau_0,\mu)\epsilon^n$, $Z_2=\sum_0^\infty b_n(\tau_0,\mu)\epsilon^n$ ($a_0=1$, $b_0=1$) are chosen order by order in ϵ to eliminate the secular divergences. Split $\tau-\tau_0$ as $(\tau-\mu)+(\mu-\tau_0)$, and then absorb the divergent part $\mu-\tau_0$ in the limit $\tau_0\to-\infty$ by redefining A_0 and B_0. Choosing $a_1=\mu-\tau_0$, $b_1=-(\mu-\tau_0)$, we get the renormalized perturbation result

$$y(\tau)=A(\mu)-\epsilon A(\mu)(\tau-\mu)+[B(\mu)+\epsilon B(\mu)(\tau-\mu)]e^{-\tau}+O(\epsilon). \tag{3.4}$$

However it is impossible that the actual solution $y(\tau)$ can depend on the arbitrary time μ which is not present in the original problem. Thus we have the renormalization group equation $\partial y/\partial\mu=0$ for any τ, which gives

$$\frac{dA}{d\mu}+\epsilon A+\left[\frac{dB}{d\mu}-\epsilon B\right]e^{-\tau}+O(\epsilon^2)=0, \tag{3.5}$$

or

$$\frac{dA}{d\mu}=-\epsilon A+O(\epsilon^2),\quad \frac{dB}{d\mu}=\epsilon B+O(\epsilon^2). \tag{3.6}$$

Notice that what we have renormalized are parameters fixed by the Cauchy data. In the oscillator examples in Sec. II, the reader may have asked why R and Θ are renormalized and not, e.g., the frequency. The question is a natural one, especially because the scaling of the time coordinate is used to remove secular terms in the so-called renormalization method, which as stressed in the introduction has no connection with RG. The basic observation on which our approach relies is a correspondence between time in the initial-value problem and length scale (or rather the logarithm of length scale) in field theory. Thus the Cauchy data can be regarded as being analogous to the bare parameters of the field theory. From this point of view, there is no ambiguity in the choice of parameters to be renormalized. Conditions in the far past are hard to observe in the same way that bare quantities at short distance scales are hard to observe. This perspective has been explored and explained in our earlier work, especially Refs. [10, 13] and [14].

Which are the appropriate quantities to renormalize can be seen clearly from the following simple example of a Wilson type RG approach. The naive perturbation solution of (3.1) gives

$$x(t)=A(0)e^{-t}-\epsilon tA(0)e^{-t}+O(\epsilon^2), \tag{3.7}$$

which is useful when $\epsilon t\ll 1$. Thus for δt satisfying $\epsilon\delta t\ll 1$ we can rely on

$$x(\delta t)=A(0)(1-\epsilon\delta t)e^{-\delta t}+O((\epsilon\delta t)^2). \tag{3.8}$$

Now we use this as the initial condition and solve (3.1) for another δt as

$$x(2\delta t)=x(\delta t)(1-\epsilon\delta t)e^{-\delta t}+O(\delta t^2). \tag{3.9}$$

In this way we eventually obtain

$$x(t)=A(0)(1-\epsilon\delta t)^{t/\delta t}e^{-t}+O(\delta t)\to A(t)e^{-t}, \quad (3.10)$$

with $A(t)=A(0)\exp(-\epsilon t)$, which is exactly the solution to (3.5). Thus the example illustrates that the Cauchy data should be renormalized, although it looks as if the decay rate has been renormalized from 1 to $1+\epsilon$.

Extending the RG calculation to the second order gives, without any ambiguity,

$$\frac{dA}{d\mu}=-(\epsilon A+\epsilon^2 A)+O(\epsilon^3), \quad \frac{dB}{d\mu}=\epsilon B+\epsilon^2 B+O(\epsilon^3). \quad (3.11)$$

Solving them, setting $\mu=\tau$ and setting back $\tau=t/\epsilon$ in (3.4), we finally obtain the uniformly valid solution

$$y(t)=C_1e^{-(1+\epsilon)t}+C_2e^{-t/\epsilon+(1+\epsilon)t}+O(\epsilon^2), \quad (3.12)$$

where C_1, C_2 are constants to be determined by the initial conditions. Clearly, the RG result to order ϵ^2 recovers exactly that obtained by the standard singular methods [1]. Notice that the equations in (3.11) are nothing but the equations of motion for a slow-time scale: the amplitude equations. Thus amplitude equations are renormalization group equations. We announced this result previously, and derived the Burgers equation as a renormalization group equation [11]. A much more complicated example illustrating this point will be given in Sec. V.

B. Example with log ϵ

The second example we consider is [5]

$$\epsilon y''+xy'-xy=0, \quad y(0)=0, y(1)=e. \quad (3.13)$$

A standard dominant-balance argument tells us that there exists a boundary layer of thickness of order $\epsilon^{1/2}$ (but *not* ϵ) at $x=0$. The complication in the conventional asymptotic matching stems from the fact that the inner expansion must contain not only powers of $\epsilon^{1/2}$ but also those terms containing combinations of ϵ and $\ln\epsilon$ to make the intermediate matching successful. Here we explicitly show that the renormalized naive inner expansion in powers of $\epsilon^{1/2}$ gives a uniformly valid asymptotic solution. This reveals that those unexpected terms containing $\ln\epsilon$ in the conventional approach are just an artifact of perturbative expansions of $x^{-\epsilon}$.

Assuming $x=\epsilon^{1/2}X$, and $y(x)=Y(X)$, we transform (3.13) into

$$\frac{d^2Y}{dX^2}+X\frac{dY}{dX}-\sqrt{\epsilon}XY=0, \quad Y(0)=0, \quad Y(1/\sqrt{\epsilon})=e. \quad (3.14)$$

Naive expansion in $\epsilon^{1/2}$, $Y(X)=Y_0(X)+\epsilon^{1/2}Y_1(X)+\epsilon Y_2(X)+\cdots$ gives

$$Y_0''+XY_0'=0, \quad Y_n''+XY_n'=XY_{n-1}, \quad (n\geq 1). \quad (3.15)$$

Thus the naive perturbation result to order ϵ is

$$Y(X)\sim A_0+B_0\int_0^X ds\ e^{-s^2/2}+\epsilon^{1/2}\left\{A_0(X-X_0)+B_0(X-X_0)\times\int_0^X ds\ e^{-s^2/2}+\mathscr{R}\right\}+\epsilon\left\{\frac{1}{2}A_0(X-X_0)^2+\frac{1}{2}B_0(X-X_0)^2\int_0^X ds\ e^{-s^2/2}-\left(\frac{2}{\sqrt{\pi}}A_0+B_0\right)\times\ln\left(\frac{X}{X_0}\right)\int_0^X ds\ e^{-s^2/2}+\mathscr{R}\right\}, \quad (3.16)$$

where A_0, B_0 are integration constants, and $\mathscr{R}$ represents regular terms finite even in the limit $X-X_0\to\infty$ and $\ln(X/X_0)\to\infty$. The divergence can be controlled by renormalizing $A_0=Z_1A(\mu)$, $B_0=Z_2B(\mu)$, where $Z_1(\mu)=\Sigma_0^\infty a_n\epsilon^{n/2}$, $a_0=1$ and $Z_2(\mu)=\Sigma_0^\infty b_n\epsilon^{n/2}$, $b_0=1$, are renormalization constants and μ is some arbitrary position. The choice $a_1=X_0-\mu$, $a_2=(1/2)(X_0-\mu)^2$ and $b_1=X_0-\mu$, $b_2=(1/2)(X_0-\mu)^2-(2\sqrt{\pi}A+B)\ln(X_0/\mu)$ successfully removes the divergences up to order ϵ, and the renormalized perturbation result is

$$Y(X)\sim\{A(\mu)+\epsilon^{1/2}A(X-\mu)+\epsilon\ \tfrac{1}{2}\ A(X-\mu)^2\}+\left\{B(\mu)+\epsilon^{1/2}B(X-\mu)+\epsilon\ \tfrac{1}{2}\ B(X-\mu)^2-\epsilon\left(\frac{2}{\sqrt{\pi}}A+B\right)\ln(X/\mu)\right\}\int_0^X ds\ e^{-s^2/2}. \quad (3.17)$$

The RG equation $\partial Y/\partial\mu=0$ gives

$$\frac{dA}{d\mu}=\epsilon^{1/2}A+O(\epsilon^{3/2}), \quad (31.18)$$

$$\frac{dB}{d\mu}=\epsilon^{1/2}B-\epsilon\left(\frac{2}{\sqrt{\pi}}A+B\right)\Big/\mu+O(\epsilon^{3/2}). \quad (3.19)$$

Solving these two equations, we obtain

$$A(\mu)=C_1e^{\epsilon^{1/2}\mu}+O(\epsilon^{3/2}\mu), \quad (3.20)$$

$$B(\mu)=-\frac{\epsilon}{1+\epsilon}\frac{2}{\sqrt{\pi}}C_1\mu e^{\epsilon^{1/2}\mu}+C_2\mu^{-\epsilon}e^{\epsilon^{1/2}\mu}+O(\epsilon^{3/2}\mu), \quad (3.21)$$

where C_1, C_2 are constants to be determined by the given boundary conditions. Setting $\mu=X$, we obtain

$$Y(X)\sim C_1e^{\epsilon^{1/2}X}+\left\{-\frac{\epsilon}{1+\epsilon}\frac{2}{\sqrt{\pi}}C_1X+C_2X^{-\epsilon}\right\}e^{\epsilon^{1/2}X}\times\int_0^X ds\ e^{-s^2/2}, \quad (3.22)$$

