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Inductive instability in conductors with a moving front of electric conductivity jump

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It is shown that motion of the boundary separating regions with different electric conductivities can cause a decrease of the inductance of the conductor. Since the effective damping resistance of the conductor is a sum of the Ohmic resistance and the time derivative of the inductance, the damping resistance can become negative at a certain velocity of the boundary motion. This work studies the effect of the spontaneous excitation of the electric current in heterogeneous conductors due to the rapid decrease of their inductance. Excitation of the instability in various geometries is analyzed using a quasistationary theory, and the velocity of the boundary motion which is required for the excitation of the instability is determined. In the case of an expanding homogeneous cylindrical conductor, an exact analytical solution of Maxwell equations describing spontaneous excitation of the electric current is derived. The exact expression for the threshold velocity coincides with the predictions of the quasistationary theory. [S1063-651X(96)12909-3]

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I. INTRODUCTION

Various physical processes in naturally occurring phenomena, or encountered in technological applications, are accompanied by motion of the boundary separating regions with different magnitudes of electric conductivity. Such a situation occurs during propagation of ionization and recombination waves [1], melting and evaporation of current-carrying conductors [2,3], melting of metallic resonators in the electromagnetic fields, z-pinch systems, star oscillations in astrophysics, etc.

In all these phenomena the velocity of motion of the boundary separating regions with different electric conductivity can be quite high. Since the effective damping resistance of the conductor is a sum of the Ohmic resistance and the time derivative of the inductance, it becomes negative at a certain velocity of the boundary motion. The goal of this study is to analyze the effect of the spontaneous excitation of the electric current in heterogeneous conductors due to the rapid decrease of their inductance.

The physical mechanism for the generation of the electromagnetic field is quite transparent. If the motion of the boundary is caused by the ponderomotive forces arising due to the electric current, such motion causes the increase of the inductance and of the effective damping resistance. However, when the motion of the boundary is caused by an external source (heating, ionization, phase transitions, etc.), this motion may cause a decrease of the inductance of the conductor. Then, due to Faraday's effect, the magnitude of the electric current can increase with time. Evidently, in the latter case the external source works against the ponderomotive forces which are generated in the conductor.

The paper is organized as follows. In Sec. II, excitation of the instability in various geometries is analyzed using a quasistationary theory when the inductance of the conductor is considered to be a function of the boundary position only. In Sec. III we derive an exact analytical solution of the Maxwell equations describing the spontaneous excitation of the electric current in the expanding cylindrical conductor.

II. CONDITIONS FOR NEGATIVE VALUE OF A DAMPING RESISTANCE IN HETEROGENEOUS CONDUCTORS

In an analysis of the feasibility of the spontaneous excitation of the electric current in heterogeneous conductors with strong spatial dispersion of conductivity, we use Ohm's law generalized for the case of a conductor with a varying inductance (see Ref. [4], Chap. 7, Sec. 61):

$$L\dot{I}+I(R+\dot{L})=U_e\,, \tag{1}$$

where U_e is an external voltage, I is the magnitude of the electric current, and L is the inductance of the conductor. Consider the case of a free conductor, i.e., $U_e=0$. Then the condition for the instability with respect to the spontaneous excitation of the electric current reads

$$\mathrm{sgn}(\dot{I})=\mathrm{sgn}(I)\quad\text{or}\quad R+\dot{L}<0. \tag{2}$$

Obviously, condition (2) is satisfied only at high rates of the inductance change $|\dot{L}|$. In heterogeneous conductors, the large magnitude of $|\dot{L}|$ can be achieved by the fast motion of the boundary between regions with different electric conductivities.

We now determine the velocity of motion of the boundary which allows us to satisfy condition (2). In this section, in the analysis of the excitation of the instability, we use the adiabatic approximation, i.e., we neglect the skin effect.

