Inverse problem with a dilated kernel containing different singularities

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In a recent paper [Phys. Lett. A **205**, 130 (1995)], we investigated the inverse problem of solving $g(x_1,\ldots,x_q)$ from the integral equation $n(y_1,\ldots,y_q)=\int K(y_1,\ldots,y_q|x_1,\ldots,x_q)g(x_1,\ldots,x_q)dx_1\cdots dx_q$, with the given integral n and kernel K by analytically dilating variable y to the complex plane. We showed, by studying the singularities and discontinuities of the dilated kernel and integral, that the unknown function g can be obtained from an algebraic relation in the case where the dilated kernel contains a simple and single-valued pole. The present paper intends to generalize this result to the case where the kernel contains higher-order and/or multivalued poles. We show that the integral equation in these more general cases can be transformed to algebraic, ordinary, or partial differential equations, depending on the type of the singularities of the kernel and the dimension of the inverse problem. Moreover, some conditions constraining the integral n, which are independent of the integrand g, are revealed when K has multivalued or high-order singularities. [S1063-651X(96)00809-4]

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I. INTRODUCTION

In physics and many other branches of science and engineering one often needs to predict or estimate certain input from the corresponding output. Examples include recovering the incident (optical or particle) beams from the scattered beams, estimating transmitted signals from the received ones, and detecting the inner structures of systems such as the energy spectra of the given matters from the measurable free energy. This kind of problem is generally called the inverse problem. In many cases the inverse problem is mathematically represented by an integral equation of the first kind

$$n(y_1,\ldots,y_q)=\int_{a_q}^{b_q}\cdots\int_{a_1}^{b_1}K(y_1,\ldots,y_q|x_1,\ldots,x_q)\times g(x_1,\ldots,x_q)dx_1\cdots dx_q, \tag{1}$$

where the kernel K is often known from our understanding of the physical mechanisms of the systems involved. The function n is also known in practice from the measurements. The function g is unknown and it is the task for the research of the inverse problem to determine this function.

Equation (1) has been investigated as a typical integral equation in mathematics for a long time [1–6]. Recently, Chen *et al.* have solved Eq. (1) for Fermi or Fermi-like kernels in the one-dimensional case by elegantly linking the kernel to the δ function [7,8]. The limiting procedure involved in [7,8] was later clarified [9]. In a recent paper [10], we suggested an approach to dilate the integral n and the kernel K into the complex plane. The solution to Eq. (1) is then obtained by investigating the singularities of the kernel K and the discontinuities of the integral n at the singular set of K. In [10] we treated only the case where K, after dilated to the complex plane, has a single-valued simple pole (or order-one pole) for each x. In this paper, we will extend the approach of [10] to more general cases where K, if dilated to the complex plane, has multiple and high-order poles. Moreover, we will show that some interesting intrinsic relations exist between n and K, which are independent of g in these more general cases.

This paper is organized as follows. In Secs. II and III we consider the one-dimensional case, for which Eq. (1) is simplified to

$$n(y)=\int_a^b K(y|x)g(x)dx. \tag{2}$$

In Sec. II we will first summarize the results of the previous paper for the case of simple poles and then generalize them to the case of multiple-valued singularities. Section III will be devoted to the high-order singularity problem. In Sec. IV we extend the results obtained in the previous sections to the multidimensional systems. Finally, we conclude the paper in Sec. V.

Before dealing with the concrete cases, let us first specify some general conditions required for our approach. As in most of practical problems, $g(x_1,\ldots,x_q)$ and $K(y_1,\ldots,y_q|x_1,\ldots,x_q)$ are assumed to be real and analytic functions of real variables $x_1,\ldots,x_q$ and $y_1,\ldots,y_q$ in their supports, which are determined by physical problems. Thus $n(y_1,\ldots,y_q)$ must also be a real and analytic function for real $y_1,\ldots,y_q$. We assume that K and n can be analytically dilated from the original real $y_1,\ldots,y_q$ space to complex $z_1,\ldots,z_q$ space. Since K is a known analytical function, its analytical continuation can be directly written down by replacing the real y's by complex variables z's. We further assume that the analytical continuation of n to complex z's is also entirely known. The crucial assumption in our approach

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is that Eqs. (1) and (2) are valid not only for real y but also for the complex z variables as well, i.e., we have

$$n(z_1,\ldots,z_q)=\int_{a_q}^{b_q}\cdots\int_{a_1}^{b_1}K(z_1,\ldots,z_q|x_1,\ldots,x_q)\times g(x_1,\ldots,x_q)dx_1\cdots dx_q \tag{3}$$

generally, or

$$n(z)=\int_a^b K(z|x)g(x)dx \tag{4}$$

in the one-dimensional case. This point can be easily verified by the above assumptions of analytic continuation.