Imposing boundary conditions $Y(0)=0$, $Y(1/\sqrt{\epsilon})=e$ gives $C_1=0$ and $C_2=\sqrt{2/\pi}\epsilon^{-\epsilon/2}=\sqrt{2/\pi}e^{-(1/2)\epsilon\ln\epsilon}$ as $\epsilon\to 0_+$. Setting back $X=x/\epsilon^{1/2}$, we obtain the final uniformly valid asymptotic result to order ϵ,

$$y(x)\sim e^x x^{-\epsilon}\left\{1-\sqrt{2/\pi}\int_{x/\sqrt{\epsilon}}^{\infty} ds\ e^{-s^2/2}\right\}. \tag{3.23}$$

Thus the terms such as $\epsilon\ln\epsilon$, which are present in the inner expansion given in, e.g., Ref. [1], are relics of the expansion of $x^{-\epsilon}$. It is worthwhile to note that the RG result is slightly different from the asymptotic matching result given by Bender and Orszag in their book [35]. To leading order, the former is

$$y_0^{\mathrm{RG}}(x)\sim e^x\left\{1-\sqrt{2/\pi}\int_{x/\sqrt{\epsilon}}^{\infty} ds\ e^{-s^2/2}\right\}, \tag{3.24}$$

while the latter is

$$y_0^{\mathrm{BO}}(x)\sim e^x-\sqrt{2/\pi}\int_{x/\sqrt{\epsilon}}^{\infty} ds\ e^{-s^2/2}. \tag{3.25}$$

Comparing with the numerical result of the original Eq. (3.13), we find that in the boundary-layer region, the RG result (3.24) is a better approximant than the standard result (3.25).

C. Nonlinear boundary-layer problem

Boundary-layer analysis applies to nonlinear as well as to linear differential equations. In this section and in the following section, we will demonstrate that the RG method can be used to solve nonlinear boundary-layer problems.

Let us consider the following illustrative nonlinear problem [36]

$$\epsilon y''+2y'+e^y=0,\quad y(0)=y(1)=0. \tag{3.26}$$

There is only one boundary layer of thickness ϵ at $x=0$. Setting $X=x/\epsilon$, $Y(X)=y(x)$ in (3.26) gives

$$\frac{d^2Y}{dX^2}+2\frac{dY}{dX}=-\epsilon e^Y. \tag{3.27}$$

Assuming an inner expansion $Y=Y_0+\epsilon Y_1+\cdots$ gives the following asymptotic result as $X\to\infty$,

$$Y(X)\sim A_0+B_0e^{-2X}-\epsilon\{\tfrac{1}{2}e^{A_0}(X-X_0)+\mathscr{R}\}+O(\epsilon^2), \tag{3.28}$$

where A_0, B_0 are integration constants and $\mathscr{R}$ represents all regular terms in the expansion finite even in the limit $X-X_0\to\infty$. The renormalized perturbation result obtained as in the previous examples is

$$Y(X)\sim A(\mu)+B(\mu)e^{-2X}-\epsilon\tfrac{1}{2}e^{A(\mu)}(X-\mu)+O(\epsilon^2). \tag{3.29}$$

The RG equation gives, to order ϵ,

$$\frac{dA}{d\mu}+\epsilon\frac{1}{2}e^A=0,\quad \frac{dB}{d\mu}=0. \tag{3.30}$$

Solving (3.30), we get

$$A(\mu)=\ln\left(\frac{2}{\epsilon\mu+C_1}\right),\quad B(\mu)=C_2, \tag{3.31}$$

where C_1, C_2 are constants of integration to be determined by the given boundary conditions. Equating μ and X in (3.29) and restoring $x=\epsilon X$, we obtain the uniformly valid asymptotic result

$$y(x)\sim\ln\left(\frac{2}{x+C_1}\right)+C_2e^{-2x/\epsilon}+O(\epsilon). \tag{3.32}$$

Imposing boundary conditions $y(0)=0$, $y(1)=0$ gives $C_1=1$, $C_2=-\ln 2$ in the limit $\epsilon\to 0_+$. Therefore the final result is

$$y(x)\sim\ln\left(\frac{2}{x+1}\right)-(\ln 2)e^{-2x/\epsilon}+O(\epsilon). \tag{3.33}$$

This RG result recovers the leading-order result from the boundary-layer analysis.

D. Nonlinear problem of carrier

In this section, we consider a first-order nonlinear model problem of Carrier [37],

$$(x+\epsilon f\,)f'+f=1,\quad f(1)=2,\quad 0\leqslant x\leqslant 1. \tag{3.34}$$

The exact solution can be obtained by integrating (3.34) once,

$$f(x,\epsilon)=-\frac{x}{e}+\left(\frac{x^2}{\epsilon^2}+\frac{2(x+1)}{\epsilon}+4\right)^{1/2}. \tag{3.35}$$

It becomes, however, a nontrivial singular perturbation problem, if we pretend that we cannot obtain the exact solution. The method of strained coordinates or the method of asymptotic matching can be applied with a rather lengthy matching. We show here how to solve the problem using RG without matching, and give the exact result, starting only from the inner expansion.

First, we apply the usual dominant balance argument to make the structure of the equation clear. We introduce $X\equiv\eta(\epsilon)x$ and $F=\delta(\epsilon)f$; the latter is needed because the equation is nonlinear. The original equation reads $[X+(\epsilon\eta/\delta)F]dF/dX+F=\delta$. $\epsilon\eta/\delta\ll 1$ corresponds to the outer limit, and $\epsilon\eta/\delta\sim 1$ is the only nontrivial alternative possibility. Hence, $\delta\sim\epsilon^\alpha$ and $\eta\sim\epsilon^{\alpha-1}$ with $\alpha\in(0,1]$ are the useful scalings. The expansion parameter becomes $\delta\sim\epsilon^\alpha$. It turns out that any choice of α is admissible in this case, and so we adopt the simplest choice $\alpha=1$.

Accordingly, we rescale f as $F=\epsilon f$ to convert the original Eq. (3.34) to

$$(x+F)F'+F=\epsilon. \tag{3.36}$$

Expanding F as $F=F_0+\epsilon F_1+\cdots$, we have

$$(x+F_0)F_0'+F_0=0, \tag{3.37}$$

whose general positive solution is

$$F_0(x)=(x^2+A_0)^{1/2}-x, \tag{3.38}$$

with A_0 a constant of integration determined by the initial condition given at some arbitrary x_0. The first-order equation is given by

$$(x+F_0)F_1'+F_1F_0'+F-1=1. \tag{3.39}$$

This linear equation has a general solution

$$F_1(x)=\frac{x-x_0}{(x^2+A_0)^{1/2}}. \tag{3.40}$$

Thus the straightforward perturbation result, to $O(\epsilon)$, is given by

$$F(x)=(x^2+A_0)^{1/2}-x+\epsilon\,\frac{(x-x_0)}{(x^2+A_0)^{1/2}}+O(\epsilon^2). \tag{3.41}$$

We see that this naive perturbation (3.41) breaks down formally for $x\gg x_0$. Actually, the domain of our problem is finite, and because x is not scaled, it is not possible that $x\gg x_0$ can occur within the domain. A better argument is as follows. Since the boundary condition is $F(1)=2\epsilon$, near $x=1$ the $O(\epsilon)$ term dominates; this is a singular perturbation, and indeed the perturbation term diverges relative to the zeroth-order term.

The secular divergence can be removed by renormalizing A_0 by $A_0(x_0)=ZA(\mu)$, and the renormalized perturbation result obtained is

$$F(x)=[x^2+A(\mu)]^{1/2}-x+\epsilon\,\frac{(x-\mu)}{[x^2+A(\mu)]^{1/2}}+O(\epsilon^2). \tag{3.42}$$

The RG equation gives, to $O(\epsilon)$,

$$dA/d\mu=2\epsilon \tag{3.43}$$

with solution

$$A(\mu)=A(0)+2\epsilon\mu. \tag{3.44}$$

Setting $\mu=x$ and $f=F/\epsilon$, we obtain the uniformly valid asymptotic result

$$f(x,\epsilon)=-\frac{x}{\epsilon}+\left(\frac{x^2}{\epsilon^2}+\frac{2x}{\epsilon}+\frac{A(0)}{\epsilon^2}\right)^{1/2}. \tag{3.45}$$

Imposing the boundary condition $f(1)=2$ gives $A(0)=2\epsilon+4\epsilon^2$. Therefore, the uniformly valid result to order ϵ, is given by

$$f(x,\epsilon)=-\frac{x}{\epsilon}+\left(\frac{x^2}{\epsilon^2}+\frac{2(x+1)}{\epsilon}+4\right)^{1/2}. \tag{3.46}$$

This happens to be the exact solution to the problem. A further calculation demonstrates that all the higher-order corrections vanish. The conventional methods can also recover the exact result, but clearly the RG is simpler.