Consider first the case of a thin layered slab $-b_1<z<b_1$ consisting of two parallel layers with electric conductivities of the internal and external layers σ_1 and σ_2, respectively (see Fig. 1). Assume that during motion of the boundary the conductor preserves its initial symmetry with respect to the $z=0$ plane. Then the time variation of the conductivity dur-

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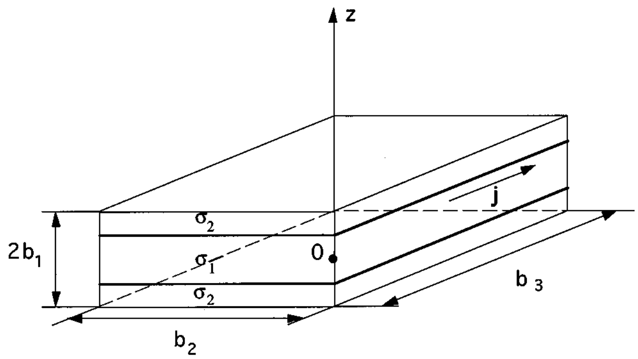

FIG. 1. A thin layered slab consisting of two parallel layers with electric conductivities σ_1 and σ_2.

ing the motion of the boundary between two regions with different conductivities can be described by the following formula:

$$\sigma=\sigma_1\theta[\overline{s(t)}-|z|]+\sigma_2\theta[|z|-\overline{s(t)}],\quad \sigma=0\quad \text{at } |z|>b_1,$$

where

$$\theta(x)=\begin{cases}1, & x\geqslant 0\\ 0, & x<0,\end{cases}$$

and $\overline{s(t)}$ is the distance between the right or left interface and a plane $z=0$.

In the adiabatic approximation the inductance of the conductor is a function of the location of the boundary only, and it can be represented as a sum $L=L_e+L_i$, where L_e is an inductance associated with a magnetic field outside the conductor, and L_i is associated with a magnetic field inside the conductor. During the motion of the internal boundary which separates regions with different electric conductivities, the magnitude of the external magnetic flux at a given magnitude of the total electric current I does not change, provided that the symmetry of the problem is preserved. Therefore, in order to analyze the effects of the inductance change, we may consider only the internal inductance L_i which is determined by the following equation:

$$\frac{L_iI^2}{2}=\int\frac{\vec{H}^2}{8\pi}d\vec{r} \tag{3}$$

where integration is performed over the volume of the slab:

$$I=b_2\int_{-b_1}^{b_1}\vec{j}(z)dz, \tag{4}$$

and b_2 is the width of the slab in the direction normal to that of the electric current.

In the adiabatic approximation the magnetic field $\vec{H}$ and the density of an electric current $\vec{j}$ are determined by the following equations:

$$\text{rot }\vec{H}=\frac{4\pi}{c}\vec{j},\quad \text{rot }\frac{\vec{j}}{\sigma}=0,\quad \text{div }\vec{j}=0 \tag{5}$$

In calculations of the distributions of the magnetic field and electric current, we used the approximation of a thin infinite slab. Then using Eqs. (3)–(5) after some algebra, we find that

$$L_i=L_0[1+f(s)],\quad L_0=\frac{2\pi}{3c^2}\frac{b_1b_3}{b_2},$$
$$f(s)=(1-\kappa)\frac{s(1-s)[\kappa+(2-\kappa)s]}{[s+\kappa(1-s)]^2}, \tag{6}$$

where b_3 is a length of the slab in the direction of the electric current, $\kappa=(\sigma_2/\sigma_1)$, and $s=(\overline{s/b_1})$.

Equation (6) is the first nonvanishing term in the expansion of the internal inductance of the slab with respect to small parameters $(b_1/b_3)\ll 1$ and $(b_1/b_2)\ll 1$. Equation (6) shows that the internal inductance L_i does not vary monotonically, and that it passes through the extremum. When the boundaries are located at $s=0$ and $s=1$, Eq. (6) recovers the value of the inductance for the homogeneous slab. According to Eqs. (2) and (6) the condition for the excitation of the instability can be written as follows:

$$\frac{2\pi}{3c^2}\frac{\partial f}{\partial s}\dot{s}<-\frac{1}{2b_1\overline{\sigma}},\quad \overline{\sigma}=\sigma_1[s+\kappa(1-s)]. \tag{7}$$

