Generation of non-Gaussian stationary stochastic processes

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A procedure is developed to generate a non-Gaussian stationary stochastic process with the knowledge of its first-order probability density and the spectral density. The procedure is applicable to an arbitrary probability density if the spectral density is of a low-pass type, and to a large class of probability densities if the spectral density is of a narrow band, with its peak located at a nonzero frequency. [S1063-651X(96)07307-2]

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I. INTRODUCTION

When investigating the response of a randomly excited dynamical system, either analytically or by Monte Carlo simulation, the modeling of the excitation process must agree with experimental evidence. In many cases, it is reasonable to assume that the excitation is a stationary stochastic process, but clearly non-Gaussian, and the experimental knowledge is usually limited to the estimates of the first-order probability density and the spectral density. Early publications (e.g., [1,2]) on simulating stationary excitation processes made use of the Fourier series expansion, an idea traceable to that of Rice [3]. However, this procedure is only suitable for matching a target spectral density, not a target non-Gaussian distribution, since such a series is asymptotically Gaussian. The matching of both a non-Gaussian first-order probability density and a spectral density is considerably more difficult. Two approaches have been proposed to meet the non-Gaussian requirement. One by Yamazaki and Shinozuka [4] is to incorporate a numerical iterative procedure into the usual Fourier-series representation. Another is to apply a zero-memory nonlinear transformation to the output of a linear filter excited by a Gaussian white noise [5–8]. The first procedure is purely numerical, whereas the success of the second procedure is dependent very much on the ability to devise a particular nonlinear transformation to suit a particular case.

In a recent paper, Kontorovich and Lyandres [9] have proposed a scheme in which the simulated process is obtained as the output of a nonlinear system under Gaussian white-noise input. In this case, the simulated process is a diffusive Markov process or a component of a Markov vector, and its probability density is governed by a Fokker-Planck equation. In the Kontorovich-Lyandres procedure, the coefficients in the equation, known as the drift and diffusion coefficients, are chosen so that the solution form agrees with that of the target probability density, and the parameters in the solution form are then determined in order to approximate the target spectral density. This procedure is attractive since the simulated process is embodied in one or a set of governing equations; thus it is more amenable to analytical treatments, in addition to being a useful tool for Monte Carlo studies.

In the present paper, an alternative procedure is developed, also on the basis of the Markov theory, in which matching of the spectral density is accomplished by adjusting the drift coefficient alone, which is then followed by adjusting the diffusion coefficient to match the probability density. The alternative scheme is more efficient and easier to apply, as demonstrated in examples.

II. LOW-PASS SPECTRAL DENSITY

Consider a stationary stochastic process $X(t)$ defined on the interval $[x_l,x_r]$, which can be either bounded or unbounded. Without loss of generality, we assume that $X(t)$ has a zero mean. Then $x_l<0$ and $x_r>0$. With the knowledge of the probability density $p(x)$ and the spectral density $\Phi_{XX}(\omega)$ of $X(t)$, we wish to establish a procedure to model the process $X(t)$.

Let the spectral density be of the following low-pass type:

$$\Phi_{XX}(\omega)=\frac{\alpha\sigma^2}{\pi(\omega^2+\alpha^2)},\quad \alpha>0, \tag{1}$$

where σ^2 is the mean-square value of $X(t)$. If $X(t)$ is also a diffusive Markov process, then it is governed by the following stochastic differential equation in the Itô sense [10]:

$$dX=-\alpha X\ dt+D(X)dB(t), \tag{2}$$

where α is the same parameter in (1), $B(t)$ is a unit Wiener process, and the coefficients $-\alpha X$ and $D(X)$ are known as the drift and the diffusion coefficients, respectively. To demonstrate that this is the case, multiply (2) by $X(t-\tau)$ and take the ensemble average to yield

$$\frac{dR(\tau)}{d\tau}=-\alpha R(\tau), \tag{3}$$

where $R(\tau)$ is the correlation function of $X(t)$, namely, $R(\tau)=E[X(t-\tau)X(t)]$. Equation (3) has a solution

$$R(\tau)=A\ \exp(-\alpha|\tau|) \tag{4}$$

in which A is arbitrary. By choosing $A=\sigma^2$, expressions (1) and (4) become a Fourier transform pair. Thus Eq. (2) generates a process $X(t)$ with a spectral density (1). Note that the diffusion coefficient $D(X)$ has no influence on the spectral density.