II. KERNEL WITH SINGLE- AND MULTIVALUED ORDER-ONE POLES

$K(y|x)$ in (2) is assumed to be analytical for the variables x and y on its physically meaningful supports. According to the Liouville theorem [11], there exist, however, singularities outside the physical region for $K(z|x)$, if $K(z|x)$ is not identically constant and bounded for large $|z|$. We make an assumption, which is reasonable in many physical systems, that $|K(z|x)|<R$ as $|z|\to\infty$ with R being a certain finite positive number. Thus $K(z|x)$ must have singularities. In this section we first consider the case where, for each x, $K(z|x)$ has only a single order-one pole at $x=h(\hat{z})$ [$\hat{z}=h^{-1}(x)$ is single valued], namely,

$$K(z|x)=\frac{\phi(x,z)}{x-h(z)}, \tag{5}$$

with $\phi(x,z)$ being analytic on the cut $\hat{z}=h^{-1}(x)$, where x belongs to $[a,b]$. From the well-known relation

$$\lim_{\epsilon\to 0}\frac{1}{x-y\pm i\epsilon}=\mathrm{P}\,\frac{1}{x-y}\mp i\pi\delta(x-y), \tag{6}$$

where P denotes the Cauchy principal value, we can express [10] Eq. (5) in the form

$$K^{[\pm]}(\hat{z}|x)=\lim_{\epsilon\to 0}\frac{1}{x-h(\hat{z})\mp i\epsilon}\,\phi(x,\hat{z})$$
$$=\phi(x,\hat{z})\mathrm{P}\,\frac{1}{x-h(\hat{z})}\pm i\pi u(\hat{z})\,\delta(x-h(\hat{z}))$$
$$u(\hat{z})=\phi(x,\hat{z}), \tag{7}$$

which is the key starting point of the analysis in [10]. Equation (7) leads to

$$n^{[\pm]}(\hat{z})\equiv\int_a^b K^{[\pm]}(\hat{z}|x)g(x)dx$$
$$=\mathrm{P}\int_a^b \phi(x,\hat{z})g(x)\,\frac{1}{x-h(\hat{z})}\,dx\pm i\pi u(\hat{z})g(x), \tag{8}$$

from which the inverse problem can be solved exactly:

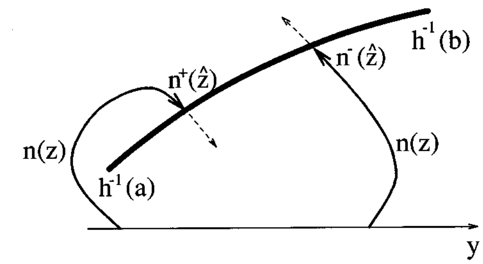

FIG. 1. Schematic representation of the analytical continuation of the integral $n(y)$ to the complex plane. The thick solid line $[h^{-1}(a),h^{-1}(b)]$ indicates the singular cut of the kernel, while the arrowed solid lines indicate the continuation. The extension across the singular cut as marked by the dashed arrows is not allowed.

$$g(x)=\frac{n^{[+]}(\hat{z})-n^{[-]}(\hat{z})}{2i\pi u(\hat{z})}, \tag{9}$$

where $\hat{z}$ can be replaced by a function of x as $\hat{z}=h^{-1}(x)$ and $n^{\pm}(\hat{z})$ are the values of $n(z)$ by taking the limits $z\to\hat{z}$ from the left and right [with respect to the oriented curve from $\hat{z}=h^{-1}(a)$ to $\hat{z}=h^{-1}(b)$; see Fig. 1] sides, respectively.

Before we present further results, it is helpful to make the following remarks to clarify result (9).

(i) The kernel $K(y|x)$ is analytic on the physical supports of x and y. However, it has singularities when $K(y|x)$ is dilated from y to complex z plane (the support of x is not changed). $K(z|x)$ is single valued for each pair of variables (z,x). This is different from the usual case studied in the context of singular integral equations, where the physical support itself contains singularities.

(ii) $K(z|x)$ has a single simple pole for each x at $\hat{z}=h^{-1}(x)$, thus it has a cut when x varies continuously on its support $[a,b]$. When z approaches $\hat{z}$ from two sides of the cut, $K(z|x)$ undergoes a δ-function jump indicated by (7).