E. Problem with multiple-boundary layers

In many situations, there exist multiple-boundary layers at one side, for which multiple calculations of inner and outer solutions and their asymptotic matchings have to be made in different separated regions to obtain a uniformly valid solution. Again it turns out that the RG method manages to produce the solution without any matching needed. Let us consider the following initial-value problem [38]

$$\epsilon^{3/2}y'''+(\epsilon^{1/2}+\epsilon+\epsilon^{3/2})y''+(1+\epsilon^{1/2}+\epsilon)y'+y=0, \tag{3.47}$$

with initial conditions $y(0)=3, y'(0)=-1-\epsilon^{-1/2}-\epsilon^{-1}$, $y''(0)=1+\epsilon^{-1}+\epsilon^{-2}$ [1]. The exact solution is $y(t)=e^{-x}+e^{-x/\epsilon^{1/2}}+e^{-x/\epsilon}$. Pretending we do not know how to solve it exactly, we resort to conventional singular perturbation methods. It turns out that the conventional perturbation calculation is very tedious and rather challenging. By dominant balance, this problem is found to have two distinguished boundary layers at $t=0$, of thickness of order $\epsilon^{1/2}$ and ϵ, respectively. Therefore one outer solution and two inner solutions must be calculated and two asymptotic matchings are necessary, if boundary layer theory is used. Starting only with the thinnest or innermost boundary layer by rescaling t by $t=\epsilon T$, and expanding $y=Y(T)$, e.g., in $\epsilon^{1/2}$, the RG method successfully recovers the exact solution without any matching.

F. Linear boundary-layer and WKB problems I: No turning points

To conclude this section, we show how linear boundary-layer and WKB problems in general forms can be treated using RG in a unified fashion. This relationship between boundary-layer theory and WKB is explained in Bender and Orszag's book [1]. The boundary-layer type problem we wish to study using RG has the following general form:

$$\epsilon^2\frac{d^2y}{dx^2}+a(x)\frac{dy}{dx}-b(x)y=0,\quad 0\leqslant x\leqslant 1,\quad \epsilon\to 0_+ \tag{3.48}$$

where we assume that $a(x)$ is differentiable and $b(x)$ is an arbitrary, not necessarily continuous function. This equation covers all linear examples we presented earlier in this section. A simple dominant-balance argument determines that in general, the boundary layer lies at $x=0$ when $a(x)\geqslant 0$ for $0\leqslant x\leqslant 1$, and that the boundary layer lies at $x=1$ when $a(x)<0$ for $0\leqslant x\leqslant 1$. Without loss of generality we will consider only the former case.

Although in a number of cases we could perform perturbative RG analysis on the original general Eq. (3.48), often it is wiser to start with the canonical form of Eq. (3.48) under the transformation

$$y(x)=\exp\left[-\frac{1}{2\epsilon^2}\int^x a(x')dx'\right]u(x), \tag{3.49}$$

converting (3.48) to

$$\epsilon^2\frac{d^2u}{dx^2}=Q(x)u(x), \tag{3.50}$$

with

$$Q(x)\equiv\frac{1}{4\epsilon^2}a^2(x)+\frac{1}{2}a'(x)+b(x). \tag{3.51}$$

This form is just the Schrödinger form, which can be solved by the WKB methods. Consequently, we can treat both the linear boundary-layer and the WKB problems in a unified way.

In the remainder of this section and in the following section, we will show how to solve Schrödinger equations using RG. Our strategy is to first introduce a natural change of the independent variable which allows one to obtain efficiently the nonperturbative part of the solution. The transformation is identical to the independent variable portion of the standard Liouville-Green transformation [39] or its natural generalization used by Langer [40], but the crucial difference is that we do not introduce the new dependent variable. This is the analogue of the geometrical-optics approximation in the WKB theory [17], and is the starting point of a renormalized perturbation series, which reproduces the physical-optics and higher-order WKB approximations. Although it may be possible to derive even the geometric-optics approximation using RG, we have not succeeded in so doing. When $Q(x)$ vanishes, its zeros lead to turning points in the standard WKB approach. The simplest WKB approximations break down there, and connection formulas are required in the conventional procedure in order to match approximations on either side. Langer [40] found that a suitable generalization of the Liouville-Green transformation can produce a uniformly valid approximation across the turning point. Again for the cases with turning points, we transform the independent variable only with a straightforward generalization of the no-turning point case. We emphasize that we are able to avoid the need to perform matching, and that the transformation of the dependent variable is produced naturally by RG. The use of RG is *not* responsible for the choice of the transformation of the independent variable, but our choice not to introduce the transformation of dependent variables in contrast to the approaches by Liouville and Green and Langer is motivated by the RG. This allows us to choose a better transformation of the dependent variable, which agrees with the conventional result in the small ϵ limit. The corrections and prefactors which accompany the zeroth-order Langer-type solution *are* calculated by the RG, and do differ from and improve upon those obtained by the standard analysis. Furthermore, we can expand our asymptotic sequence in ϵ to reproduce the standard textbook results.

The remainder of this section concerns Schrödinger problems with no turning points. The following section discusses the case with one-turning point, and gives an outline of how the methods can be generalized for higher numbers of turning points and for multiple-boundary-layer linear problems as well.

We will first rederive the well-known physical-optics approximation using the RG theory, valid when the function $Q(x)$ has no zeroes in the interval of interest. Following Liouville and Green, we introduce a new independent variable $t=f(x)$ implicitly determined as $dt=\sqrt{Q}dx/\epsilon$. The choice is natural from the perturbation point of view, because even when $du/d(x/\epsilon)$ is significant, $dQ(x)/dx$ is not, so $Q(x)$ can be regarded as a constant to order $O(\epsilon^0)$. Equation (3.50) is thus converted to

$$\frac{d^2u}{dt^2}-u=2\epsilon S(x)\frac{du}{dt}, \tag{3.52}$$

where $S(x)\equiv-(1/4)Q^{-3/2}Q'(x)$ is assumed to be a slowly varying function on the time scale t, of order unity, and $S(x)\neq0$ for $0\leqslant x\leqslant1$.

Naively expanding u as $u(t)=u_0(t)+\epsilon u_1(t)+\cdots$, we get the bare perturbation result

$$\begin{aligned}u(t)=e^t\Bigg\{&A_0+\epsilon A_0\int_{t_0}^{t}S(x(t'))dt'-\epsilon A_0e^{-2t}\\&\times\int_{t_0}^{t}S(x(t'))e^{2t'}dt'\Bigg\}+e^{-t}\Bigg\{B_0+\epsilon B_0\int_{t_0}^{t}S(x(t'))dt'\\&-\epsilon B_0e^{2t}\int_{t_0}^{t}S(x(t'))e^{-2t'}dt'\Bigg\}+O(\epsilon^2),\end{aligned} \tag{3.53}$$

where A_0,B_0 are integration constants. The corresponding renormalized result is

$$\begin{aligned}u(t)=&e^t\Bigg\{A(\mu)+\epsilon A(\mu)\int_{\mu}^{t}S\,dt'\Bigg\}\\&+e^{-t}\Bigg\{B(\mu)+\epsilon B(\mu)\int_{\mu}^{t}S\,dt'\Bigg\}+O(\epsilon),\end{aligned} \tag{3.54}$$

where $O(\epsilon)$ refers to all regular terms of order ϵ which remain finite even as $t-t_0\to\infty$. The RG equation $\partial u/\partial\mu\equiv0$ gives

$$\frac{dC}{d\mu}+\epsilon\,\tfrac{1}{4}Q^{-3/2}Q'[x(\mu)]C=O(\epsilon^2), \tag{3.55}$$

where $C=A$ or B. Again, Eq. (3.55) corresponds to the amplitude equation or slow motion equation. Setting $\mu=t$ and using $dt=Q^{1/2}dx/\epsilon$, we get

$$A(x)\sim Q^{-1/4}(x),\quad B(x)\sim Q^{-1/4}(x). \tag{3.56}$$

This is exactly the adiabatic invariant $A(x)Q^{1/4}(x)=A(0)Q^{1/4}(0)=$const. The physical-optics approximation for the WKB Eq. (3.50) is recovered

$$\begin{aligned}u(x)\sim&C_1Q^{-1/4}(x)\exp\Bigg[\frac{1}{\epsilon}\int^{x}dx'\sqrt{Q(t')}\Bigg]\\&+C_2Q^{-1/4}(x)\exp\Bigg[-\frac{1}{\epsilon}\int^{x}dx'\sqrt{Q(t')}\Bigg],\end{aligned} \tag{3.57}$$

as $\epsilon\to0$.

The uniformly valid asymptotic result $y(x)$ for the general linear boundary-layer problem (3.48) is given by (3.49). For numerical evaluation of (3.33), we do not need any further expansion, because (3.49) is the uniformly valid result we want. To compare, however, with the conventional results due to asymptotic matching methods, let us make asymptotic expansions of $Q(x)$.

As a simple check, let us assume that $a(x),b(x)$ are some analytic functions, and $a(x)>0$ for $0\leqslant x\leqslant1$, and $a(0)\neq0$. Obviously, in the whole region $0\leqslant x\leqslant1$, as $\epsilon\to0$, the term

$a^2(x)/4\epsilon^2$ is the dominant term, compared to $a'(x)/2$ and $b(x)$. Simply Taylor expanding as

$$\sqrt{Q(x)} \simeq \frac{a(x)}{2\epsilon} + \frac{\epsilon}{2}\frac{a'(x)}{a(x)} + \epsilon\frac{b(x)}{a(x)}, \tag{3.58}$$

and imposing boundary conditions $y(0)=A, y(1)=B$, we obtain

$$y(x) \sim B e^{\int_1^x [b(\xi)/a(\xi)]d\xi} + \frac{a(0)}{a(x)} \times [A - B e^{\int_1^0 [b(\xi)/a(\xi)]d\xi}] e^{-\int_0^x \{[a(\xi)/\epsilon^2] + [b(\xi)/a(\xi)]\}d\xi}. \tag{3.59}$$

This expression can be simplified further, because the second term contributes appreciably only when $x = O(\epsilon^2)$ $(\epsilon \to 0)$. Thus

$$y(x) \sim B \exp\left[\int_1^x \frac{b(\xi)}{a(\xi)}\, d\xi\right] + \left[A - B \exp\int_1^0 \frac{b(\xi)}{a(\xi)}\, d\xi\right] e^{-a(0)x/\epsilon^2}. \tag{3.60}$$

This is exactly the same as the uniformly valid leading boundary layer or WKB result.