Now we shall determine $D(X)$ so that $X(t)$ possesses a given stationary probability density $p(x)$. The Fokker-Planck equation, governing the probability density $p(x)$ of $X(t)$ in the stationary state, is obtained from (2) as follows:

$$\frac{d}{dx}G = -\frac{d}{dx}\left\{\alpha x p(x) + \frac{1}{2}\frac{d}{dx}[D^2(x)p(x)]\right\} = 0, \quad (5)$$

where G is known as the probability flow. Since $X(t)$ is defined on $[x_l, x_r]$, G must vanish at the two boundaries $x=x_l$ and $x=x_r$. In the present one-dimensional case, G must vanish everywhere; consequently, Eq. (5) reduces to

$$\alpha x p(x) + \frac{1}{2}\frac{d}{dx}[D^2(x)p(x)] = 0. \quad (6)$$

Integration of (6) results in

$$D^2(x)p(x) = -2\alpha\int_{x_l}^{x} up(u)du + C, \quad (7)$$

where C is an integration constant. To determine the integration constant C, two cases are considered. If $x_l=-\infty$, or $x_r=\infty$, or both, then $p(x)$ must vanish at the infinite boundary; thus $C=0$ from (7). If both x_l and x_r are finite, then the drift coefficient $-\alpha x_l$ at the left boundary is positive, and the drift coefficient $-\alpha x_r$ at the right boundary is negative, indicating that the average probability flows at the two boundaries are directed inward. However, the existence of a stationary probability density implies that all sample functions must remain within $[x_l, x_r]$, which requires additionally that the drift coefficient vanish at the two boundaries, namely, $D^2(x_l)=D^2(x_r)=0$. This is satisfied only if $C=0$. In either case,

$$D^2(x) = -\frac{2\alpha}{p(x)}\int_{x_l}^{x} up(u)du. \quad (8)$$

Function $D^2(x)$, computed from Eq. (8), is non-negative, as it should be, since $p(x)\geqslant 0$ and the mean value of $X(t)$ is zero. Thus we have proved that the stochastic process $X(t)$ generated from (2) with $D(X)$ given by (8) possesses a given stationary probability density $p(x)$ and the spectral density (1).

The Itô type stochastic differential equation (2) may be converted to that of the Stratonovich type as follows:

$$\dot{X} = -\alpha X - \frac{1}{4}\frac{dD^2(X)}{dX} + \frac{D(X)}{\sqrt{2\pi}}\,\xi(t), \quad (9)$$

where $\xi(t)$ is a Gaussian white noise with a unit spectral density. Equation (9) is better suited for simulating sample functions. Some illustrative examples are given below.

Example 1. Assume that $X(t)$ is uniformly distributed, namely,

$$p(x) = \frac{1}{2\Delta}, \quad -\Delta \leqslant x \leqslant \Delta. \quad (10)$$