(iii) The function $n(z)$ satisfying Eq. (4) is also single valued. It can be obtained from the analytical dilatation from the measured function $n(y)$ without crossing the singular cut of $K(z|x)$ (see Fig. 1 for a schematic representation). One can certainly find a discontinuity of $n(z)$ on the cut. The difference between the left and right limits of $n(z)$, $[n^+(\hat{z})-n^-(\hat{z})]$, divided by the residue of the kernel at the corresponding pole, gives the solution $g(x)$ in (9).

(iv) It is to be emphasized that $n(y)$ can be often analytically dilated to cross the cut without experiencing any discontinuity. However, its continuous dilatation in the region crossing the cut does not satisfy Eq. (4) (see the dashed lines in Fig. 1) and cannot be used for solving the inverse problem. Actually, the dilatation of $n(y)$ must have multiple-sheet structure. The unique sheet giving $n(z)$ of (4) should be reached by the paths from y to the given z. The paths may turn around the branch points $\hat{z}=h^{-1}(a)$ and $h^{-1}(b)$, but they can never cross over the cut as indicated in Fig. 1.

(v) The result (9) is surprisingly compact. Now the inverse problem is reduced to a simplest algebraic computation. The main work for solving the inverse problem is then to seek the location of the singular point of $K(z|x)$, $(x,\hat{z})$, and to specify its residue $u(\hat{z})$ and the discontinuity of $n(z)$ on this singular set $n^+(\hat{z})-n^-(\hat{z})$.

Now we generalize the above results to the case where kernel $K(z|x)$ may have more than one singular point in the z plane for each x, namely,

$$x=h(\hat{z}(1))=h(\hat{z}(2))=\cdots=h(\hat{z}(\mu)). \tag{10}$$

In this case, one obtains the same $g(x)$ solution from each cut according to Eq. (9),

$$g(x)=\frac{n^{[+]}(\hat{z}(l))-n^{[-]}[\hat{z}(l)]}{2i\pi u(\hat{z}(l))}, \quad l=1,2,\ldots,\mu. \tag{11}$$

Therefore, this solution gives rise to some relations between the discontinuities of $n(z)$ and the residues of $K(z|x)$ at different singular points $z(l)$, $l=1,2,\ldots,\mu$,

$$\frac{n^{[+]}(\hat{z}(l))-n^{[-]}(\hat{z}(l))}{2i\pi u(\hat{z}(l))}=\frac{n^{[+]}(\hat{z}(1))-n^{[-]}(\hat{z}(1))}{2i\pi u(\hat{z}(1))},$$

$$l=2,3,\ldots,\mu. \tag{12}$$

They serve as constraints governing the integral $n(z)$ [as well as $n(y)$]. We point out that these constraints are independent of the integrand $g(x)$ and they are purely due to the distribution of the singularities of the kernel $K(z|x)$.

III. KERNEL WITH HIGHER-ORDER POLES

Now we consider the case where $K(z|x)$ has a higher-order pole in the complex z plane

$$n(z)=\int_a^b \frac{\phi(x,z)g(x)}{[x-h(z)]^{\nu+1}}\,dx, \tag{13}$$

with $\phi(x,z)$ being analytic on the cut $\hat{z}=h^{-1}(x)$, $x=[a,b]$. First, let us study the case of order-two pole ($\nu=1$) in detail. Equation (13) is reduced to

$$n(z)=\frac{A_1(z)}{a-h(z)}-\frac{B_1(z)}{b-h(z)}+m(z), \tag{14}$$

with

$$A_1(z)=\phi(a,z)g(a), \quad B_1(z)=\phi(b,z)g(b) \tag{15}$$

$$m(z)=\int_a^b \frac{[\phi(x,z)g(x)]'}{[x-h(z)]}\,dx, \tag{16}$$

where the prime denotes the partial derivative with respect to x. Now the integration of Eq. (16) takes the same form as Eq. (4). An important implication of Eqs. (14) and (15) is, if the kernel $K(z|x)$ has order-two singularity, the dilated $n(z)$ must have simple-pole singularities at the two boundaries $\hat{z}_a=h^{-1}(a)$ and $\hat{z}_b=h^{-1}(b)$. From them we immediately determine

$$g(a)=\frac{\lim_{z\to\hat{z}_a}[a-h(z)]n(z)}{\phi(a,\hat{z}_a)}, \tag{17}$$

$$g(b)=-\frac{\lim_{z\to\hat{z}_b}[b-h(z)]n(z)}{\phi(b,\hat{z}_b)}. \tag{18}$$

Inserting (17) and (18) into (14) and (15) we can specify $A_1(z)$, $B_1(z)$, and $m(z)$. Equation (16) can be solved following the procedures from (4)–(9), leading to

