Optical solitons with power-law asymptotics

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It is shown that self-guided optical beams with power-law asymptotics, i.e., *algebraic optical solitons*, can be regarded as a special case of sech-type solitons (i.e. solitons with exponentially decaying asymptotics) in the limit where the beam propagation constant coincides with the threshold for linear wave propagation. This leads to the conjecture that algebraic optical solitons should be *inherently unstable* due to interactions with linear waves, even in cases when the corresponding family of sech-type solitons is stable. This conjecture is verified numerically for a wide class of optical solitons described by the generalized nonlinear Schrödinger equation with two competing nonlinearities. [S1063-651X(96)11908-5]

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I. INTRODUCTION

There is growing interest in the subject of self-guided nonlinear waves (spatial solitons) in uniform nonlinear media [1–9]. Since the appearance of the classical paper by Chiao *et al.* [1], attention has concentrated mainly on the (self-focusing or self-defocusing) Kerr medium. However, practical materials often display physical effects, such as saturation, which can only be described by more generalized models of the nonlinear refractive index. For such non-Kerr materials theoretical predictions of new nonlinear effects, including multistability [2] and nonlinear switching and steering [8], are very important. In particular, it has been shown recently [9] that, for certain forms of the generalized nonlinear Schrödinger (NLS) equation with two nonlinear terms of opposite sign, e.g., cubic-quintic nonlinearity, there exist *weakly-localized* solitary waves, i.e., solitary waves with power-law asymptotics, the so-called *algebraic solitons*.

We note that the existence of different types of solitary waves with power-law asymptotics has been already established from other models of nonlinear physics, e.g., [10–18]. In their application to nonlinear optics, such weakly-localized solitary waves are often treated as a class of separate solutions contrasting with the sech-type solitons, i.e., solitons with exponentially decaying tails (see, e.g., [9] and discussions therein).

The main objective of our paper is threefold. First, we show that the existence of algebraic optical solitons is a generic property of the generalized NLS models with two nonlinear terms of opposite sign but arbitrary power and we find these solutions in an explicit form. Second, we demonstrate that the algebraic solitons appear as the special limit of more general, sech-type solitons when the propagation constant coincides exactly with the threshold for periodic wave propagation. Third, we point out that the origin of this kind of algebraic solitons automatically implies that they should be *inherently unstable,* but, as we show here, the character of this instability depends on the power of the nonlinearity. As we demonstrate numerically, if the amplitude of the algebraic soliton is decreased initially by a small amount, then the perturbation grows, and finally the algebraic soliton decays into diffractive linear waves. Otherwise, if the amplitude of the algebraic soliton is initially increased, the algebraic soliton evolves into a sech-type soliton. The instability becomes exponentially growing and manifests itself even more strongly when the algebraic solitons belong to an unstable branch of the sech-type solitons.

The paper is organized as follows. Section II presents our model, which leads to the generalized NLS equation with two power-law nonlinearities. The different types of bright soliton solutions to this equation, including the special case of nonlinear periodic waves, are analyzed in Sec. III. Then, in Sec. IV, we discuss algebraic solitons and their properties. In particular, we demonstrate that algebraic solitons can be regarded as a special limit of sech-type solitons, a property which was *not* noted in Ref. [9]. This limit corresponds exactly to a threshold between solitary waves and continuous waves. Such an observation allows us to understand the character of the instability of algebraic solitons and also to predict their behavior in collisions. Some general discussions about the link between algebraic solitons and guided modes of graded-index planar waveguides are presented in Sec. V, and in Sec. VI we make some concluding remarks.

II. MODEL

We consider the propagation of a monochromatic scalar electric field E in a medium with an intensity-dependent refractive index, $n=n_0+n_{\rm nl}(|E|^\sigma)$, where n_0 is the uniform refractive index of the unperturbed medium, and $n_{\rm nl}(|E|^\sigma)$ describes the variation in the index due to the field intensity $|E|$, where σ is a positive constant. For small field intensities, we expand $n_{\rm nl}$ as a power series in $|E|^\sigma$, and retain the first two terms,

$$n_{\rm nl}\approx a|E|^\sigma+b|E|^{2\sigma}, \tag{1}$$

where a and b are constants. In the case of nonlinear saturation, we must have $ab<0$. Then, within the weak-guidance approximation, solutions of the governing Maxwell's equation can be presented in the form

$$E(X,Z;t)=\mathcal{E}(X,Z)e^{i\beta_0 Z-i\omega t}+\text{c.c.}, \tag{2}$$

where c.c. denotes complex conjugate, ω is the source frequency, and $\beta_0=2\pi n_0/\lambda$ is the plane-wave propagation constant for the uniform background medium, in terms of the

TABLE I. Classification of the soliton solutions (6) and results of the corresponding stability analysis.