Therefore the magnitude of the velocity which is required for the excitation of the instability depends upon the location of the boundary between regions with different electric conductivities, and upon the electrical conductivities of both regions. The values of this velocity in several limiting cases are presented below:

$$s\ll\kappa\ll 1,\quad f(s)=\frac{s}{\kappa},\quad \frac{\dot{s}}{c}<-\frac{3c}{4\pi b_1\sigma_1}, \tag{8}$$

$$s^2\ll\kappa\ll s\ll 1,\quad f(s)=2-\frac{3\kappa}{s},\quad \frac{\dot{s}}{c}<-\frac{cs}{4\pi b_1\sigma_1\kappa}, \tag{9}$$

$$\kappa\ll s^2,\quad s\ll 1,\quad f(s)=2(1-s),\quad \frac{\dot{s}}{c}>\frac{3c}{8\pi b_1\sigma_1 s}, \tag{10}$$

$$\kappa(1-s)\ll 1,\quad 1-s\ll 1,\quad f(s)=2(1-s)(1-\kappa),$$
$$\frac{\dot{s}}{c}>\frac{3c}{8\pi b_1\sigma_1(1-\kappa)}. \tag{11}$$

The inductance of the conductor attains maximum ($\kappa<1$) or minimum ($\kappa>1$) values at some position of the boundary between regions with different electric conductivities s^*. Therefore a negative damping resistance can occur during propagation of the boundary toward the surface of the conductor, or during its propagation in the opposite direction. Assume that at the position of the boundary s^*, the inductance attains the maximum magnitude. Then for $s(t)<s^*$ the negative damping resistance can occur when the boundary moves inwards. In the opposite case, i.e., when $s(t)>s^*(t)$, the negative damping resistance can occur when the boundary moves outwards.

Thus the general condition for the occurrence of the negative damping resistance, at some stage of the stratification of the conductor, is the motion of the boundary separating regions with different electric conductivities causing the reduction of the inductance. The estimate of the absolute value of the threshold velocity when the damping resistance becomes negative reads

$$\frac{|\dot{s}|}{c} \geqslant \frac{3c}{4\pi b_1 \max(\sigma_1, \sigma_2)}. \tag{12}$$

In the stratified conductor which comprises a layer with a high electric conductivity $\sigma \sim 10^{15}$–10^{17} s^{-1} and $b \sim 10^n$ cm, $|\dot{s}| \geqslant 2 \times (10^3 - 10^5) 10^{-n}$ cm/s.

The above model was presented in order to demonstrate the possibility of the occurrence of negative damping resistance in conductors with moving boundaries separating regions with different electric conductivities. Nevertheless it must be noted that such a situation can occur during shock compression of metal conductors.

In derivations of the asymptotic relations (8)–(11), it was assumed that the electric conductivity of the internal layer is larger than the electric conductivity of the external layer, i.e., $\sigma_1 > \sigma_2$. However Eqs. (6) and (7) allow us to analyze the opposite situation which is not discussed here.

Consider now the applicability of the equations of magnetostatics (5), which yield Eq. (6) and the asymptotic Eqs. (8)–(11). Equations of magnetostatics (5) and a quasistationary circuit equation (1) can be applied when the characteristic diffusion time of a magnetic field τ_m into a slab with a thickness b_1,

$$\tau_m = \frac{4\pi\sigma b_1^2}{c^2}, \tag{13}$$

satisfies the condition $\tau_m \ll \tau$, where τ is the characteristic time of the excitation of the instability. Since during excitation of the instability $|\dot{L}| \geqslant R$, Eq. (1) allows us to estimate the time of the excitation of the instability,

$$\tau \sim \frac{L}{\dot{L}} \sim \frac{b_1}{\dot{s}}.$$

According to Eqs. (12) and (13), $\tau \geqslant \tau_m$, and, therefore, equations of magnetostatics (5) and the quasistationary circuit equation (1) are not applicable. Therefore it is of interest to analyze the problem using Maxwell equations with appropriate boundary conditions, and to compare the solution with that obtained using the approximate theory. However, the solution of Maxwell equations with a moving internal boundary is quite involved. Nevertheless, in some special cases, e.g., $\kappa = 0$, an exact analytical solution of the Maxwell equations can be determined. The condition $\kappa = 0$ corresponds to the case of the expanding homogeneous conductor surrounded by an ideal dielectric. This exact analytical solution can be derived in several geometries, e.g., for an infinite thin slab and for a homogeneous long cylinder. In this study we consider only the expanding homogeneous cylindrical conductor, and apply the results to an analysis of the electric explosion of conductors.