$$2i\pi[\phi(x,z)g(x)]'|_{z=\hat{z}}=m^+(\hat{z})-m^-(\hat{z}), \tag{19}$$

where again we have $\hat{z}=h^{-1}(x)$ and $m^{\pm}(\hat{z})$ have the same meanings as $n^{\pm}(\hat{z})$ in Eq. (9). It is interesting that an integral equation (13) is now transformed to a first-order differential equation

$$D(x)g'+S(x)g=M(x), \tag{20}$$

with $D(x)=2\pi i\phi[x,h^{-1}(x)]$, $S(x)=2\pi i\partial\phi(x,z)/\partial x|_{z=h^{-1}(x)}$, and $M(x)=m^+[h^{-1}(x)]-m^-[h^{-1}(x)]$. The explicit solution of (20) can be easily worked out as

$$g(x)=C\exp\left(-\int_a^x dx\,\frac{S(x)}{D(x)}\right)+\exp\left(-\int_a^x dx\,\frac{S(x)}{D(x)}\right)\times\int_a^x dx\,\frac{M(x)}{D(x)}\exp\left(\int_a^x dx\,\frac{S(x)}{D(x)}\right), \tag{21}$$

where the constant C is fixed by the boundary condition at $x=a$ in (17)

$$C=g(a)=\frac{\lim_{z\to\hat{z}_a}[a-h(z)]n(z)}{\phi(\hat{z}_a)}. \tag{22}$$

A point particularly important is that relation (18) has not been used for the boundary condition in solving (20). Thus this serves as a constraint for the function $n(y)$ itself, namely, we have

$$-\frac{\lim_{z\to\hat{z}_b}[b-h(z)]n(z)}{\phi(b,\hat{z}_b)}=\frac{\lim_{z\to\hat{z}_a}[a-h(z)]n(z)}{\phi(a,\hat{z}_a)}\exp\left(-\int_a^b dx\,\frac{S(x)}{D(x)}\right)+\exp\left(-\int_a^b dx\,\frac{S(x)}{D(x)}\right)\int_a^b dx\,\frac{M(x)}{D(x)}\times\exp\left(\int_a^b dx\,\frac{S(x)}{D(x)}\right). \tag{23}$$

Now we can briefly summarize what we have achieved in the above analysis in this section. If the dilatation of the kernel $K(z|x)$ has an order-two pole, the dilatation of the integral $n(z)$ should have two kinds of singularities: two order-one poles at the two boundaries of the integration [i.e., at $\hat{z}_a=h^{-1}(a)$, $h^{-1}(a)$, in our notation] and a discontinuity along the cut $x=h(\hat{z})$ with the branches at $\hat{z}_{a,b}$. The residues of $n(z)$ at the poles are associated with the boundary values of the unknown function $g(x)$ [$g(a)$ and $g(b)$ in our case] and the gap of the discontinuity is given by the derivative of the integrand. These arguments allow the original integral equation to be transformed to a much simpler and solvable first-order differential equation with well defined boundary conditions. Moreover, the two kinds of singularities are related to each other, namely, the two residues of the poles should be linked by the integration along the cut through Eq. (23), which describes the intrinsic links between the singularities of the kernel and the integral, and should be satisfied by $n(z)$ for arbitrary analytical $g(x)$.

The above method of solution can be extended to more general integral equation (13). Equation (14) can be written generally in the form

$$n(z)=\sum_{k=1}^{k=\nu}\frac{(-1)^k}{\nu(\nu-1)\cdots(\nu+1-k)}\left(\frac{[\phi(b,z)g(x)]^{(k-1)}}{[b-h(z)]^{\nu+1-k}}-\frac{[\phi(a,z)g(x)]^{(k-1)}}{[a-h(z)]^{\nu+1-k}}\right)+\int_a^b dx\,\frac{[\phi(x,z)g(x)]^{(\nu)}}{x-h(z)}, \tag{24}$$

where we use the notations

$$[\phi(x,z)g(x)]^{(k)}=\frac{\partial^k[\phi(x,z)g(x)]}{\partial x^k}.$$

Now we can fix all the boundary conditions of $g(a,b)^{(k-1)}$, $k=1,2,\ldots,\nu$. For instance, we have

$$g(a)=\nu\,\frac{\lim_{z\to z} a[a-h(z)]^{\nu}n(z)}{\phi(a,h^{-1}(a))} \tag{25}$$

and

$$g(a)^{(1)}=-\left\{\left[\nu(\nu-1)\lim_{z\to h^{-1}(a)}[a-h(z)]^{\nu-1}\times\left(n(z)-\nu\,\frac{\phi(a,z)g(a)}{[x-h(z)]^{\nu}}\right)\right]-g(a)\phi(a,h^{-1}(x))^{(1)}\right\}\Big/\phi(a,h^{-1}(x)). \tag{26}$$