It is known that the case with $a(0)=0$ is subtle. For simplicity, we consider only the cases $a(x)=x^\alpha, b(x)=1$, where $\alpha > -1$ so that there exists a boundary layer at $x=0$.

When $\alpha > 1$, the thickness of boundary layer is of order $\delta = O(\epsilon)$. When $x \gg \epsilon$, the term $a^2(x)/4\epsilon^2$ dominates over other two terms, $a'(x)/2$ and $b(x)$, in $Q(x)$. However, when $x \sim O(\epsilon)$, we have to be careful with the asymptotic expansion of $Q(x)$, since the dominant term now is $b(x)=1$. Thus, as $\epsilon \to 0$, the leading term of $\sqrt{Q(x)}$ is 1. The final uniformly valid approximation is

$$y(x) \sim B \exp\left[\frac{1}{\alpha - 1}(1 - x^{1-\alpha})\right] + A \exp[-x/\epsilon]. \tag{3.61}$$

When $|\alpha| \leqslant 1$, it is straightforward to check that the boundary layer is of thickness of order $\delta \sim \epsilon^{2/(1+\alpha)}$, and that the first and second term in $Q(x)$ are of the same order, when $x \sim \delta(\epsilon)$. The uniformly valid expression turns out to be

$$y(x) = B \exp\left[\int_1^x \frac{b(\xi)}{a(\xi)}\, d\xi\right] + \left[A - B \exp\int_1^0 \frac{b(\xi)}{a(\xi)}\, d\xi\right] \times \left\{\frac{Q(0)}{Q(x)}\right\}^{-1/4} \exp\left[-\int_0^x d\xi \left\{\frac{a(\xi)}{2\epsilon^2} + \frac{\sqrt{Q(\xi)}}{\epsilon}\right\}\right]. \tag{3.62}$$

Expanding the above leading uniformly valid result obtained with the aid of RG recovers the outer and inner solutions due to boundary-layer theory and asymptotic matching. Note that the above results are obtained from the ''inner expansion'' alone without ever having to perform any asymptotic matching. This is practically important as we will see in the next section.

G. WKB analysis II: Turning points

In order to complete this section, we begin by presenting a general discussion of Schrödinger equations and one-turning-point WKB problem, and at the end of this section, we generalize the case to multiple-turning-point and multiple-boundary-layer problems.

The Schrödinger equation which we will consider in this section is

$$\epsilon^2 \frac{d^2u}{dx^2} = Q(x)u(x), \quad u(+\infty) = 0. \tag{3.63}$$

when Q in (3.63) vanishes or changes its sign, the approach in the preceding subsection fails as can easily be seen from the presence of the factor $Q^{-1/4}$. If Q has an isolated zero at $x=0$ of order $\alpha > 0$, we can write locally $Q(x) = x^\alpha \psi(x)$ with a positive definite function ψ without any loss of generality. A natural choice of the counterpart of the Liouville-Green transformation $x \to t$ is to remove the zeros from dt/dx: we introduce a new independent variable $t = f(x)$ implicitly determined as $dt = (Q/t^\alpha)^{1/2} dx/\epsilon$ giving

$$t(x) = \left(\frac{2+\alpha}{2\epsilon} \int_0^x dx' [Q(x')]^{1/2}\right)^{2/(2+\alpha)}. \tag{3.64}$$

The original Eq. (3.50) is transformed into

$$\frac{d^2u}{dt^2} = t^\alpha u + \epsilon S(t(x)) \frac{du}{dt}, \tag{3.65}$$

where $S \equiv d[(t^\alpha/Q)^{1/2}]/dx$. Since $t \sim x$ as $x \to 0$, S is a bounded function even near $x=0$. Notice that in contrast to the conventional approaches due to Liouville and Green or Langer, we do not introduce the transformation for the dependent variable, because it will be produced by the RG procedure. Here we work out the simplest case $\alpha = 1$.

Expanding naively u in powers of ϵ as $u = u_0 + \epsilon u_1 + \epsilon^2 u_2 + \cdots$, we obtain the bare perturbation result to order ϵ,

$$u = C_0 \,\mathrm{Ai}(t) - \epsilon C_0 \pi \left\{ \mathrm{Ai}(t) \int_{t_0}^t dt' S(t') \mathrm{Ai}'(t') \mathrm{Bi}(t') - \mathrm{Bi}(t) \int_{t_0}^t dt' S(t') \mathrm{Ai}(t') \mathrm{Ai}'(t') \right\}, \tag{3.66}$$

where Ai,Bi are two linearly independent Airy functions, and the $\mathrm{Bi}(t)$ function is already discarded in the zeroth-order solution, since it does not satisfy the physical condition $u(+\infty)=0$. In the limit $t - t_0 \to +\infty$, the second term of the first-order perturbation $\mathrm{Bi}(t)\int_{t_0}^t dt' S(t') \mathrm{Ai}(t') \mathrm{Ai}'(t')$ remains finite. However, the term $\int_{t_0}^t dt' S(t') \mathrm{Ai}'(t') \mathrm{Bi}(t')$ diverges and must be renormalized, giving the renormalized perturbation series

$$u = \mathrm{Ai}(t) \left(C(\mu) - \epsilon C(\mu) \pi \int_\mu^t dt' S(t') \mathrm{Ai}'(t') \mathrm{Bi}(t') \right) + O(\epsilon), \tag{3.67}$$

where $C(\mu)$ is the counterpart of the bare amplitude $C_0(t_0)$, and $O(\epsilon)$ refers to all finite regular terms of order ϵ even in the limit $t-t_0\to\infty$. The RG equation $du/d\mu\equiv 0$ gives

$$\frac{dC(\mu)}{d\mu}+\epsilon C(\mu)\pi S(\mu)\mathrm{Ai}'(\mu)\mathrm{Bi}(\mu)=O(\epsilon^2). \tag{3.68}$$

Integrating (3.68) and setting $\mu=t$, we get

$$C(t)=C(0)\exp\left\{-\pi\int_0^t dt'\,\mathrm{Ai}'(t')\mathrm{Bi}(t')\times\frac{d}{dt'}\{\ln[(t'/Q)^{1/2}]\}\right\}, \tag{3.69}$$

where C_0 is a constant of integration to be determined by the boundary condition at $t=0$. Thus we have arrived at the adiabatic invariant

$$C(t)\exp\left\{\pi\int_0^t dt'\,\mathrm{Ai}'(t')\mathrm{Bi}(t')\,\frac{d}{dt'}\{\ln[(t'/Q)^{1/2}]\}\right\}, \tag{3.70}$$

which differs from that usually obtained, leading to the final uniformly valid solution

$$u=C(0)\exp\left\{-\pi\int_0^t dt'\,\mathrm{Ai}'(t')\mathrm{Bi}(t')\,\frac{d}{dt'}\times(\ln[(t'/Q)^{1/2}])\right\}\mathrm{Ai}(t), \tag{3.71}$$

where $t(x)=([3/2\epsilon]\int_0^x dx'\,\sqrt{Q(x')})^{2/3}$.

The RG result (3.69) differs from the standard Langer formula, since (3.69) involves Airy functions Ai and Bi. Note that the new variable t given in (3.64) is a function of ϵ, and that as $\epsilon\to 0$ for fixed x, $t\to\infty$. In this limit, we can resort to the asymptotic properties of the Airy functions $\mathrm{Ai}(t)$ and $\mathrm{Bi}(t)$ for $t\to\infty$, and find that $\mathrm{Ai}'(t)\mathrm{Bi}(t)\sim -1/2\pi$, as $t\to\infty$. Thus (3.69) recovers the standard result

$$C(t(x))=C(0)(t/Q)^{1/4}. \tag{3.72}$$

However, the RG Eq. (3.68) is valid not only for relatively large μ, but also for small μ. For this reason, we expect that (3.71) is a better uniformly valid approximant than the standard Langer formula, for small and intermediate values of t, or for relatively large (or not small) ϵ cases. This is verified and can be clearly seen in Fig. 1, where we compare the RG result (3.71), the standard Langer formula, and the exact numerical solution of Eq. (3.50) for several values of ϵ. Thus the RG results (3.71) without asymptotic matching improve upon those obtained by the standard analysis.