Substituting (10) into (8),

$$D(x) = \alpha(\Delta^2 - x^2). \quad (11)$$

In this case, the desired Itô equation is given by

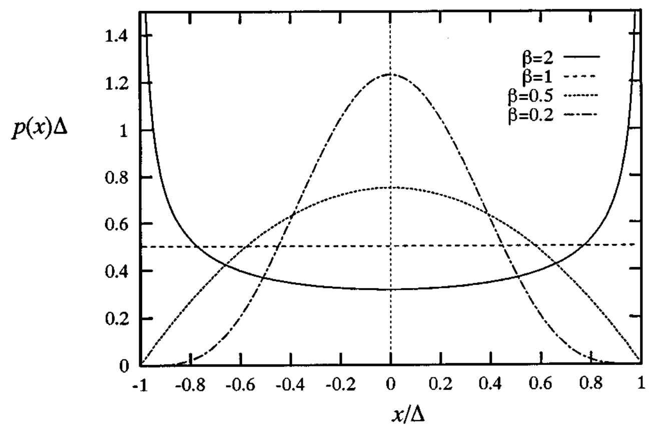

FIG. 1. Stationary probability density of $X(t)$ generated from (13).

$$dX = -\alpha X dt + \sqrt{\alpha(\Delta^2 - X^2)}dB(t). \quad (12)$$

It is of interest to note that a family of stochastic processes [This family of stochastic processes was discovered in a discussion with W. Wedig.] may be obtained from the following generalized version of (12):

$$dX = -\alpha X dt + \sqrt{\alpha\beta(\Delta^2 - X^2)}dB(t). \quad (13)$$

The stationary probability densities of $X(t)$ generated from (13) are shown in Fig. 1 for several β values. Their appearances are strikingly diverse, yet they share the same spectral density (1).

Example 2. Let $X(t)$ be governed by a Rayleigh distribution

$$p(x) = \gamma^2 x\,\exp(-\gamma x), \quad \gamma > 0, \quad 0 \leqslant x < \infty. \quad (14)$$

Its centralized version $Y(t)=X(t)-2/\gamma$ has a probability density

$$p(y) = \gamma(\gamma y + 2)\exp(-\gamma y + 2), \quad -\frac{2}{\gamma} \leqslant y < \infty. \quad (15)$$

From Eq. (8),

$$D^2(y) = \frac{2\alpha}{\gamma}\left(y + \frac{2}{\gamma}\right). \quad (16)$$

The Itô equation for $Y(t)$ is

$$dY = -\alpha Y dt + \left[\frac{2\alpha}{\gamma}\left(Y + \frac{2}{\gamma}\right)\right]^{1/2} dB(t) \quad (17)$$

and the corresponding equation for $X(t)$ in the Stratonovich form is

$$\dot{X} = -\alpha X + \frac{3\alpha}{2\gamma} + \left(\frac{\alpha}{\pi\gamma}X\right)^{1/2}\xi(t). \quad (18)$$

Note that the spectral density of $X(t)$ contains a delta function $(4/\gamma^2)\delta(\omega)$ due to the nonzero mean $2/\gamma$.

Example 3. Consider a family of probability densities, which obeys an equation of the form

$$\frac{d}{dx}\,p(x)=J(x)p(x). \tag{19}$$

Equation (19) can be integrated to yield

$$p(x)=C_1\exp\left(\int J(x)dx\right), \tag{20}$$

where C_1 is a normalization constant. In this case,

$$D^2(x)=-2\alpha\,\exp[-J(x)]\int x\,\exp[J(x)]dx. \tag{21}$$

Several special cases may be noted. Let

$$J(x)=-\gamma x^2-\delta x^4,\quad -\infty<x<\infty, \tag{22}$$

where γ may be arbitrary if $\delta>0$. Substitution of (22) into (8) leads to

$$D^2(x)=\frac{\alpha}{2}\sqrt{\pi/\delta}\,\exp\left[\delta\left(x^2+\frac{\gamma}{2\delta}\right)^2\right]\mathrm{erfc}\left[\sqrt{\delta}\left(x^2+\frac{\gamma}{2\delta}\right)\right], \tag{23}$$

where erfc(y) is the complementary error function defined as

$$\mathrm{erfc}(y)=\frac{2}{\sqrt{\pi}}\int_y^\infty e^{-t^2}dt. \tag{24}$$

The case of $\gamma<0$ and $\delta>0$ corresponds to a bimodal distribution, and the case of $\gamma>0$ and $\delta=0$ corresponds to a Gaussian distribution.