$g(b)$, $g(b)^{(1)}$, and all higher derivatives $g(a)^{(k)}$ and $g(b)^{(k)}$, $k=2,3,\ldots,\nu-1$, can be obtained in a similar way order by order. Finally, the function $m(z)$ can be specified and we arrive at

$$[\phi(x,z)g(x)]^{(\nu)}_{z=\hat{z}}=\frac{m^{+}(\hat{z})-m^{-}(\hat{z})}{2i\pi}. \tag{27}$$

Similar to the case of an order-two pole, we have transformed the integral equation to a well defined differential equation, with the boundary conditions $g(a)$, $g(a)^{(1)},\ldots,$ $g(a)^{(k-1)}$, which are computed from the residues of the various orders of singularities of $n(z)$. The value $g(b)$ and all the known derivatives at the boundary b have not been used in solving (27). They serve, like Eq. (23), as the constraints restricting the integral $n(z)$. These constraints are, however, independent of the function $g(x)$.

The discussions in Sec. II and in this section can be extended to even more general cases where the kernel is written as

$$K(z|x)=\frac{\phi(x,z)}{\prod_{i=1}^{i=\mu}[x-h(z(i))]^{\nu_i+1}}, \tag{28}$$

where $\phi(x,z)$ is analytical on all the cuts $x=h(\hat{z}(i))$, $i=1,\ldots,\mu$. $K(z|x)$ is singular on μ cuts, from each cut we can uniquely solve the inverse problem, and the solutions from all cuts should identically give the same function $g(x)$. Thus we have $\mu-1$ constraints for the integral $n(z)$. Moreover, for the ith singularity (which is a pole of order ν_i+1), we have ν_i constraints. All together we have

$$L=\mu-1+\sum_{i=1}^{i=\mu}\nu_i \tag{29}$$

constraints for the function $n(z)$ [and also for $n(y)$, of course]. These constraints describe the relations between the structure of the integral $n(z)$ and the singularities of the kernel $K(y|x)$, which exist without regard to the form of the function $g(x)$.

IV. MULTIDIMENSIONAL CASES

The multidimensional inverse problem is generally much more complicated than the one-dimensional one. It is therefore extremely desirable to have a practical method to deal with the problem. The above approach can be systematically extended to the multidimensional cases. The general idea of treating a multidimensional problem can be clearly manifested in two-dimensional cases. Therefore, in this section we focus on the two-dimensional problem. The applications to higher-dimensional problems will be briefly discussed at the end of this section. The corresponding integral equation is written as

$$n(y_1,y_2)=\int_{a_2}^{b_2}\int_{a_1}^{b_1}K(y_1,y_2|x_1,x_2)g(x_1,x_2)dx_1dx_2. \tag{30}$$

The functions K and n are analytic in their physical supports and both the function and the equation can be analytical dilated to complex planes

$$n(z_1,z_2)=\int_{a_2}^{b_2}\int_{a_1}^{b_1}K(z_1,z_2|x_1,x_2)g(x_1,x_2)dx_1dx_2. \tag{31}$$

The dilated kernel $K(z_1,z_2|x_1,x_2)$ is assumed to have singularities on the nonphysical supports. After certain coordinate and functional transformations we can get the following simple form for the singularity:

$$K(z_1,z_2|x_1,x_2)=\frac{\phi(x_1,x_2,z_1,z_2)}{[x_1-h_1(z_1,z_2)]^{\nu_1+1}[x_2-h_2(z_1,z_2)]^{\nu_2+1}}, \tag{32}$$

with ϕ being analytic on the sets

$$x_1=h_1(z_1,z_2), \quad x_2=h_2(z_1,z_2). \tag{33}$$

Note that $K(z_1,z_2|x_1,x_2)$ is single valued for each pair of (x,z); then $n(z_1,z_2)$ must also be single valued on the complex (z_1,z_2) support. For clarity and simplicity we analyze the cases $\nu_1=\nu_2=0$; $\nu_1=1$, $\nu_2=0$; and $\nu_1=\nu_2=1$ in detail. The cases of general singularities (32) will be mentioned afterward.