To conclude this section, we briefly outline the recipe to generalize the methods for multiple-turning-point and linear multiple-boundary-layer problems. [For linear cases, with the help of the transformation (3.49) both problems can be transformed into the canonical form and can be treated in a unified way.] We need only consider the case in which $Q(x)$ in (3.50) has multiple-turning points. Without loss of generality, we assume Q has the form: $Q(x)=f(x)\psi(x)$, where

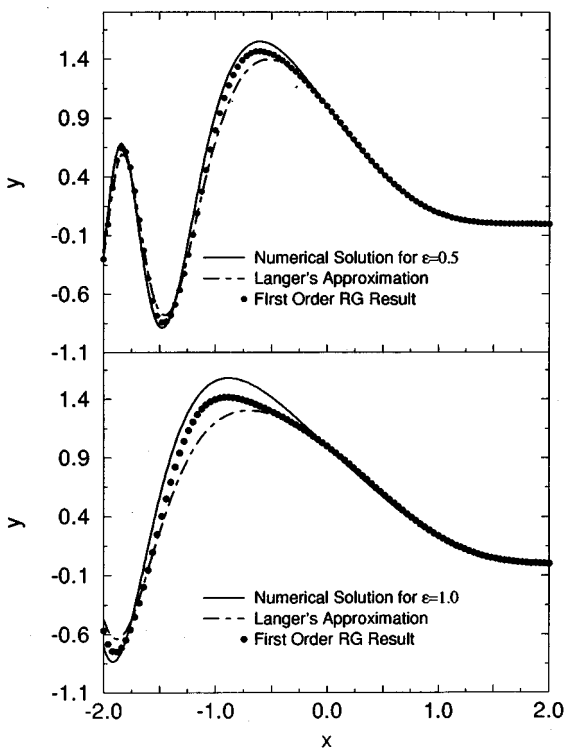

FIG. 1. Comparison of the RG result (3.71), the standard Langer formula, and the numerical solution of Eq. (3.50) for $\epsilon=0.5$ and $\epsilon=1.0$.

$f(x)=(x-x_1)(x-x_2)\cdots(x-x_n)$, $n>1$ is a polynomial of x with n zeros $x_1<x_2<\cdots<x_n$, and $\psi(x)>0$ has no zeros. The general strategy is first to introduce a new independent variable t defined implicitly as $dt/\sqrt{Q/f(t)}dx/\epsilon$, where f is chosen to cancel all the zeros of Q. Then we develop the straightforward perturbation series for the resultant equation, and renormalize the integration constant to absorb the secular divergence. This procedure avoids performing multiple connection formulas matching and leads to a uniformly valid approximation. For higher-order WKB problems or linear boundary-layer problems, the generalization of the methods given here is straightforward.

IV. SWITCHBACK PROBLEMS

In previous sections, we have already seen that the RG approach not only has conceptual, but also technical advantages compared with various conventional methods. In this section, we will demonstrate this further, by studying, with the aid of the RG more complicated problems which involve the so-called ''switchback.'' In switchback problems, as conventionally treated, only through subtle analysis in the course of actually solving the problem is it possible to realize the need for, e.g., unexpected order terms to make asymptotic matching consistent.

A. Example 1: Stokes-Oseen caricature

A model example is a caricature of the Stokes-Oseen singular-boundary-layer problem, which describes the low Reynolds number viscous flow past a sphere of unit radius. The main result of this problem has been presented in Ref.

[18], and in the following, we will only briefly summarize the final results and make some additional comments.

The equation is [5]

$$\frac{d^2u}{dr^2}+\frac{2}{r}\frac{du}{dr}+\epsilon u\frac{du}{dr}=0,\quad u(1)=0,\quad u(\infty)=1, \tag{4.1}$$

where ϵ, the Reynolds number, is a small non-negative constant. This is a very delicate singular boundary-layer problem, with complicated asymptotic expansions and matching, involving unexpected orders such as $\epsilon\ln(1/\epsilon)$.

Since there exists a boundary layer of thickness $\delta=O(\epsilon)$ near $r=\infty$, setting $x=\epsilon r$ transforms (4.1) into the following ''inner'' equation:

$$\frac{d^2u}{dx^2}+\frac{2}{x}\frac{du}{dx}+u\frac{du}{dx}=0,\quad u(x=\epsilon)=0,\quad u(x=\infty)=1. \tag{4.2}$$

Using RG theory, the final uniformly valid result is found to be, to order $\lambda_1=1/e_2(\epsilon)$,

$$u(r;\epsilon)=1-e_2(\epsilon r)/e_2(\epsilon)+O\{[1/e_2(\epsilon)]^2\}, \tag{4.3}$$

where the exponential integral $e_2(t)=\int_t^\infty d\rho\rho^{-2}e^{-\rho}$, whose asymptotic expansion as $t\to 0$ is given by $e_2(t)\sim 1/t+\ln t+(\gamma-1)-t/2+O(t^2)$ with Euler's constant $\gamma\simeq 0.577\ldots$.

The result from asymptotic matching is given by the following expression [5]. For r fixed, we have, to $O(\epsilon^2\ln^2(1/\epsilon))$,

$$u(r)\sim\left(1-\frac{1}{r}\right)+\epsilon\ln(1/\epsilon)\left(1-\frac{1}{r}\right)+\epsilon\left[-\ln r+(1-\gamma)\left(1-\frac{1}{r}-\frac{\ln r}{r}\right)\right], \tag{4.4}$$

while for $\rho=\epsilon r$ fixed, to $O(\epsilon^2\ln(1/\epsilon))$,

$$u(\rho)\sim 1-\epsilon e_2(\rho). \tag{4.5}$$

Accordingly, examining the asymptotic result of (4.3) in the limit $\epsilon\to 0$, by expanding both $e_2(\epsilon r)$ and $e_2(\epsilon)$ for r fixed, and $e_2(\epsilon)$ only for $\rho=\epsilon r$ fixed, respectively, it is found that the resulting asymptotic solution using RG is correct to $O[\epsilon\ln(1/\epsilon)]$ and agrees with that obtained by asymptotic matching. Note that in our method, the $\epsilon\ln\epsilon$ term appears naturally from the asymptotic expansion of $e_2(\epsilon)$, whereas some artistry is required to obtain this term conventionally. To recover the $O[\epsilon]$ term with $(\ln r)/r$, we have to extend the RG calculation to order $O([1/e_2(\epsilon)]^2)$. Thus the result to $O(\epsilon)$ given by asymptotic matching [5] is obtained from the renormalized perturbation expansion to $O([1/e_2(\epsilon)]^2)$.

This fact may suggest that our RG result is inferior to the conventional one. It is important to notice, however, that neither the asymptotic expansion (4.4) augmented with the $(\ln r)/r$ term of order ϵ nor (4.5) is uniformly valid in its variable r or ρ, respectively. In contrast, it seems that our full result $1-e_2(\epsilon r)/e_2(\epsilon)$ to order $\lambda_1=1/e_2(\epsilon)$ is uniformly valid as is clearly seen in Fig. 2.

As discussed in the preceding paragraph, (4.3) is not an asymptotic series in powers of ϵ; thus one might conclude

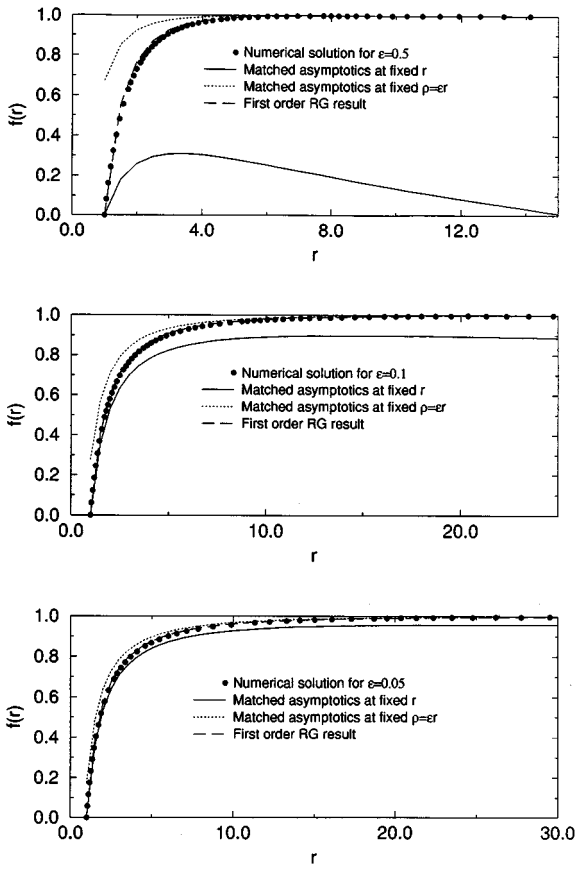

FIG. 2. Comparison between the numerical solution of Eq. (4.2) for several values of ϵ, the first-order RG result $1-e_2(\epsilon r)/e_2(\epsilon)$, and two matched asymptotic expansions (one at fixed r, the other at fixed $\rho\equiv r\epsilon$), as derived in Ref. [5].

that our result is not even an asymptotic series in any sense. Recall, however, that the asymptotic expansion of a function is unique only when an asymptotic sequence of functions is fixed. The choice of the sequence is a question of vital importance, if one wishes to have a useful asymptotic series. In the conventional singular perturbation methods, an asymptotic sequence is selected by the matching conditions. However there is no compelling reason to believe that the selected sequence is practically the best asymptotic sequence (of course, it should be the most convenient one for the matching procedure). As we have seen, the RG approach also produces an asymptotic sequence $\{\lambda_i(\epsilon)\}$ from the requirement to satisfy the boundary condition order by order. Therefore we propose the point of view that a consistent and presumably better asymptotic expansion (starting with $\lambda_1=1/e_2(\epsilon)$ in the present problem) may be obtained by the RG. The standard ϵ expansion may well be an inferior asymptotic expansion to our expansion. In addition, the superiority to the RG approach can also be seen from the fact that a closed expression uniformly valid for the whole (infinite) interval has been obtained for the problem, which is not the case for the standard asymptotic matching method.