The Pearson family of probability distributions (e.g., [11]) corresponds to

$$J(x)=\frac{a_1x+a_0}{b_2x^2+b_1x+b_0}. \tag{25}$$

In the special case of $a_0+b_1=0$,

$$D^2(x)=-\frac{2\alpha}{a_1+2b_2}\,(b_2x^2+b_1x+b_0). \tag{26}$$

III. NARROW-BAND SPECTRAL DENSITY

To generate a narrow-band stochastic process with the spectrum peak located at a nonzero frequency, a two-dimensional system is required. A large class of two-dimensional system is described by the following pair of Itô equations:

$$\begin{aligned} dX_1&=(a_{11}X_1+a_{12}X_2)dt+D_1(X_1,X_2)dB_1(t),\\ dX_2&=(a_{21}X_1+a_{22}X_2)dt+D_2(X_1,X_2)dB_2(t), \end{aligned} \tag{27}$$

where $B_1(t)$ and $B_2(t)$ are two independent unit Wiener processes. Note that the drift coefficients are assumed to be linear in X_1 and X_2, with constants a_{ij} to be determined along with the functional forms for the diffusion coefficients $D_1(X_1,X_2)$ and $D_2(X_1,X_2)$. For the system to be stable and possess a stationary probability density, it is required that $a_{11}<0$, $a_{22}<0$, and $a_{11}a_{22}-a_{12}a_{21}>0$. Multiplying (27) by $X_1(t-\tau)$ and taking the ensemble average, we have

$$\begin{aligned} \frac{d}{d\tau}R_{11}(\tau)&=a_{11}R_{11}(\tau)+a_{12}R_{12}(\tau),\\ \frac{d}{d\tau}R_{12}(\tau)&=a_{12}R_{11}(\tau)+a_{22}R_{12}(\tau), \end{aligned} \tag{28}$$

where

$$\begin{aligned} R_{11}(\tau)&=E[X_1(t-\tau)X_1(t)],\\ R_{12}(\tau)&=E[X_1(t-\tau)X_2(t)]. \end{aligned} \tag{29}$$

Equation set (28) is solved subject to the initial conditions

$$R_{11}(0)=m_{11}=E[X_1^2],\quad R_{12}(0)=m_{12}=E[X_1X_2]. \tag{30}$$

In order to obtain directly the spectral density $\Phi_{11}(\omega)$ of $X_1(t)$, define the following integral transformation:

$$\overline{R_{ij}}(\omega)=\mathscr{F}[R_{ij}(\tau)]=\frac{1}{\pi}\int_0^\infty R_{ij}(\tau)e^{-i\omega\tau}d\tau. \tag{31}$$

Then $\Phi_{11}(\omega)$ can be obtained as

$$\Phi_{11}(\omega)=\frac{1}{2\pi}\int_{-\infty}^\infty R_{11}(\tau)e^{-i\omega\tau}d\tau=\mathrm{Re}[\overline{R_{11}}(\omega)], \tag{32}$$

where Re denotes the real part. Since $R_{ij}(\tau)\to 0$ as $\tau\to\infty$, it can be shown that

$$\mathscr{F}\left(\frac{dR_{ij}(\tau)}{d\tau}\right)=i\omega\overline{R_{ij}}(\omega)-\frac{1}{\pi}\,R_{ij}(0). \tag{33}$$

Differential equations (28) in the time domain can be transformed into algebraic equations in the frequency domain as follows, using (31) and (33):

$$\begin{aligned} i\omega\overline{R_{11}}-\frac{m_{11}}{\pi}&=a_{11}\overline{R_{11}}+a_{12}\overline{R_{12}},\\ i\omega\overline{R_{12}}-\frac{m_{12}}{\pi}&=a_{21}\overline{R_{11}}+a_{22}\overline{R_{12}}. \end{aligned} \tag{34}$$