Consider first the conditions for the negative damping resistance of a stratified cylindrical conductor consisting of two concentric cylindrical layers, an internal layer $0 < \rho < \rho_F$ with electric conductivity σ_1, and an external layer $\rho_F < \rho < \rho_0$ with electric conductivity σ_2 (see Fig. 2). In the case of a moving boundary $\rho_F(t)$, condition (2) yields

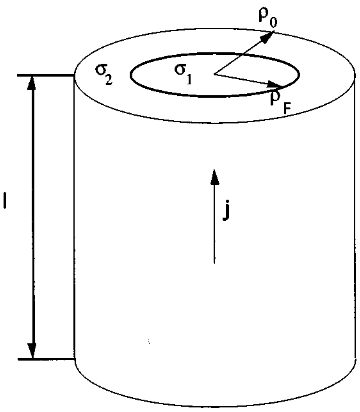

FIG. 2. Cylindrical conductor consisting of two concentric cylindrical layers with electric conductivities σ_1 and σ_2.

$$\frac{2}{c^2} \frac{\partial f}{\partial y} \dot{\rho}_F > \frac{1}{2\pi \overline{\sigma} \rho_F(t)}.$$

The equation for $f(y)$ was derived using a magnetostatic approximation in our previous study [2],

$$f(y) = \xi^2(y) y [y \ln(y) + \overline{a}(1-y)],$$

where $\xi = 1/(\overline{a} - y)$, $\overline{\sigma} = \sigma_1[y + \kappa(1-y)]$, $\overline{a} = \kappa/(\kappa - 1)$, and $y = (\rho_F^2/\rho_0^2)$.

In the range $\kappa \ll y \ll 1$, $f(y) = \ln(y)$, and the condition for the excitation of the instability reads

$$\dot{\rho}_F > \frac{c^2}{2\pi \rho_F(t) \sigma_1}. \tag{14}$$

In the range $y \ll \kappa \ll 1$, $f(y) = -(y/\kappa)$, $\overline{\sigma} = \sigma_1 \kappa$, and the condition for the excitation of the instability reads

$$\dot{\rho}_F < -\frac{c^2}{2\pi \rho_F(t) \sigma_1}. \tag{15}$$

Thus the problem is quite similar to that of the thin slab which was analyzed above. Since in this case the characteristic time of the excitation of the instability $\tau \sim (\rho_F/\dot{\rho}_F) < \tau_m \cong (4\pi\sigma_1\rho_0^2/c^2)$, and the magnetostatistics equations are not applicable as in the case of a slab.

In order to solve the problem using Maxwell equations, it must be simplified. Therefore in Sec. III we consider the case with $\sigma_2 = 0$. In this case the moving boundary coincides with the external boundary.

III. EXCITATION OF ELECTRIC CURRENT IN A RAPIDLY EXPANDING CYLINDRICAL CONDUCTOR

Consider a cylindrical conductor with the initial radius $\rho_F(0) \equiv \rho_0$ and length l. Neglect the edge effects, i.e., assume the axial symmetry of the problem. Then Maxwell equations determining the dynamics of electric $\vec{E}$ and magnetic $\vec{H}$ fields read

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho H_\varphi)=\frac{4\pi}{c}\sigma\left(E_z+\frac{v(\rho,t)H_\varphi}{c}\right), \qquad (16)$$

$$\frac{\partial E_z}{\partial\rho}=\frac{1}{c}\frac{\partial H_\varphi}{\partial t}, \qquad (17)$$

where H_φ and E_z are the axial and longitudinal components of the magnetic and electric fields, respectively, and $v(\rho,t)$ is a radial component of the velocity. Since in the problem considered all other components of the electric and magnetic fields vanish, hereafter we omit indices near these fields in the equations.