B. Example 2: Difficulty with asymptotic matching

To illustrate that the RG method is generally simpler to use, and yields practically better approximants than other

methods, let us next consider a ''terrible'' problem whose model equation can be written as [41]

$$\frac{d^2u}{dr^2}+\frac{1}{r}\frac{du}{dr}+\alpha\left(\frac{du}{dr}\right)^2+\epsilon u\frac{du}{dr}=0,\quad u(1)=0,\quad u(\infty)=1, \tag{4.6}$$

where ϵ is a small non-negative constant, and $\alpha=0$ or 1. For $\alpha=1$, the asymptotic matching is notoriously difficult, because an infinite number of terms must be calculated before even the leading order can be matched successfully. We will see how the RG avoids such difficulties in obtaining the leading-order result uniformly valid for the entire interval $1\leqslant r<\infty$.

Since there exists a boundary layer of thickness $\delta=O(\epsilon)$ near $r=\infty$, setting $x=\epsilon r$ transforms (4.6) into the following ''inner'' equation:

$$\frac{d^2u}{dx^2}+\frac{1}{x}\frac{du}{dx}+\alpha\left(\frac{du}{dx}\right)^2+u\frac{du}{dx}=0,\quad u(x=\epsilon)=0,$$

$$u(x=\infty)=1. \tag{4.7}$$

As in other boundary-layer problems, let us first look for the general form of the solution, and then impose the required boundary conditions to determine the constants of integration left in the solution. To do so, we solve (4.2) as an initial-value problem, given an initial condition $u(x_0)=A_0$ at some arbitrary point $x=x_0$, where A_0 is a finite constant. Assuming a naive expansion $u(x;\epsilon)=u_0(x)+\lambda_1(\epsilon)u_1(x)+\lambda_2(\epsilon)u_2(x)+\cdots$ with initial conditions $u_0(x_0)=A_0$, $u_i(x_0)=0$, $i=1,2,\ldots$, where the asymptotic sequence $\lambda_i(\epsilon)$, $i=1,2,\ldots$ are to be determined later, we obtain

$$\frac{d^2u_0}{dx^2}+\frac{1}{x}\frac{du_0}{dx}+\alpha\left(\frac{du_0}{dx}\right)^2+u_0\frac{du_0}{dx}=0. \tag{4.8}$$

The *finite* uniform solution can be guessed as $u_0(x)=A_0$, because the uniform field should not be affected appreciably by the distant disturbance source. Thus, the goal is to find out the small perturbation effect on this uniform field in the presence of a distant disturbance.

The equation for u_1 is

$$\frac{d^2u_1}{dx^2}+\left(\frac{1}{x}+A_0\right)\frac{du_1}{dx}=0. \tag{4.9}$$

We easily see that the equation satisfied by u_2 which is significantly different from (4.9) (i.e., with a forcing term) appears only if $\lambda_1^2/\lambda_2=O(1)$. We will show that indeed the choice $\lambda_2=\lambda_1^2$ works. The nontrivial equation at order $\lambda_2=\lambda_1^2$ can be written as

$$\frac{d^2u_2}{dx^2}+\left(\frac{1}{2}+A_0\right)\frac{du_2}{dx}=-\alpha\left(\frac{du_1}{dx}\right)^2-u_1\frac{du_1}{dx}. \tag{4.10}$$

The perturbation result is given by

$$\begin{aligned}u(x)=&A_0+\lambda(\epsilon)A_{10}[e_1(A_0x_0)-e_1(A_0x)]+\lambda^2(\epsilon)\\&\times\{A_2[e_1(A_0x_0)-e_1(A_0x)]-\alpha\tfrac{1}{2}A_{10}^2[e_1(A_0x_0)\\&-e_1(A_0x)]^2-A_{10}^2A_0^{-1}[e_0(A_0x)e_1(A_0x)\\&-2e_1(2A_0x)-e_1(A_0x_0)e_0(A_0x)+e_0(A_0x_0)e_1(A_0x)\\&-e_0(A_0x_0)e_1(A_0x_0)+2e_1(2A_0x_0)]\}+O[\lambda^3(\epsilon)],\end{aligned} \tag{4.11}$$

where the exponential integral $e_1(t)=\int_t^\infty \rho^{-1}e^{-\rho}$, $\lambda_1(\epsilon)$ is already replaced by $\lambda(\epsilon)$, and A_{10}, A_2 are constants of integration. When x_0 is very small and $x-x_0$ is large, the divergence arises from those terms containing $e_1(A_0x)$ or $e_1(A_0x_0)$, but *not* $e_0(A_0x)$ or $e_0(A_0x_0)$. To remove the divergence from these cross terms of e_0 and e_1, presumably both A_0 and A_1 must be renormalized. The renormalized perturbation result reads

$$\begin{aligned}u(x)=&A(\mu)+\lambda(\epsilon)A_1(\mu)[e_1(A\mu)-e_1(Ax)]\\&+\lambda^2(\epsilon)\{A_2[e_1(A\mu)-e_1(Ax)]-\alpha\tfrac{1}{2}A_1^2[e_1(A\mu)\\&-e_1(Ax)]^2-A_1^2A^{-1}[e_0(Ax)(e_1(Ax)-e_1(A\mu))\\&-2e_1(2Ax)+2e_1(2A\mu)]\}+O(\lambda^3(\epsilon)),\end{aligned} \tag{4.12}$$

where $A(\mu),A_1(\mu)$ are finite counterparts of $A_0(x_0),A_{10}(x_0)$, and μ is some arbitrary length scale. The RG equation $du/d\mu=0$ gives

$$\frac{dA_1}{d\mu}=-\lambda(\epsilon)\alpha A_1^2\mu^{-1}e^{-A\mu}+O(\lambda^2(\epsilon)), \tag{4.13}$$

$$\begin{aligned}\frac{dA}{d\mu}=&-\lambda(\epsilon)A_1\mu^{-1}e^{-A\mu}-\lambda^2(\epsilon)A_1^2A^{-1}\mu^{-1}e^{-2A\mu}\\&+\lambda^2(\epsilon)A_2\mu^{-1}e^{-A\mu}+O(\lambda^3(\epsilon)).\end{aligned} \tag{4.14}$$

Now we discuss the $\alpha=0$ and $\alpha=1$ cases separately. For $\alpha=0$ (4.13) suggests that A_1 can be treated as a constant and there is no need to renormalize it. Solving (4.14) to order $\lambda(\epsilon)$ and setting $\mu=x$ and $x=\epsilon r$ in (4.12), we obtain

$$u(r)=1-\lambda(\epsilon)A_1e_1(\epsilon r)+\lambda^2(\epsilon), \tag{4.15}$$

where use is already made of the boundary condition $u(r=\infty)=1$. Imposing $u(r=1)=0$ determines $\lambda(\epsilon)A_1=1/e_1(\epsilon)$ from which $\lambda(\epsilon)$ can be chosen as $\lambda(\epsilon)=1/e_1(\epsilon)$, whose asymptotic expansion in the limit $\epsilon\to 0_+$, is $\lambda(\epsilon)\sim 1/\ln(1/\epsilon)+\gamma/\ln^2(1/\epsilon)+\cdots$, giving all necessary orders required in the asymptotic matching. Accordingly $A_1=1$. Thus the uniformly valid asymptotic result can be written in a single expression as

$$u(r)\sim 1-e_1(\epsilon r)/e_1(\epsilon)+O([1/e_1(\epsilon)]^2). \tag{4.16}$$

For $\alpha=1$ solving (4.13) and (4.14) to order $\lambda(\epsilon)$, we get

$$A_1(\mu)=\frac{A_1(\infty)}{1-\lambda(\epsilon)A_1(\infty)e_1(\mu)}+O(\lambda^2(\epsilon)), \tag{4.17}$$

$$A(\mu)=\ln\{1-\lambda(\epsilon)A_1(\infty)e_1(\mu)\}+A(\infty)+O(\lambda^2(\epsilon)), \tag{4.18}$$

where $A_1(\infty), A(\infty)$ are constants of integration to be determined by the boundary conditions. Setting $\mu=x$ and $x=\epsilon r$ in (4.12) we have

$$u(r)=\ln\{1-\lambda(\epsilon)A_1(\infty)e_1(\epsilon r)\}+A(\infty)+O(\lambda^2(\epsilon)). \tag{4.19}$$

Using boundary conditions $u(r=\infty)=1$ and $u(r=1)=0$ produces $A(\infty)=1$ and $\lambda(\epsilon)A_1(\infty)=(1-1/e)/e_1(\epsilon)$. Again we may choose $\lambda(\epsilon)=1/e_1(\epsilon)$, and then $A_1(\infty)=1-1/e$. Finally the uniformly valid asymptotic result is given by

$$u(r)\sim\ln[1+(e-1)e_1(\epsilon r)/e_1(\epsilon)]+O([1/e_1(\epsilon)]^2). \tag{4.20}$$

Comparing the RG results (4.16) and (4.20) and the corresponding asymptotic matching results, again we find the RG results are more accurate.

V. REDUCTIVE PERTURBATION THEORY AND RG

In previous examples, we have already mentioned the idea that amplitude or phase equations are RG equations. We will demonstrate that the RG theory is a general and systematic method to derive slow motion equations, even for those complicated problems for which no explicit analytic zeroth-order solutions are known. In previous reports we already discussed the one-dimensional Swift-Hohenberg equation [18] and the Burgers equation [11] as renormalization group equations. Center manifold theory can be considered from the reductive perturbation point of view, because it also extracts slow motion equations on the manifold. Thus we may expect that the center manifold theory can also be interpreted as an application of the renormalization approach as well.