	$\alpha<0$	$\alpha>0$
$\gamma>0$	$B=B_-$ $\sigma\leqslant 1$, stable $1<\sigma<2$, stable for $\Omega>\Omega_{\rm th}^{(2)}$ $\sigma>2$, unstable	$B=B_+$ $\sigma\leqslant 2$, stable $2<\sigma<4$, stable for $\Omega<\Omega_{\rm th}^{(1)}$ $\sigma>4$, unstable
$\gamma<0$	Bright solitons do not exist	$B=B_+$ Stable solitons exist for $\Omega<\Omega_{\rm cr}\equiv\alpha^2(1+\sigma)/\lvert\gamma\rvert(2+\sigma)^2$

source wavelength $\lambda=2\pi c/\omega$, c being the free-space speed of light. We have assumed a two-dimensional model, so that the Z-axis is parallel to the direction of propagation, and the X-axis is in the transverse direction.

The function $\mathcal{E}(X,Z)$ describes the wave envelope; in the absence of nonlinear and diffraction effects $\mathcal{E}$ would be a constant. If we substitute Eq. (2) into the two-dimensional, scalar wave equation

$$\left(\frac{\partial^2}{\partial X^2}+\frac{\partial^2}{\partial Z^2}+k^2n^2\right)E=0, \tag{3}$$

then set $\beta_0=kn_0$, and retain only the first-order longitudinal derivative of the envelope function, since it is assumed to be slowly varying in Z, we obtain the normalized (dimensionless) generalized NLS equation,

$$i\frac{\partial u}{\partial z}+\frac{\partial^2 u}{\partial x^2}+\alpha|u|^{\sigma}u+\gamma|u|^{2\sigma}u=0, \tag{4}$$

where $u(x,z)$ denotes the normalized field envelope corresponding to $\mathcal{E}(X,Z)$, z and x are normalized coordinates, and α and γ are constants proportional to a and b, respectively. This equation covers all cases considered previously, including, in particular, $\sigma=1$ (quadratic-cubic nonlinearity) [9], $\sigma=2$ (cubic-quintic nonlinearity) [4,6,9,14], and general integer values of σ [19].

III. SOLITON SOLUTIONS

Spatially localized solutions (bright solitons) of Eq. (4), which describe self-guiding, exponentially-decaying modes in a medium with the generalized nonlinearity of Eq. (1), can be derived in an explicit analytic form. We look for the solution in the form $u(x,z)=v(x)\exp(i\Omega z)$, Ω being constant, and for the real function $v(x)$ we obtain from Eq. (4) an ordinary differential equation which has the first integral (for convenience, we assume $v>0$),

$$\left(\frac{dv}{dx}\right)^2=\left[\Omega v^2-\frac{2\alpha}{(\sigma+2)}v^{\sigma+2}-\frac{\gamma}{(\sigma+1)}v^{2(\sigma+1)}+C\right], \tag{5}$$

where C is a constant. It is implicit that the bright soliton solutions must satisfy the conditions $v=0$ and $dv/dx=0$ for $|x|\to\infty$, in which case $C=0$. Consequently, Eq. (5) can be further integrated by reduction to a standard integral with the change of variable $\xi=v^{-\sigma}$. As a result, we obtain the soliton solution $u_s(x,z;\Omega)$ in the following form:

$$u_s(x,z;\Omega)=\left[\frac{A}{\cosh(Dx)+B}\right]^{1/\sigma}e^{i\Omega z}, \tag{6}$$

where A, D, and B are real parameters defined by the expressions

$$A=\frac{(2+\sigma)B\Omega}{\alpha},\quad D=\sigma\sqrt{\Omega}, \tag{7}$$

$$B\equiv B_{\pm}=\pm\left[1+\frac{(2+\sigma)^2}{(1+\sigma)}\frac{\gamma}{\alpha^2}\Omega\right]^{-1/2}. \tag{8}$$

The arbitrary constant Ω characterizes the nonlinearity-induced shift of the propagation constant from its unperturbed value kn_0 corresponding to the homogeneous linear medium.

Solution (6) describes self-guided nonlinear waves in the medium with the generalized nonlinear refractive index (1) which depends on the values of Ω, α, and γ. We note that if α and γ are both nonzero, Eq. (4) can be normalized in such a way that only the signs of the coefficients α and γ are important. Thus, in analyzing the qualitative behavior of the different types of these solitons, it is sufficient to investigate the cases $\alpha=\pm 1$ and $\gamma=\pm 1$. The analysis of these four cases is summarized in Table I. It follows from Table I that the solution (6) for bright solitons exists provided at least one of the parameters α and γ is positive, i.e., at least one of the nonlinearities is focusing. If the leading nonlinearity (i.e., nonlinearity with higher power) is positive, the bright soliton solution exists for $\Omega>0$. In each case, the soliton solution (6) exists only for one of two possible values of B, i.e., either for $B=B_+$ or for $B=B_-$.