Solving for $\overline{R_{11}}(\omega)$ and taking its real part, we obtain

$$\Phi_{11}(\omega)=\frac{-(a_{11}m_{11}+a_{12}m_{12})\omega^2+A_2(a_{12}m_{12}-a_{22}m_{11})}{\pi[\omega^4+(A_1^2-2A_2)\omega^2+A_2^2]}, \tag{35}$$

where $A_1=a_{11}+a_{22}$, and $A_2=a_{11}a_{22}-a_{12}a_{21}$. Expression (35) is quite general for a narrow-band spectral density. The constants a_{11}, a_{12}, a_{21}, and a_{22} can be adjusted to obtain a best fit for a target spectrum.

The Fokker-Planck equation for the joint probability density of $X_1(t)$ and $X_2(t)$ in the stationary state is given by

$$\frac{\partial}{\partial x_1}\left((a_{11}x_1+a_{12}x_2)p-\frac{1}{2}\frac{\partial}{\partial x_1}[D_1^2(x_1,x_2)p]\right)+\frac{\partial}{\partial x_2}\left((a_{21}x_1+a_{22}x_2)p-\frac{1}{2}\frac{\partial}{\partial x_2}[D_2^2(x_1,x_2)p]\right)=0. \tag{36}$$

Our objective is to determine the non-negative functions $D_1^2(x_1,x_2)$ and $D_2^2(x_1,x_2)$ for a given $p(x_1,x_2)$. If such $D_1^2(x_1,x_2)$ and $D_2^2(x_1,x_2)$ functions can be found, then the equations for simulation in the Stratonovich form are given by

$$\dot{X}_1=a_{11}X_1+a_{12}X_2-\frac{1}{4}\frac{\partial}{\partial x_1}D_1^2(X_1,X_2)+\frac{D_1(X_1,X_2)}{\sqrt{2\pi}}\xi_1(t),$$
$$\dot{X}_2=a_{21}X_1+a_{22}X_2-\frac{1}{4}\frac{\partial}{\partial x_2}D_2^2(X_1,X_2)+\frac{D_2(X_1,X_2)}{\sqrt{2\pi}}\xi_2(t), \tag{37}$$

where $\xi_1(t)$ and $\xi_2(t)$ are two independent unit Gaussian white noises.

Example 4. Consider two independent uniformly distributed stochastic process $X_1(t)$ and $X_2(t)$, namely,

$$p(x_1,x_2)=\frac{1}{4\Delta_1\Delta_2},$$
$$-\Delta_1\leqslant x_1\leqslant\Delta_1,\quad -\Delta_2\leqslant x_2\leqslant\Delta_2. \tag{38}$$

Substituting (38) into (36), we obtain

$$a_{11}-\frac{1}{2}\frac{\partial^2}{\partial x_1^2}D_1^2+a_{22}-\frac{1}{2}\frac{\partial^2}{\partial x_2^2}D_2^2=0, \tag{39}$$

which is satisfied if

$$D_1^2=-a_{11}(\Delta_1^2-x_1^2),\quad D_2^2=-a_{22}(\Delta_2^2-x_2^2). \tag{40}$$

The two equations in (37) are now

$$\dot{X}_1=\frac{1}{2}a_{11}X_1+a_{12}X_2+\left(-\frac{a_{11}}{2\pi}(\Delta_1^2-X_1^2)\right)^{1/2}\xi_1(t),$$
$$\dot{X}_2=a_{21}X_1+\frac{1}{2}a_{22}X_2+\left(-\frac{a_{22}}{2\pi}(\Delta_2^2-X_2^2)\right)^{1/2}\xi_2(t), \tag{41}$$

which generate a uniformly distributed stochastic process $X_1(t)$ with a spectral density given by (35).