A. Newell-Whitehead equation

The example we consider here is the two-dimensional Swift-Hohenberg equation widely used as a simple model of Rayleigh-Benard convection [42].

$$\frac{\partial u}{\partial t}=\epsilon u-u^3-\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+k^2\right)^2 u, \tag{5.1}$$

where ϵ is a control parameter or a reduced Rayleigh number, a measure of the degree of convective instability of the stationary state $u=0$. For small positive ϵ, the system exhibits a supercritical bifurcation. Since we wish to treat $\epsilon u-u^3$ as a perturbative term, to be consistent ϵu and u^3 must be of the same order. We scale u as $\sqrt{\epsilon}u$, and denote the new u with the same symbol. Then, the original equation reads

$$\frac{\partial u}{\partial t}=\epsilon(u-u^3)-\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+k^2\right)^2 u. \tag{5.2}$$

We consider this in the whole plane for all positive t. As a zeroth-order solution, we choose the roll solution along the y axis: $Ae^{ikx}+$complex conjugate, where A is a complex numerical constant. We expand u around this solution as $u=Ae^{ikx}+\epsilon u_1+\cdots+$complex conjugate. The first-order correction obeys

$$\frac{\partial u_1}{\partial t}+\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+k^2\right)^2 u_1=(1-3|A|^2)Ae^{ikz}. \tag{5.3}$$

Here, to study only the singular behavior of u_1, e^{3ikx} and similar nonresonant terms are ignored. We rewrite this equation as

$$[L_1+L_2+L_3+L_4]u_1=(1-3|A|^2)Ae^{ikx}, \tag{5.4}$$

where the operators are given defined as

$$L_1\equiv\frac{\partial}{\partial t},\quad L_2\equiv\left(\frac{\partial^2}{\partial x^2}+k^2\right)^2,\quad L_3\equiv 2\left(\frac{\partial^2}{\partial x^2}+k^2\right)\frac{\partial^2}{\partial y^2},$$

$$L_4\equiv\frac{\partial^4}{\partial y^4}. \tag{5.5}$$

We must look for space-time secular terms in the solution. Secular terms which modify the global system behavior appear only in the special solution of the equation consistent with the inhomogeneous term. In order to find (space-time secular) special solutions of (5.4) we have only to solve $L_i u_{Si}=(1-3|A|^2)Ae^{ikx}$ separately, and to make the linear combination of their solutions as $\Sigma\mu_i u_{Si}$ with $\Sigma\mu_i=1$. This is because all four operators L_i commute, and $L_j e^{ikx}=0$, so that $L_i L_j u_{Si}=0$. Thus the space-time secular behavior of $L_j u_{Si}$ is less severe than that of u_{Si}, so that we may ignore this. That is, without affecting the divergence structure of the inhomogeneous solution, we may set $L_j u_{Si}=0$. A trivial calculation gives

$$u_{S1}=tA(1-3|A|^2)e^{ikx}. \tag{5.6}$$

u_{S2} is governed by

$$\left(\frac{\partial^2}{\partial x^2}+k^2\right)^2 u_{S2}=\left(\frac{\partial}{\partial x}+ik\right)^2\left(\frac{\partial}{\partial x}-ik\right)^2 u_{S2}$$
$$=(1-3|A|^2)Ae^{ikz}. \tag{5.7}$$

That is,

$$\left(\frac{\partial}{\partial x}-ik\right)^2 u_{S2}=-\frac{1}{4k^2}A(1-3|A|^2)e^{ikx}. \tag{5.8}$$

Here we do not pay attention to inhomogeneous terms nonresonant with the operator. Hence, the *most* singular part is

$$u_{S2}=-\frac{x^2}{8k^2}A(1-3|A|^2)e^{ikx}. \tag{5.9}$$

Similarly, we get

$$u_{S3}=\frac{xy^2}{8ik}A(1-3|A|^2)e^{ikx}, \tag{5.10}$$

and

$$u_{S4}=\frac{y^4}{4!}A(1-3|A|^2)e^{ikx}. \tag{5.11}$$

In this way we get the following perturbation result,

$$u=Ae^{ikx}+\epsilon\left(\mu_1 t-\mu_2\frac{x^2}{8k^2}+\mu_3\frac{xy^2}{8ik}+\mu_4\frac{y^4}{4!}\right)\times A(1-3|A|^2)e^{ikx}+\text{c.c.}+\cdots . \tag{5.12}$$

Here all the less singular terms (e^{ikx} times $1,x,y,xy,y^2,y^3$), higher-order terms and nonsecular terms (those terms which do not grow indefinitely far away or in the long future) are omitted. These terms will not contribute to the final result, as shown in the argument below. Now, the secular terms are absorbed into the redefinition of the amplitude A as follows. We introduce regularization points X, Y and T and split, for example, x^α as $x^\alpha-X^\alpha+X^\alpha$ (for some exponent α), and absorb X^α into A. Thus we get,

$$u=A(X,Y,T)e^{ikx}+\epsilon\left(\mu_1(t-T)-\mu_2\frac{(x^2-X^2)}{8k^2}+\mu_3\frac{(xy^2-XY^2)}{8ik}+\mu_4\frac{(y^4-Y^4)}{4!}\right)\times A(1-3|A|^2)e^{ikx}+\cdots . \tag{5.13}$$

Since u should not depend on X, Y or T, the renormalization group equation, to $O(\epsilon)$, reads $\partial^{\alpha+\beta+\gamma}u/\partial T^\alpha\partial X^\beta\partial Y^\gamma=0$ for any positive integers α,β,γ with $\alpha\beta\gamma\neq 0$, where values of α,β,γ are chosen in such a way that the universal slow motion equation we are seeking is independent of any system details. Thus we have

$$\frac{\partial A}{\partial T}-\epsilon\mu_1 A(1-3|A|^2)=0,$$

$$\frac{\partial^2 A}{\partial X^2}+\epsilon\mu_2\frac{1}{4k^2}A(1-3|A|^2)=0,$$

$$\frac{\partial^3 A}{\partial X\partial Y^2}-\epsilon\mu_3\frac{1}{4ik}A(1-3|A|^2)=0,$$

$$\frac{\partial^4 A}{\partial Y^4}-\epsilon\mu_4 A(1-3|A|^2)=0. \tag{5.14}$$

Obviously, μ_i are still almost arbitrary and must be fixed by the auxiliary conditions. Therefore, to get an auxiliary condition free equation of motion, we use $\Sigma\mu_i=1$ to arrive at the following RG equation after equating X,Y,T and x,y,t, respectively:

$$\frac{\partial A}{\partial t}+\left(-4k^2\frac{\partial^2}{\partial x^2}+4ik\frac{\partial^3}{\partial x\partial y^2}+\frac{\partial^4}{\partial y^4}\right)A=\epsilon A(1-3|A|^2). \tag{5.15}$$

Thus we have arrived at the Newell-Whitehead equation.

Let us compare this derivation with the conventional method, for which a summary may be found in the Appendix to the review article by Cross and Hohenberg [43]. Perhaps the most notable point is that no scaling of spatial variables like $x\to\epsilon^{1/2}x$, $y\to\epsilon^{1/4}y$ is needed. Furthermore, the expansion is a straightforward one in terms of ϵ instead of $\epsilon^{1/2}$. That is, the result is almost automatically obtained from the global well definedness of the perturbation result.

If there are no spatial degrees of freedom, each step of the standard reductive perturbation [7] using the solvability condition and that in the RG derivation above are in one-to-one correspondence. However, if there are spatial degrees of freedom, the standard reductive perturbation regards the spatial derivatives as a perturbation if the zeroth-order solution is space independent, or uses the multiple-scale analysis if the zeroth-order solution is spatially varying. In contrast, in our RG approach, spatial and time coordinates are treated on an equal footing, and the correct scalings of variables are given automatically.

As the reader may have realized, kinetic equations are expected to be derivable as slow motion equations from the Bogoliubov-Born-Green-Kirkwood-Yvon hierarchy. For example, the Boltzmann equation can be derived by an RG method. Thus we suggest that it is a rule that slow motion equations are RG equations.

B. Center manifold and RG

In this section, we discuss briefly the general relationship between RG theory and center manifold theory [6]. In the general theory of reduction, we wish to know the slow manifold (e.g., inertial manifold, center manifold) which attracts all the long-time asymptotic solutions, and the equation of motion on the manifold. It is well known that the center manifold reduction and normal form theory [6] have played a significant role in studying instabilities and bifurcations encountered in dynamical systems and fluid dynamics. In many circumstances, this approach provides a greatly simplified picture of complicated dynamics by reducing the dimension of the system without losing essential information concerning the instability and bifurcation. In addition, the local dynamics on the center manifold constructed in this way is invariant or universal, in the sense that the structure of the reduced system is independent of specific physical models under consideration. Thus a variety of different phenomena can have the same type of bifurcation, belonging to the same universality class in the parlance of RG. Although the center manifold fits in the RG picture clearly, the general correspondence between them has not yet been established. In certain cases such as the weakly nonlinear stability of fluid motion, the equivalence of the method of center manifold, the method of multiple scales, and the method of amplitude expansion has been established explicitly by applying these methods to the derivation of the Landau equation from the Navier-Stokes equation to the seventh order [44].