An important property of solitary waves is their stability to perturbations, as only stable self-guided waves can be realized experimentally and used in practical applications. To analyze the linear stability of the soliton solution (6), we employ the general stability criterion for spatially localized solutions of the generalized NLS equations first formulated by Vakhitov and Kolokolov [20] and verified in numerous cases. According to this criterion, a localized mode is unstable provided $dP/d\Omega<0$, where P is the total beam power,

$$P(\Omega)=\int_{-\infty}^{+\infty}|u_s(x,z;\Omega)|^2dx, \tag{9}$$

and Ω is the nonlinear-induced shift of the propagation constant in Eq. (6). Figures 1(a)–1(c) plot the dependence of P on Ω for the three cases (a) $\alpha,\ \gamma>0$, (b) $\alpha>0$ and $\gamma<0$, and (c) $\alpha<0$ and $\gamma>0$, respectively. Besides the results which have been discussed in the literature (for example, the existence of the blow-up instability for $\sigma>2$ and $\gamma>0$), we report some additional features in the soliton stability analysis.

(i) Normally, as $\Omega\rightarrow+0$, the soliton power $P\rightarrow 0$, but for $\alpha=-1$ and $\gamma=+1$, the power remains finite as $\Omega\rightarrow+0$ [see Fig. 1(c)]. As we show below, this explains the origin of algebraic solitons.

(ii) For $\gamma>0$, the soliton stability in the region $2<\sigma<4$ (for $\alpha>0$) and in the region $1<\sigma<2$ (for $\alpha<0$) depends on the parameter Ω. The stability criterion is determined by the slope of the function $P(\Omega)$. Thus the solitons are unstable for $\Omega>\Omega_{\rm th}^{(1)}$ [$\alpha>0$, see Fig. 1(a)] or for $0<\Omega<\Omega_{\rm th}^{(2)}$ [$\alpha<0$, see Fig. 1(c)], but are stable otherwise. The insets in Figs. 1(a) and 1(c) show the dependence of the threshold values $\Omega_{\rm th}^{(1)}$ and $\Omega_{\rm th}^{(2)}$ on the nonlinearity power σ. These results arise from the competition between the two power-law nonlinearities in Eq. (4), which is especially important for the case $\alpha=-1$ and $\gamma=+1$, i.e., ''defocusing +focusing'' nonlinearity. All results of the stability analysis for the soliton solution (6) are also summarized in Table I.

It is known that a necessary condition for solitons to exist is the absence of resonances between a soliton and linear waves (see, e.g., Ref. [21]). For the case considered here, this means that solitons can only exist for $\Omega>0$, which ensures that D in Eq. (7) is real, whereas linear (nonlocalized) waves with fields $\propto\exp(i\Omega z-iqx)$ satisfying the dispersion relation $\Omega=-q^2$ exist only for $\Omega<0$. Accordingly, if a soliton emits radiation due to an initial perturbation (transition radiation), interaction between the soliton and radiation is *nonresonant*. The other necessary condition for soliton robustness in conservative systems is the existence of *a family* of localized solutions, which, in our case, is characterized by the free parameter Ω. Indeed, as we show below, after emitting radiation, an asymptotic soliton has a renormalized amplitude and, generally speaking, this situation resembles the case of the integrable NLS equation (see, e.g., Ref. [22]).

Below we are primarily interested in the case $\alpha<0$, $\gamma>0$ where algebraic solitons occur. To display the solutions described by Eqs. (6)–(8) for the case $\alpha=-1$, $\gamma=+1$, we plot the maximum amplitude $|u_s(0)|=[A/(1+B)]^{1/\sigma}$ as a function of the propagation constant Ω in Fig. 2. Spatially localized solutions, i.e., solitary waves with exponentially decaying tails, or sech-type solitons, exist for any $\Omega>0$, whereas the region $\Omega<0$ corresponds to periodic waves with the asymptotic behavior $\sim\exp(i\Omega z-iqx)$ and satisfying the dispersion relation $\Omega=-q^2$.

An interesting feature of the solution of Eq. (6), for $\alpha<0$, $\gamma>0$, is that it can also describe a special class of *nonlinear periodic waves*. If the solution (6) is continued analytically into the region $\Omega<0$, where D is pure imaginary, then $\cosh(i|D|x)\rightarrow\cos(|D|x)$, and the solution (6) de-