Example 5. Consider a joint stationary probability density of $X_1(t)$ and $X_2(t)$ in the form

$$p(x_1,x_2)=\rho(\lambda),\quad \lambda=\frac{1}{2}x_1^2-\frac{a_{12}}{2a_{21}}x_2^2,$$
$$-\infty<x_1,x_2<\infty, \tag{42}$$

where the ratio a_{12}/a_{21} is assumed to be negative, and $\rho(\lambda)$ is an arbitrary function such that $p(x_1,x_2)$ is both non-negative and normalizable. Substitution of (42) into (36) leads to

$$\frac{\partial}{\partial x_1}\left(a_{11}x_1p-\frac{1}{2}\frac{\partial}{\partial x_1}[D_1^2(x_1,x_2)p]\right)+\frac{\partial}{\partial x_2}\left(a_{22}x_2p-\frac{1}{2}\frac{\partial}{\partial x_2}[D_2^2(x_1,x_2)p]\right)=0. \tag{43}$$

Equation (43) is satisfied by two non-negative functions,

$$D_1^2(x_1,x_2)=\frac{2a_{11}}{p(x_1,x_2)}\int_{x_{1l}}^{x_1}up(u,x_2)du,$$
$$D_2^2(x_1,x_2)=\frac{2a_{22}}{p(x_1,x_2)}\int_{x_{2l}}^{x_2}\nu p(x_1,\nu)d\nu. \tag{44}$$

One useful form for $\rho(\lambda)$ is

$$p(x_1,x_2)=\rho(\lambda)=C_1\exp(-b_1\lambda-b_2\lambda^2),\quad b_2>0. \tag{45}$$

The marginal probability density for $X_1(t)$ is then

$$p(x_1)=C_1f(x_1)\exp(-\tfrac{1}{2}b_1x_1^2-\tfrac{1}{4}b_2x_1^4), \tag{46}$$

where

$$f(x_1)=\int_{-\infty}^{\infty}\exp\left(\frac{a_{12}}{2a_{21}}(b_1+b_2x_1^2)u^2-\frac{b_2a_{12}^2}{4a_{21}^2}u^4\right)du. \tag{47}$$

The equation set (37) with D_1^2 and D_2^2 given by (44) may be used to generate a stochastic process $X_1(t)$ which has a probability density of the form (46) and a spectral density of the form (35). Constants a_{ij} in (35) can be adjusted to match a target spectral density, while constants b_i in (46) can be adjusted to match a target probability density.

Another useful form for $\rho(\lambda)$ is given by

$$p(x_1,x_2)=\rho(\lambda)=C_1(\lambda+b)^{-\delta},\quad b>0,\quad \delta>1. \tag{48}$$

In this case,

$$D_1(x_1,x_2)=-\frac{2a_{11}}{\delta-1}(\lambda+b),$$
$$D_2(x_1,x_2)=\frac{2a_{22}a_{12}}{a_{12}(\delta-1)}(\lambda+b), \tag{49}$$

and

$$p(x_1)=C_1\int_{-\infty}^{\infty}\left(\frac{1}{2}x_1^2-\frac{a_{12}}{2a_{21}}u^2+b\right)^{-\delta}du. \tag{50}$$

A large class of probability densities may be fitted in the form of (46) or (50). Once the parameters b_1 and b_2 in (46) or b and δ in (50) are determined, D_1 and D_2 can be calculated from (44).

IV. CONCLUSION

A systematic procedure is developed to model a non-Gaussian stochastic process as a diffusive Markov process, or a component of a diffusive Markov vector, with the knowledge of its probability density and spectral density. The key step is to obtain the drift and diffusion coefficients in the associated Fokker-Planck equation; the former are determined from the given spectral density and the latter from the probability density. It is shown that, if the given spectral density is of a low-pass type, then a one-dimensional Markov model is adequate regardless of the type of the probability distribution. In the case of a narrow-band spectral density with its peak located at a no-zero frequency, a Markov vector model is required, and the target probability density may be fitted within a large class of non-Gaussian distributions. Since the stochastic process model so obtained is described by stochastic differential equations, it can be used in analytical investigations or as a basis for Monte Carlo simulation.

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