For $\nu_1=\nu_2=0$,

$$n(z_1,z_2)=\int_{a_2}^{b_2}\int_{a_1}^{b_1}dx_1dx_2 \times\frac{g(x_1,x_2)\phi(x_1,x_2,z_1,z_2)}{[x_1-h_1(z_1,z_2)][x_2-h_2(z_1,z_2)]}. \tag{34}$$

As in the one-dimensional case the study in this case has a fundamental significance for the general multidimensional inverse problem. Comparing with (7), we should define $K^{[++]}$, $K^{[+-]}$, $K^{[-+]}$, and $K^{[--]}$ for Eq. (34) instead of $K^{[\pm]}$ in Eq. (7) for (4)

$$K^{[\pm\pm]}(\hat{z}_1,\hat{z}_2|x_1,x_2) = \lim_{\epsilon_1\to 0}\lim_{\epsilon_2\to 0}\frac{1}{x_1-h_1(\hat{z}_1,\hat{z}_2)\mp i\epsilon_1} \times\frac{1}{x_2-h_2(\hat{z}_1,\hat{z}_2)\mp i\epsilon_2}\,\phi(x_1,x_2,\hat{z}_1,\hat{z}_2), \tag{35}$$

where the set $(\hat{z}_1,\hat{z}_2)$ satisfies both equations $x_1=h_1(\hat{z}_1,\hat{z}_2)$ and $x_2=h_2(\hat{z}_1,\hat{z}_2)$. Correspondingly, we have

$$n^{[\pm\pm]}(\hat{z}_1,\hat{z}_2)=\int_{a_2}^{b_2}\int_{a_1}^{b_2}K^{[\pm\pm]}g(x_1,x_2)dx_1dx_2. \tag{36}$$

$n^{[\pm\pm]}(\hat{z}_1,\hat{z}_2)$ can be obtained from the analytic continuation of $n(y_1,y_2)$ to the given point of complex (z_1,z_2) without crossing any singular sets of (33). It is obvious that Eq. (36) is nothing but Eq. (31) in various limiting cases. Applying the same procedures as those from Eqs. (7)–(9), we have

$$[n]=n^{[++]}+n^{[--]}-n^{[-+]}-n^{[+-]} =-4\pi^2u(\hat{z}_1,\hat{z}_2)g(x_1,x_2), \tag{37}$$

where we have $u(\hat{z}_1,\hat{z}_2)=\phi(x_1,x_2;\hat{z}_1,\hat{z}_2)$, which finally produces

$$g(x_1,x_2)=-\frac{n^{[++]}+n^{[--]}-n^{[+-]}-n^{[-+]}}{4\pi^2u(\hat{z}_1,\hat{z}_2)}, \tag{38}$$

with the right-hand side being a function of x_1 and x_2 through the identities $x_{1,2}=h_{1,2}(\hat{z}_1,\hat{z}_2)$.

For $\nu_1=1$, $\nu_2=0$ we have

$$n(z_1,z_2)=\int_{a_2}^{b_2}\int_{a_1}^{b_1}\frac{g(x_1,x_2)\phi(x_1,x_2,z_1,z_2)}{[x_1-h_1(z_1,z_2)]^2[x_2-h_2(z_1,z_2)]} \times dx_1dx_2$$

$$=f_{a_1}(z_1,z_2)-f_{b_1}(z_1,z_2)+m(z_1,z_2), \tag{39}$$

with f_{a_1}, f_{b_1}, and m being given by

$$f_{a_1}(z_1,z_2)=\frac{1}{a_1-h_1(z_1,z_2)} \times\int_{a_2}^{b_2}dx_2\,\frac{\phi(a_1,x_2,z_1,z_2)g(a_1,x_2)}{x_2-h_2(z_1,z_2)}, \tag{40}$$

$$f_{b_1}(z_1,z_2)=\frac{1}{b_1-h_1(z_1,z_2)} \times\int_{a_2}^{b_2}dx_2\,\frac{\phi(b_1,x_2,z_1,z_2)g(b_1,x_2)}{x_2-h_2(z_1,z_2)}, \tag{41}$$

$$m(z_1,z_2)=\int_{a_2}^{b_2}\int_{a_1}^{b_1}dx_1dx_2\,\frac{[\phi(x_1,x_2,z_1,z_2)g(x_1,x_2)]_{x_1}}{(x_1-h_1)(x_2-h_2)}, \tag{42}$$