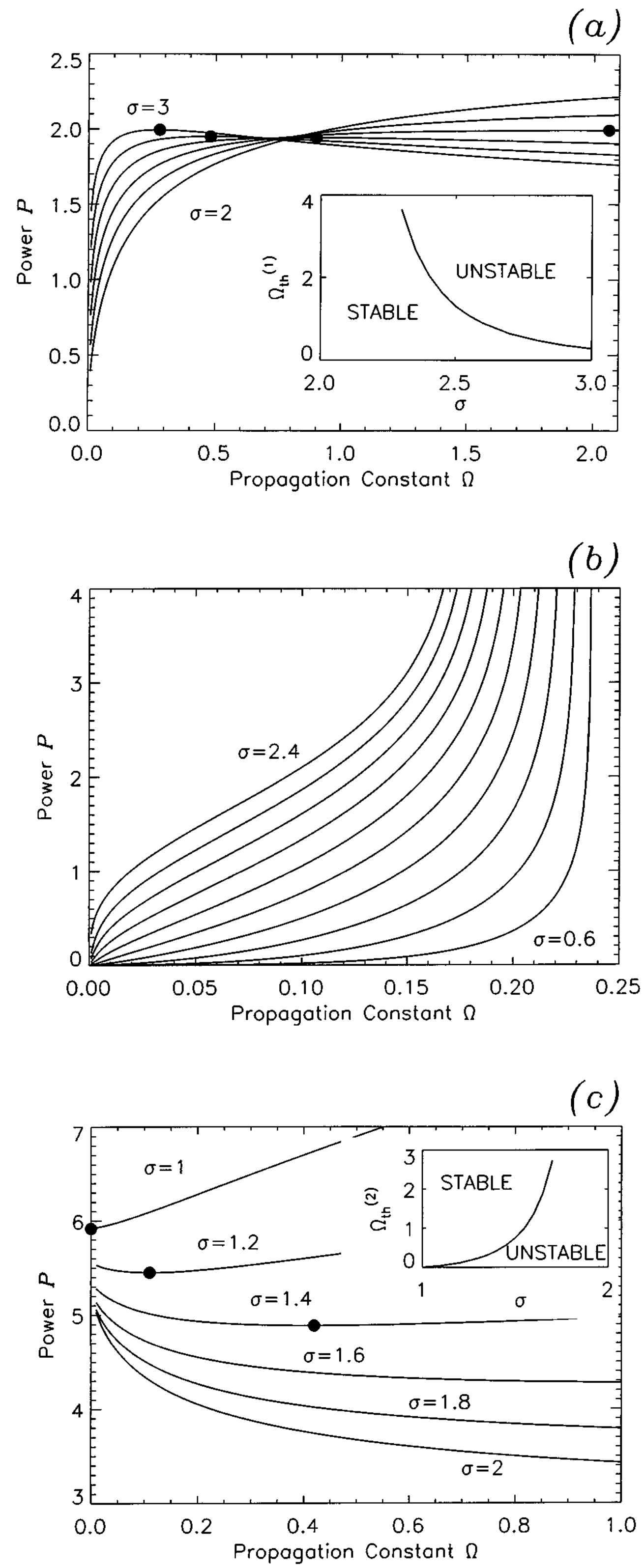

FIG. 1. Dependence of the soliton power P on the propagation constant Ω for different cases described by the solution (6): (a) $\alpha=+1$ and $\gamma=+1$, (b) $\alpha=+1$ and $\gamma=-1$, and (c) $\alpha=-1$ and $\gamma=+1$. Numbers next to the curves indicate the corresponding value of σ. The insets in (a) and (c) show the corresponding dependence of the threshold values $\Omega_{\rm th}^{(1)}$ and $\Omega_{\rm th}^{(2)}$ of the propagation constant Ω which separate stable and unstable solitons. Solid circles denote points where $dP/d\Omega=0$.

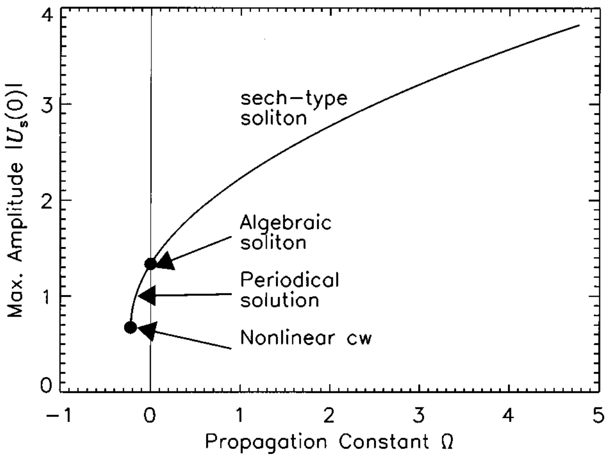

FIG. 2. Maximum amplitude of the soliton solution (6) for $\sigma=1$ ($\alpha=-1$, $\gamma=+1$) as a function of the nonlinearity-induced shift of the propagation constant Ω. The case $\Omega=0$ corresponds to algebraic solitons (10).

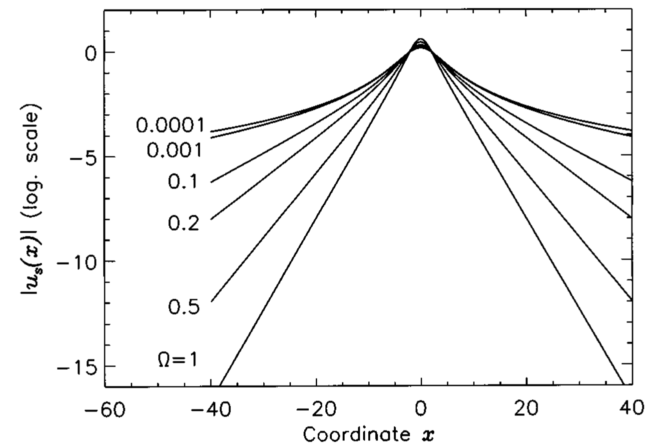

FIG. 4. Transformation of the sech-type soliton (6) into the algebraic soliton (10) for $\Omega\rightarrow+0$ and $\sigma=0.5$. Numbers next to the curves indicate the corresponding values of Ω, and the ordinate has a logarithmic scale.