To illustrate the relevance of RG, let us consider the following set of equations:

$$\frac{dx}{dt}=f(x,y),$$

$$\frac{dy}{dt}=-y+g(x,y), \tag{5.16}$$

where f and g are higher order in the sense that $f(\lambda x,\lambda y)$ or $g(\lambda x,\lambda y)$ is $O(\lambda^2)$ for small λ. Thus the variable y decays

quickly but x does not. Hence, the long-time behavior of the system is expected to be confined close to a local one manifold near the origin. This local manifold is the center manifold (not unique), and the long-time behavior of the system is governed by the equation of motion defined on this manifold. Thus, as discussed at the beginning of this section, the problem of finding a center manifold and the equation on it is a problem of extracting slow motion behavior of the system. In this sense, this problem and the general reductive perturbation can be treated in a unified fashion. Since we are interested in the local center manifold, we may rescale the variables as $x\to\lambda x$ and $y\to\lambda y$, and may assume that λ is small. Therefore, instead of the original system (5.16), we study

$$\frac{dx}{dt}=\lambda f(x,y),$$

$$\frac{dy}{dt}=-y+\lambda g(x,y). \tag{5.17}$$

We assume the following formal expansions:

$$f(x,y)=f_{20}x^2+f_{11}xy+f_{02}y^2+\lambda(f_{30}x^2+f_{21}x^2y+f_{12}xy^2+f_{03}y^3)+\cdots,$$

$$g(x,y)=g_{20}x^2+g_{11}xy+g_{02}y^2+\lambda(g_{30}x^3+g_{21}x^2y+g_{12}xy^2+g_{03}y^3)+\cdots. \tag{5.18}$$

The standard approach goes as follows: Let $y=h(x)$ be the formula for a center manifold. Then we get the following differential equation for h:

$$-h(x)+\lambda g(x,h(x))=\lambda h'(x)f(x,h(x)). \tag{5.19}$$

This equation is usually solved by perturbation: $h(x)=\lambda h_2x^2+\lambda^2h_3x^3+\cdots$. The result is

$$y=\lambda g_{20}x^2+\lambda^2[g_{20}(g_{11}-2f_{20})+g_{30}]x^3+O(\lambda^3). \tag{5.20}$$

The equation of motion on the center manifold is obtained by substituting y with $h(x)$ in the equation for dx/dt.

Our RG program starts with the construction of a power series expansion of the solution for (5.17) in terms of λ as $x=x_0+\lambda x_1+\lambda^2x_2+\cdots$, and $y=y_0+\lambda y_1+\lambda^2y_2+\cdots$. Paquette has also pursued the same line independently [45]. A lengthy but straightforward calculation gives

$$x=A+\lambda f_{20}A^2t+\lambda^2(f_{20}^2A^3t^2+f_{11}g_{20}A^3t+f_{30}A^3t)+\lambda^3\{t^3f_{20}^3A^4+\tfrac{5}{2}(f_{11}f_{20}g_{20}+f_{30}f_{20})A^4t^2+[-2g_{20}f_{20}f_{11}+f_{11}g_{20}g_{11}+f_{11}g_{30}+f_{20}g_{20}^2+f_{21}g_{20}+f_{40}]A^4t+CT\}+O(\lambda^4),$$

$$y=\lambda g_{20}A^2+\lambda^2[2g_{20}f_{20}A^3(t-1)+g_{20}g_{11}A^3+g_{30}A^3], \tag{5.21}$$

where CT denotes the constant terms and A is the initial condition for x. Here we have discarded all the exponentially decaying terms. For example, to the first order the full solution reads

$$x_1=f_{20}A_0^2t-f_{11}A_0B_0e^{-t}-\tfrac{1}{2}f_{02}B_0^2e^{-2t}+A_1,$$

$$y_1=g_{20}A_0^2+g_{11}A_0B_0te^{-t}-g_{02}B_0^2e^{-2t}+B_1e^{-t}, \tag{5.22}$$

where A_0,B_0,B_1 are numerical constants dependent on the initial data. The exponentially decaying terms do not contribute to the secular behavior of perturbation series. We absorb the secular terms proportional to the powers of T into the redefined A by splitting t as $t^\alpha-T^\alpha+T^\alpha$, where α is an appropriate integer. That this can be achieved consistently must be checked order by order. The simplest way may be to introduce the renormalized counterpart A_R of A as $A=A_R(1+\lambda\omega_1+\lambda^2\omega_2+\lambda^3\omega_3+\cdots)$, where ω_i are determined to remove the powers of T from the perturbation result for x after splitting t. The renormalization condition can be written as

$$A_R(1+\lambda\omega_1+\lambda^2\omega_2+\lambda^3\omega_3+\cdots)+\lambda f_{20}(1+\lambda\omega_1+\lambda^2\omega_2)^2A^2t+\lambda^2(f_{20}^2t^2+f_{11}g_{20}t+f_{30}t)A^3(1+\lambda\omega_1)^3+\lambda^3[\cdots]=A_R. \tag{5.23}$$

From this, order by order in powers of λ, we can fix ω_i as

$$\omega_1=-f_{20}At,$$

$$\omega_2=f_{20}^2A^2t^2-f_{11}g_{20}A^2t-f_{30}A^2t,$$

$$\omega_3=A^3\{-f_{20}^3t^3+\tfrac{5}{2}(f_{11}f_{20}g_{20}+f_{30}f_{20})t^2+[2g_{20}f_{20}f_{11}t-f_{11}g_{20}g_{11}-f_{11}g_{30}-f_{20}g_{20}^2-f_{21}g_{20}-f_{40}]t\}. \tag{5.24}$$

The renormalization group equation reads

$$\frac{dA_R}{dT}=\frac{d}{dT}\{A_R(T)(1+\lambda\omega_1+\lambda^2\omega_2+\lambda^3\omega_3+\cdots)\}=0. \tag{5.25}$$

Introducing the explicit forms of ω_i into this equation, we experience almost miraculous cancellations of all the terms containing powers of t explicitly to have

$$\frac{dA_R}{dt}=\lambda A_R^2f_{20}+\lambda^2A_R^3(f_{11}g_{20}+f_{30})+\lambda^3A_R^4(f_{11}g_{30}+f_{20}g_{20}^2+f_{21}g_{20}+f_{40}-f_{20}f_{11}g_{20})+O(\lambda^4), \tag{5.26}$$

where t is identified with T. This agrees with the conventional result. For y, after renormalization, all the explicitly t dependent terms disappear to order λ^2, and

$$y=\lambda g_{20}A_R^2+\lambda^2(g_{20}g_{11}+g_{30}-2f_{20}g_{20})A_R^3+O(\lambda^3). \tag{5.27}$$

This also agrees with the result given above.

The formal solution (5.21) is order by order in λ obtained from the true solution by discarding the transcendentally small terms in the large t limit. Notice that in x_n the highest power of t is n (for y_n it is less), so that up to a given order n, by choosing λ such that $\lambda t=1$, we can make the contribution of the sum of the transcendental terms (such as $e^{-1/\lambda}$) less than any small positive number for sufficiently large t. In this way, locally up to any finite order in λ, the series

obtained as the singular (or nondecaying) terms describes the asymptotic behavior of the system. Therefore, if the system has a unique solution to the initial value problem (near the origin), then we can uniquely determine these series, and they give a parametric representation of an approximate center manifold. In the present context, renormalizability means that the motion on the approximate center manifold is autonomous. The renormalization reorganizes the expansion so that dx/dt is not explicitly time dependent.

The RG procedure given above is actually much more tedious than the conventional approach. However, the obtained center manifold by RG need not be expandable in terms of x. Thus the RG method works in some cases even when the conventional approach is not applicable [46].

VI. SUMMARY

In this paper, we have demonstrated that various singular perturbation methods and reductive perturbation methods may be understood in a unified fashion from the renormalization group point of view. Amplitude equations and phase equations describing slow motion dynamics in nonequilibrium phenomena are RG equations. The RG method seems to be more efficient and simpler to use than standard methods in the sense that it avoids the necessity to perform asymptotic matching, and generates its own problem-adapted asymptotic sequence. The approximations generated by the RG work well over the entire interval of interest, and better than the conventional approximations in the cases that we have studied. Formally expanding the approximation obtained by the RG yields a conventional perturbation expansion, but one that is of lower order than that obtained by the standard techniques, because the latter uses both inner and outer expansions. However, as is demonstrated by an example in Sec. IV, the RG result, which is apparently lower order than the standard one, may be numerically much superior to the latter. Also, as in this example, if we wish, we can recover the conventional perturbation expansion result by expansion of the appropriate order of the RG expansion.

Probably the most outstanding question is to justify mathematically the general renormalized perturbation approach developed in this paper. The rigorous and constructive renormalization group approaches of Bricmont and Kupiainen and our formal perturbative approaches have almost no common technical ground, although their philosophy is identical. Consequently, we do not have even a hint as to how to rigorize, or estimate the errors of our approach.

The Wilson-style RG [10,47] and Bricmont and Kupiainen's related constructive renormalization group approaches [15] can be implemented numerically. We have examined similarity and traveling wave solutions [48] and have developed an interpolation-resampling scheme which produces a ''virtual continuum'' to allow smooth scaling of any function on a discrete grid [49]. Finally, a completely different approach to the numerical solution of a PDE is to construct a sequence of coarse-grained approximations to the solution as opposed to the conventional method of constructing a sequence of sampled points from the solution. Whereas the sequence of sampled points obey the usual finite-difference equations, and are supposed to converge to the solution in the continuum limit, the sequence of coarse-grained approximations obey a renormalized version of the original PDE, which can in some cases be found explicitly using the RG techniques [50]. We hope to report on these developments in future publications.

Note added in proof. R. Graham has recently shown [R. Graham, Phys. Rev. Lett. **76**, 2185 (1996)] that a rotationally invariant form of the amplitude equation proposed by Gunaratne *et al.* [G. H. Gunaratne *et al.*, Phys. Rev. E **50**, 2802 (1994)] can be derived using the methods of Sec. V, by retaining all singular terms rather than the most singular terms.

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