where we use $[Q(x_1,x_2)]_{x_i}=\partial Q(x_1,x_2)/\partial x_i$. In (39) the terms f_{a_1} and f_{b_1} have first-order divergences on the planes $a_1=h_1(z_1,z_2)$ and $b_1=h_1(z_1,z_2)$, respectively, and the third term m has no divergence on the sets (33), while it has discontinuity on the set $[x_1=h_1(\hat{z}_1,\hat{z}_2),\ x_2=h_2(\hat{z}_1,\hat{z}_2)]$. In the spirit deriving Eqs. (37) and (20) we can reduce Eq. (42) to a partial differential equation for the function $g(x_1,x_2)$

$$D(x_1,x_2)g_{x_1}+S(x_1,x_2)g=M(x_1,x_2), \tag{43}$$

with $D(x_1,x_2)=-4\pi^2\phi(x_1,x_2,\hat{z}_1,\hat{z}_2)$, $S(x_1,x_2)=-4\pi^2\{[\phi(x_1,x_2,z_1,z_2)]_{x_1}\}|_{z_1,z_2=\hat{z}_1,\hat{z}_2}$, and $M(x_1,x_2)=[m]=(m^{++}-m^{+-}+m^{--}-m^{-+})$. This equation can be explicitly solved as

$$g(x_1,x_2)=C(x_2)\exp\left(-\int_{a_1}^{x_1}dx_1\,\frac{S}{D}\right)+\exp\left(-\int_{a_1}^{x_1}dx_1\,\frac{S}{D}\right) \times\int_{a_1}^{x_1}dx_1\,\frac{M}{D}\,\exp\left(\int_{a_1}^{x_1}dx_1\,\frac{S}{D}\right). \tag{44}$$

In (44) there are two unknown functions $C(x_2)$ and $M(x_1,x_2)$. $C(x_2)=g(a_1,x_2)$ can be determined from (40). Defining

$$G_{a_1}(z_1,z_2)=\lim_{h_1(z_1,z_2)\to a_1}[a_1-h_1(z_1,z_2)]n(z_1,z_2), \tag{45}$$

we have

$$G_{a_1}(z_1,z_2)=\int_{a_2}^{b_2}\frac{g(a_1,x_2)\phi(a_1,x_2,z_1,z_2)}{x_2-h_2(z_1,z_2)}\,dx_2, \tag{46}$$

where z_1 and z_2 satisfy a constraint $a_1=h_1(z_1,z_2)$. Since all functions G, ϕ, and h_2 are known from n and K, we can immediately solve (46) by applying the approach for one-dimensional inverse problem

$$g(a_1,x_2)=\frac{G^+_{a_1}-G^-_{a_1}}{2i\pi\phi(a_1,x_2,\hat{z}_{1a_1},\hat{z}_{2a_1})}. \qquad (47)$$

Here $\hat{z}_{1a_1}$ and $\hat{z}_{2a_1}$ are solved from $a_1=h_1(z_1,z_2)$ and $x_2=h_2(z_1,z_2)$. The function $g(b_1,x_2)$ can be specified in the same way as

$$g(b_1,x_2)=\frac{G^-_{b_1}-G^+_{b_1}}{2i\pi\phi(b_1,x_2,\hat{z}_{1b_1},\hat{z}_{2b_1})}, \qquad (48)$$

with $\hat{z}_{1b_1}$ and $\hat{z}_{2b_1}$ being solved from the set of equations $b_1=h_1(z_1,z_2)$ and $x_2=h_2(z_1,z_2)$ and G_{b_1} obtained from Eq. (45) by replacing a_1 by b_1. Inserting (47) and (48) into (39), we can uniquely fix the function $m(z_1,z_2)$ as well as $M(z_1,z_2)$ in (43). Then the solution (44) is explicit with $g(a_1,x_2)$ and $M(x_1,x_2)$ specified.

The relation (48) has not been used as the boundary condition in determining Eq. (44). From (44), there is an identity

$$g(b_1,x_2)=g(a_1,x_2)\exp\left(-\int_{a_1}^{b_1}dx_1\,\frac{S}{D}\right)+\exp\left(-\int_{a_1}^{b_1}dx_1\,\frac{S}{D}\right)\int_{a_1}^{b_1}dx_1\,\frac{M}{D}\times\exp\left(\int_{a_1}^{x_1}dx_1\,\frac{S}{D}\right). \qquad (49)$$

Inserting (47) and (48) into (49), we get a constraint for $n(z_1,z_2)$, independent of $g(x_1,x_2)$. This constraint is a nontrivial extension of Eq. (23).