scribes a periodic wave. This periodic wave exists for $-|\Omega_\star|<\Omega<0$, where $|\Omega_\star|=[\alpha^2(1+\sigma)/\gamma(2+\sigma)^2]$. In Fig. 2 we show the dependence of the maximum wave amplitude for the periodic solution in the region $\Omega<0$, while Fig. 3 plots the profiles of the the periodic solutions at the maximum as Ω varies. For $\Omega\rightarrow-0$, the period $\tau=2\pi/|D|$ becomes unbounded, and the solution resembles a periodic train of well-separated solitons. As $|\Omega|$ increases, the period τ decreases, and the difference between the minimum and maximum amplitudes vanishes, as is evident from Fig. 3.

IV. ALGEBRAIC SOLITONS AND THEIR PROPERTIES

A. Algebraic solitons

In the limit case $\Omega\rightarrow+0$, for $\alpha<0$, it may be verified that the soliton solution (6) reduces to

$$u_a(x)\equiv\lim_{\Omega\rightarrow+0} u_s(x,z;\Omega)$$
$$=\left[\frac{2(2+\sigma)(1+\sigma)/|\alpha|}{\sigma^2(1+\sigma)x^2+(2+\sigma)^2(\gamma/\alpha^2)}\right]^{1/\sigma}, \qquad (10)$$

corresponding to *an algebraic soliton*, which varies as $u_a(x)\sim|x|^{-2/\sigma}$ for $|x|\rightarrow\infty$. This solution is a generalization of two particular solutions found in Ref. [9] for $\sigma=1$ and $\sigma=2$. Algebraic solitons are also associated with other equations as a special limit of more general soliton families (see, e.g., Ref. [13]).

To show how the sech-type soliton (6) transforms into the algebraic soliton as $\Omega\rightarrow+0$, we plot in Fig. 4 the profile of the soliton of Eq. (6) for different values of Ω. Note that for the ''defocusing+focusing'' nonlinearity (i.e., $\alpha<0$, $\gamma>0$) considered here, the center of the algebraic soliton lies in the region of the focusing nonlinearity, while the soliton tails correspond to the region of the defocusing nonlinearity, as is clear from Fig. 5. This may be a necessary condition for the existence of such weakly-localized self-guided waves.

Hayata and Koshiba [9] claim that the solitary waves described by Eq. (10) for $\sigma=1$ and $\sigma=2$ are stable because ''. . . they remain unchanged even after propagation over sufficiently long times that attain ten soliton units.'' This is not surprising because these solutions are exact, and conclusions about stability should be made only on analyzing the effect of small perturbations. As algebraic solitons correspond to

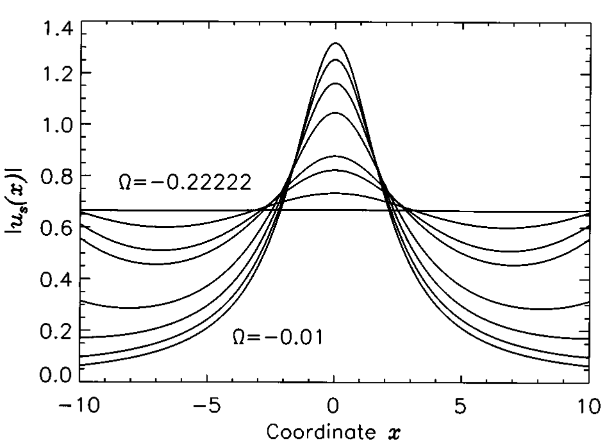

FIG. 3. Characteristic profiles of the periodic solutions for $\Omega<0$ near the maximum amplitude.

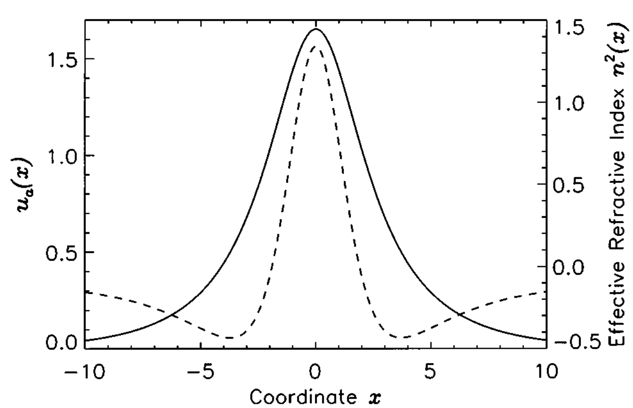

FIG. 5. The profile of the algebraic soliton (10) for $\sigma=0.5$ (solid curve) and the corresponding profile of the effective refractive index $n^2(x)$ defined by Eq. (14) (dashed curve).