Assume also that the material density distribution is homogeneous over the cross section of the conductor. Then its density $\gamma(t)$ varies with time as $\gamma(t)=\gamma_0[\rho_0^2/\rho_F^2(t)]$, and continuity equation $\dot{\gamma}+\mathrm{div}(\gamma\vec{v})=0$ yields

$$v(\rho,t)=\frac{\dot{\rho}_F\rho}{\rho_F}.$$

Equations (16) and (17) must be supplied with the following boundary conditions:

$$\left.\frac{\partial E}{\partial\rho}\right|_{\rho=0}=H|_{\rho=0}=0, \qquad (18)$$

$$E(\rho_F(t))+\frac{2\dot{I}}{c^2}\ln\left(\frac{l}{\rho_F(t)}\right)=0. \qquad (19)$$

Boundary condition (18) follows from the symmetry of the problem, while boundary condition (19) is derived using Eq. (17) and the condition of the continuity of the electric field at the conductors's surface (see, e.g., Ref. [5]). The total electric current I is determined by the following formula:

$$I(t)=2\pi\sigma\int_0^{\rho_F(t)}\left(E(\rho',t)+\frac{\dot{\rho}_F}{\rho_F}\frac{\rho H}{c}\right)\rho' d\rho'. \qquad (20)$$

Hereafter in Eq. (19) we neglect the inductance change in comparison with the initial inductance. Then Eq. (19) can be rewritten as

$$E(\rho_F(t))=-\frac{2\dot{I}}{c^2}p, \quad p=\ln\left(\frac{l}{\rho_0}\right). \qquad (21)$$

We seek the solution of the boundary value problem (16)–(20) in the following form:

$$E(\rho,t)=E(\rho_F(t))\Phi(x), \quad x=\frac{\rho}{\rho_F(t)} \qquad (22)$$

and

$$H(\rho,t)=H(\rho_F(t))f(x), \qquad (23)$$

where the magnetic field at the surface of the conductor $H(\rho_F(t))=[2I(t)/c\rho_F(t)]$. Functions $\Phi(x)$ and $f(x)$ satisfy the following conditions:

$$\Phi(1)=f(1)=1, \quad f(0)=\left.\frac{\partial\Phi}{\partial x}\right|_{x=0}=0. \qquad (24)$$

Self-similarity conditions (22) and (23), which express the absence of the explicit dependence on time, and Eq. (17) with boundary condition (21), yield

$$\frac{\rho_F(t)\dot{I}(t)}{c\dot{\rho}_F(t)I(t)}=\mathrm{const}=\lambda, \qquad (25)$$

The condition $\lambda=$const is necessary for the existence of the solution as given by Eqs. (22) and (23). This condition yields the electric current behavior in a self-similar case:

$$I(t)=I(0)\left(\frac{\rho_F(t)}{\rho_0}\right)^\lambda. \qquad (26)$$

Thus conditions (22) and (23) yield a power law dependence of the electric current upon the instantaneous radius of the conductor. Eliminating the electric field in Eqs. (16) and (17), we arrive at the following equation:

$$\frac{\partial^2 f}{\partial x^2}+\frac{1}{x}\frac{\partial f}{\partial x}-\frac{1}{x^2}f=\lambda\nu f, \quad \nu=\frac{4\pi}{c^2}\dot{\rho}_F\rho_F\sigma, \qquad (27)$$

which implies the second condition for the self-similarity:

$$\dot{\rho}_F\rho_F=\mathrm{const}.$$

Provided that function $f(x)$ is known, function $\Phi(x)$ can be determined from Eq. (17), which can be rewritten using (27) as

$$p\Phi'(x)=\frac{1}{\lambda}(xf)'-f. \qquad (28)$$

Solutions of Eqs. (27) and (28) satisfying conditions (24) can be represented as follows:

$$f(x)=\frac{I_1(\beta x)}{I_1(\beta)},$$

$$\Phi(x)=1-\frac{1}{\lambda p}+\frac{I_0(\beta)-I_0(\beta x)}{p\beta I_1(\beta)}+\frac{1}{\lambda}\frac{xI_1(\beta x)}{pI_1(\beta)}, \qquad (29)$$

where $\beta=(\nu\lambda)^{1/2}$, and $I_0(\beta x)$ and $I_1(\beta x)$ are modified Bessel functions:

$$I_k(\beta x)=\sum_{n=0}^{\infty}\left(\frac{\beta x}{2}\right)^{2n+k}\frac{1}{n!(n+k)!} \qquad (30)$$