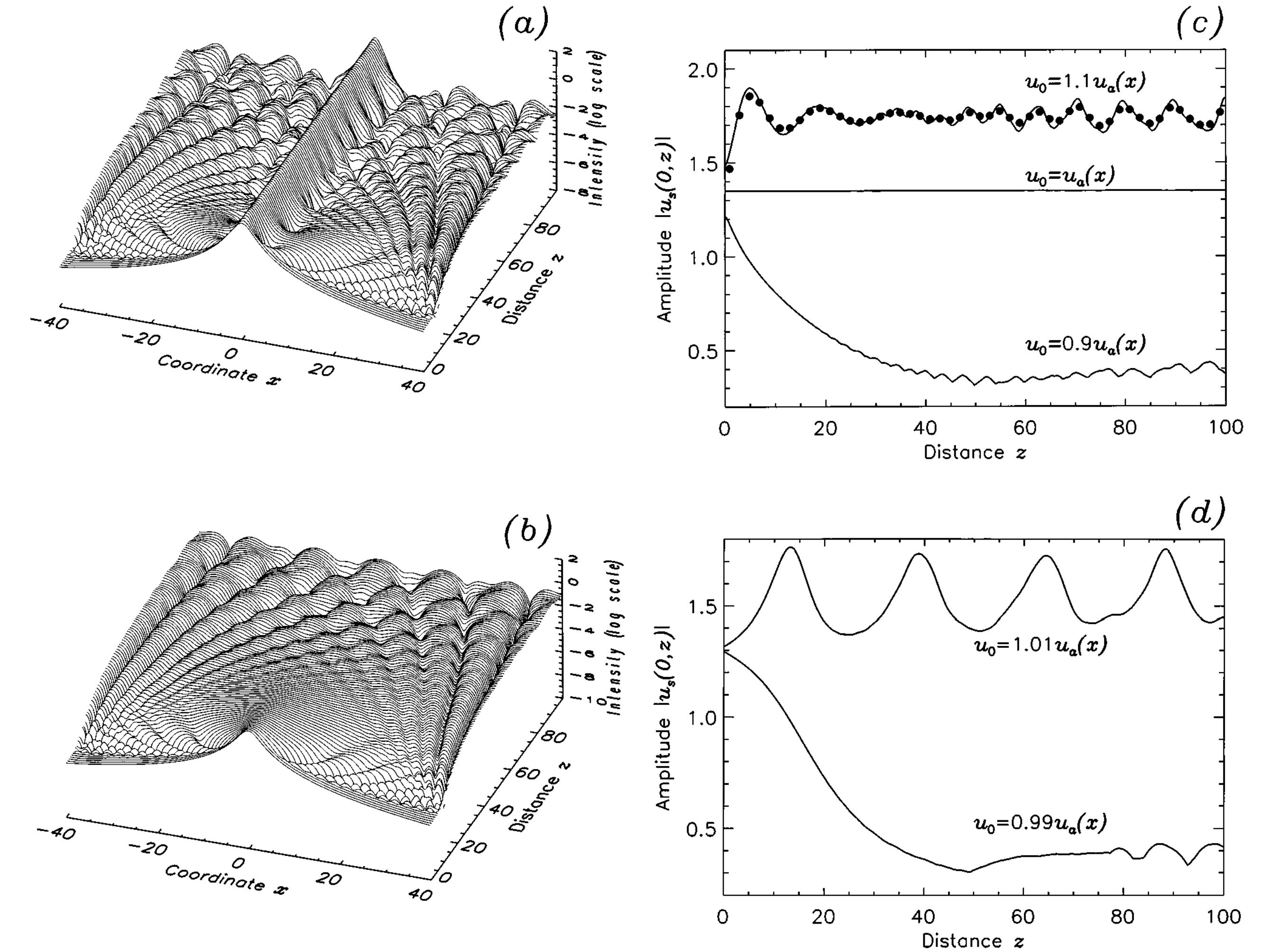

FIG. 6. (a),(b) Evolution of the perturbed algebraic soliton (10) for $\sigma=0.9$ for two types of the perturbed input profiles (in logarithmic scale): $u(x,0)=(1.1)u_a(x)$, $u(x,0)=(0.9)u_a(x)$, where $u_a(x)$ is given by Eq. (10). (c) Corresponding dynamics of the maximum beam intensity for the two scenarios. Note that in case (a) the beam profile oscillates around an effective ''sech-type'' soliton. (d) Same as case (c) but for $\sigma=1.2$. Switching to a (stable) sech-type soliton of larger amplitude is observed.

the threshold for linear wave propagation, as follows from the amplitude dependence of Fig. 2, they would be expected to be unstable due to interactions with infinitely-periodic linear waves. Further, as shown above, the algebraic solitons do not have an internal parameter, i.e., they do not form a family of localized solutions characterized by an arbitrary parameter, which is another indirect indication that they are likely to be unstable. This conclusion has been confirmed by the numerical simulations in Figs. 6(a)–6(d), where we observe that the algebraic solitons are unstable in all three main regions of the nonlinearity power σ. However the character of this instability is different for different values of σ.

B. Numerical perturbation analysis

To investigate the stability of the algebraic solitons, we solve Eq. (4) numerically, taking the initial condition at $z=0$ as a superposition of the exact solution, Eq. (10), and a *small perturbation* defined by

$$u_0(x,0)=(1+\delta)u_a(x), \tag{11}$$

where the perturbation ($|\delta|\ll 1$) is chosen in the form of a scaled soliton.

We found a *qualitative difference* between the dynamics of the initially perturbed algebraic soliton for $\sigma<1$, where the sech-type solitons are *always stable*, for $\sigma>2$, where all sech-type solitons are *always unstable*, and for the intermediate region, $1\leqslant\sigma\leqslant 2$, where the algebraic soliton is a limit case of an unstable branch of the sech-type solitons. In the first case, when $\sigma<1$, a perturbation with $\delta>0$ in Eq. (11) leads to a slow transformation of the algebraic soliton into a sech-type soliton, and is accompanied by the corresponding shift of the propagation constant to the domain $\Omega>0$. This process is shown in Fig. 6(a), where the formation of exponentially-decaying tails of the sech-type soliton is clearly seen with the help of the logarithmic scale. The shift of the propagation constant is proportional to the amplitude of the initial perturbation applied, so that a larger perturbation (10%) was used to make the soliton transformation more visible.

To confirm that the pulse so formed belongs to the family of the sech-type soliton solutions, Eq. (6), we plot the numerical value of the soliton amplitude (calculated through the numerical data for the propagation constant) as a function of z for the upper curve shown in Fig. 6(c), using Eqs. (6)–(8), together with the values of the soliton amplitude determined directly from the numerical data. One can see that the two dependencies (solid curve and circles) almost coincide, thereby justifying our conclusion that for $\delta>0$ the algebraic soliton transforms into the sech-type soliton described by Eq. (6) for $\Omega>0$.

Without perturbation, the algebraic soliton does not change, corresponding to straight lines in Fig. 6(c). This was observed by Hayata and Koshiba [9]. However, if an initial perturbation is applied such that, $\delta<0$ in Eq. (11), the soliton amplitude decreases while the propagation constant Ω becomes *negative*. The latter result corresponds to diffractive linear waves for the solution family shown in Fig. 2. If the propagation distance is large enough, the algebraic soliton diffracts and *completely disappears*, as shown in Figs. 6(b) and 6(c) (lower curve).

For the case when $1\leqslant\sigma\leqslant 2$, the algebraic soliton belongs to a branch of the unstable sech-like solitons as follows in Fig. 1(c) $(dP/d\Omega<0)$. As a result, an initially perturbed algebraic soliton with $\delta>0$ [see Eq. (11)] undergoes a switching to a stable branch of the sech-type solitons. This transformation does not depend on the perturbation amplitude, and is accompanied by the large-amplitude oscillations shown in Fig. 6(d). However, the stable branch of sech-type solitons disappears for $\sigma>2$, so that instead of switching the algebraic soliton collapses after a finite time. We note that, because of the exponentially growing perturbations, for $\sigma\geqslant 1$ much smaller initial perturbations are required to observe the instability effects over the same distance, as follows from Fig. 6(c). For negative perturbations [i.e., $\delta<0$ in Eq. (11)], the algebraic soliton always diffracts for any value of σ.

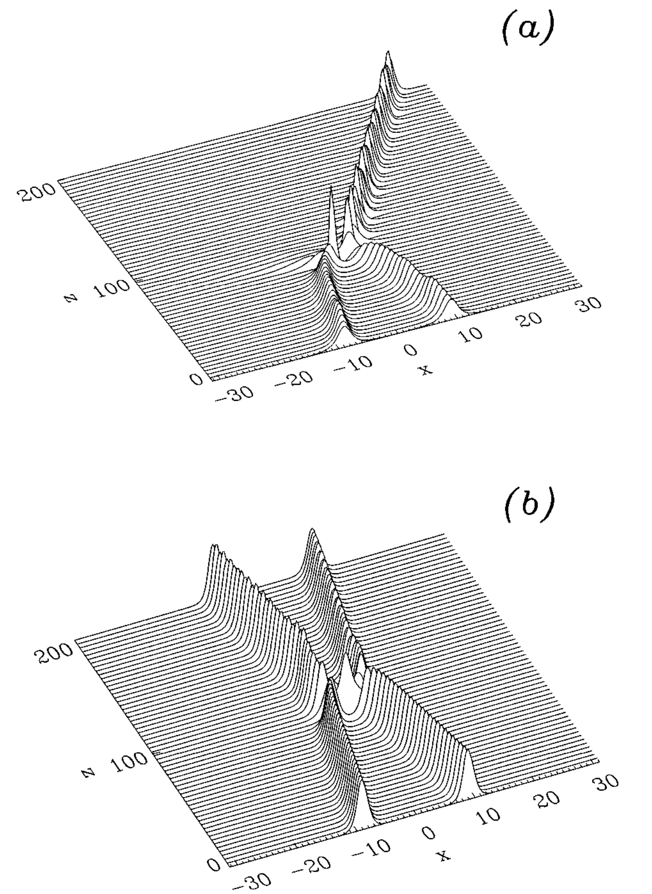

FIG. 7. (a) Destructive interactions between two algebraic solitons of the model (4) for $\sigma=1$. For comparison, in (b) we show the collision between two sech-type solitons under the same conditions but for $\Omega=1.0$.