Substituting Eq. (29) into Eq. (20) yields the dispersion equation for determining parameter $\lambda(\nu)$:

$$\nu(1-p\lambda)I_1(\beta)-\beta I_0(\beta)=0. \qquad (31)$$

In the vicinity of the threshold of the instability excitation, $\lambda\ll 1$, and, consequently, $\beta\ll 1$. Then Eqs. (30) and (31) yield

$$\lambda=\frac{\nu-2}{\nu p}.$$

Thus the condition for the excitation of instability $\nu>2$ coincides with condition (14) derived in the adiabatic approximation. Therefore the adiabatic approximation can be used at least for the qualitative analysis. When $\nu<2$, the electric current decays, and distributions of the electric and magnetic fields correspond to the skin effect [4].

Poynting vector $\vec{S}$ in this problem is determined by the following formula [6]:

$$\vec{S}=-\frac{c}{4\pi}EH\vec{n},$$

where $\vec{n}$ is the external unit vector normal to the conductor's surface. When $\nu>2$, the Pointing vector is directed outside the conductor, and the electromagnetic field outside the conductor increases. When $\nu<2$ the Poynting vector is directed outside the conductor, and the electromagnetic field outside the conductor decays.

The main goal of this study is to analyze the general effects occurring during the fast variation of the spatial dispersion of the electric conductivity in the conductors. However, it is of interest to determine the range of parameters when the instability is excited in the case when an expansion of the conductor is caused by fast heating. If β is a volumetric expansion coefficient, the velocity of expansion caused for homogeneous heating $\dot{\rho}_F$ is given by the following formula:

$$\dot{\rho}_F=\frac{\beta\rho_F^3\dot{T}}{\rho_0^2}\sim\beta\rho_F\dot{T}.$$

Thus according to Eq. (7) the characteristic heating rate

$$\dot{T}>\frac{c^2}{2\pi\rho_F^2\sigma\beta}.$$

For a metal conductor with radius $\rho_F\sim10^{-1}$ cm, $\beta\sim10^{-4}$ K^{-1}, and $\dot{T}>(10^9-10^{11})$ K s^{-1}. Such heating rates can occur in conductors with very high density of electric current, e.g., during an electric explosion of conductors [7].

Relatively slow (power law) excitation of the inductive instability does not constitute the major difficulty for experimental observation of the effect, since the real excitation rate may be different from that determined in this study because of deviations from the self-similar regime. Ponderomotive forces cause such changes in the geometry that result in an increase of inductance with time, but not its decrease. In some circumstances the effect of the ponderomotive forces can prevail. Thus, e.g., corrugation of the moving boundary separating regions with different electric conductivity can damp the inductive instability.

It must be noted that the magnetohydrodynamic instabilities occurring in the expanding conductors were extensively studied in the past (see, e.g., [8]). However in the traditional approach to an analysis of magnetohydrodynamic instabilities, the electric current is considered to be a given (external) variable. To the best of our knowledge, the inductance instability analyzed in this study was not investigated before.

III. CONCLUSIONS

It is shown that during fast motion of boundary separating regions with different electric conductivities in the heterogeneous conductor, the conductor becomes unstable with respect to the spontaneous excitation of the electric current. The cause for this effect is the decrease of the inductance during the motion of the boundary. Since the effective damping resistance of the conductor is a sum of the Ohmic resistance and the time derivative of the inductance, it becomes negative at a certain velocity of the boundary motion. The threshold velocity for the excitation of the instability was determined using both the quasistationary theory and the exact analytical solution of the Maxwell equations. Although the effect was considered only for several model examples, there are reasons to believe that it has a general character.

It is also of interest to investigate the excitation of the inductive instability in systems encountered in geophysics and astrophysics, where, due to the large sizes, the instability can occur at relatively low velocities of the interface motion. It is also conceivable that the inductive instability can cause polarization and charge separation in finite size systems.

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