C. Soliton collisions

An important property of algebraic solitons, namely that they correspond to the threshold for linear wave propagation, allows us to make a conjecture, and also to confirm numerically interesting features of these solitons when they collide. Consider, for example, the collision of two solitons which are initially well-separated and whose tails weakly overlap one another. If the relative phase of the two solitons is nonzero, not an integer multiple of π, some energy redistribution after the soliton collision can usually be observed. Further, the final amplitudes of the asymptotic solitons will differ slightly from one another [22]. For algebraic solitons, such a collision must produce a decay of one of the solitons, because a soliton with an amplitude below the threshold amplitude for an algebraic soliton will diffract. In Fig. 7(a), we display such an inelastic interaction between two algebraic solitons. For comparison Fig. 7(b) plots the corresponding interaction between two sech-type solitons under the same conditions.

D. Classification of algebraic solitons

We can also make some general remarks about the classification of different types of algebraic solitons, previously reported using different models. The *first* type of algebraic solitons always corresponds to a nonlocal interaction in the (1+1)-dimensional model, which produces the power-law asymptotic property of the localized solutions. This nonlocality of the linear part of the equation appears to be due to the different physical properties of the system, such as those in the Benjamin-Ono equation [10,11], where algebraic solitons exist as *families* of localized solutions. As a result, these solitons are stable and also display elastic collisions (see, e.g., [11]). Similar soliton solutions are also known in some (2+1)-dimensional models, such as the Kadomtsev-Petviashvili equation [23]. The *second* type of algebraic solitons has been described above for the particular case, Eq. (4), but similar types of solitons are known from other models, e.g., the so-called ''derivative NLS equation'' [12,13] and ''gap'' solitons [15,16]. The main property of these solitons, which does not depend on the integrability of the corresponding nonlinear equations, is that the algebraic solitons appear as a special limit of the sech-type solitons, when the propagation constant *coincides exactly* with the threshold for linear wave propagation. All solitons of this second type are expected to be at least *weakly unstable*, even if the corresponding family of the sech-type solitons is found to be stable.

V. GRADED-INDEX PLANAR WAVEGUIDES

An interesting link to our results can be made with the theory of guided modes of graded-index planar waveguides [24]. Equation (4) and its stationary solutions can be written as

$$u(x,z)=\psi(x)e^{i\Omega z}, \tag{12}$$

$$\frac{d^2\psi}{dx^2}+[n^2(x)-\beta^2]\psi=0, \tag{13}$$

where the effective index profile, $n(x)$, and the effective propagation constant, β, are defined by

$$n^2(x)=\alpha[\psi(x)]^{\sigma}+\gamma[\psi(x)]^{2\sigma};\quad \beta=\sqrt{\Omega}. \tag{14}$$

In this form, Eq. (14) coincides with the governing equation describing propagation of TE modes in a linear dielectric waveguide with the graded refractive index profile defined by $n(x)$ and propagation constant $\sqrt{\Omega}$ [24]. According to linear waveguide theory, $\Omega=0$ corresponds to the transition between propagating (wavelike) modes, with $\Omega>0$, and nonpropagating (evanescent) modes, with $\Omega<0$. Thus, the algebraic solutions given in (4) also provide the solution for the field $\psi(x)$ of the linear modes of the graded-index waveguide. In particular, the solution given by Eq. (10) for $\Omega=0$ and $\alpha<0$ corresponds to linear bounded modes exactly at the delineation between propagating and evanescent modes. In this case, the graded-index profile has a special shape shown by the dashed curve in Fig. 5, and the guided-mode field decreases algebraically, $F\sim|x|^{-2/\sigma}$, as $|x|\to\infty$. This kind of slowly decaying guided modes has been recently discussed in Ref. [25] (cf. Fig. 1 of Ref. [25] with Fig. 5 above).

VI. CONCLUSIONS

We have demonstrated that self-guided nonlinear waves with power-law asymptotics, i.e., algebraic optical solitons, can be explained as a special limit of exponentially-decaying (sech-type) nonlinear waves, when the propagation constant coincides with the threshold for linear wave propagation. This unique property of algebraic solitary waves differentiates them from other types of nonlinear waves with hyperbolic secant and power-law asymptotics. Furthermore, this special condition for the existence of algebraic solitons indicates their inherent instability, even in the case when the corresponding family of sech-type solitons is stable. We have demonstrated the property of algebraic solitons for a rather general example of the nonlinear Schrödinger equation with two power-law competing nonlinearities.

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