For $\nu_1=\nu_2=1$ we have

$$n(z_1,z_2)=\int_{a_2}^{b_2}\int_{a_1}^{b_1}\frac{g(x_1,x_2)\phi(x_1,x_2,z_1,z_2)}{[x_1-h_1(z_1,z_2)]^2[x_2-h_2(z_1,z_2)]^2}\times dx_1dx_2$$
$$=f^2_{a_1,a_2}-f^2_{a_1,b_2}-f^2_{b_1,a_2}+f^2_{b_1,b_2}+f^1_{a_1}-f^1_{b_1}+f^1_{a_2}-f^1_{b_2}+m(z_1,z_2), \qquad (50)$$

with

$$f^2_{e_1,e_2}=\frac{g(e_1,e_2)\phi(e_1,e_2,z_1,z_2)}{[e_1-h_1(z_1,z_2)][e_2-h_2(z_1,z_2)]}, \qquad (51)$$

$$f^1_{e_1}=\frac{1}{e_1-h_1(z_1,z_2)}\int_{a_2}^{b_2}dx_2\,\frac{[g(e_1,x_2)\phi(e_1,x_2,z_1,z_2)]_{x_2}}{x_2-h_2(z_1,z_2)},$$

$$f^1_{e_2}=\frac{1}{e_2-h_2(z_1,z_2)}\int_{a_1}^{b_1}dx_1\,\frac{[g(x_1,e_2)\phi(x_1,e_2,z_1,z_2)]_{x_1}}{x_1-h_1(z_1,z_2)}, \qquad (52)$$

$$m(z_1,z_2)=\int_{a_2}^{b_2}\int_{a_1}^{b_1}dx_1dx_2\,\frac{[\phi(x_1,x_2,z_1,z_2)g(x_1,x_2)]_{x_1x_2}}{(x_1-h_1)(x_2-h_2)}, \qquad (53)$$

where we have $e_1=a_1,b_1$ and $e_2=a_2,b_2$. Equation (53) can be reduced to a partial differential equation

$$D_0g_{x_1x_2}+D_1g_{x_1}+D_2g_{x_2}+Sg=M, \qquad (54)$$

with

$$D_0=-4\pi^2\phi(x_1,x_2,\hat{z}_1,\hat{z}_2),$$

$$D_{1,2}=-4\pi^2\{[\phi(x_1,x_2,z_1,z_2)]_{x_{2,1}}\}|_{z_1=\hat{z}_1,z_2=\hat{z}_2},$$

$$S=-4\pi^2\{[\phi(x_1,x_2,z_1,z_2)]_{x_1x_2}\}|_{z_1=\hat{z},z_2=\hat{z}_2},$$

$$M=[m(z_1,z_2)]. \qquad (55)$$

To solve this equation we still need to know the explicit form of $M(x_1,x_2)$ and the boundary conditions $g(a_1,x_2)$ and $g(x_1,a_2)$. First, we can determine the g values at the four corners

$$g(e_1,e_2)=(-1)^p\lim_{h_1(z_1,z_2)\to e_1}\lim_{h_2(z_1,z_2)\to e_2}\times\frac{[e_1-h_1(z_1,z_2)][e_2-h_2(z_1,z_2)]n(z_1,z_2)}{\phi(e_1,e_2,z_1,z_2)}, \qquad (56)$$

where $p=0$ when even number of e take b and $p=1$ otherwise. Then we can compute the g values on various boundary lines. First we compute $g(a_1,x_2)$. From (50) and (56) we can get

$$G_{a_1}(z_1,z_2)=\lim_{h_1(z_1,z_2)\to a_1}[a_1-h_1(z_1,z_2)][n-f^2_{a_1,a_2}+f^2_{a_1,b_2}+f^2_{b_1,a_2}-f^2_{b_1,b_2}], \qquad (57)$$

which leads to

$$G_{a_1}(z_1,z_2)=\int_{a_2}^{x_2}dx_2\,\frac{[g(a_1,x_2)\phi(a_1,x_2,z_1,z_2)]_{x_2}}{x_2-h_2(z_1,z_2)}. \qquad (58)$$

Note that z_1,z_2 in this equation are restricted by the condition $h_1(z_1,z_2)=a_1$. Equation (58) can be again reduced to a first-order differential equation for $g(a_1,x_2)$,

$$D_{a_1} g(a_1,x_2)_{x_2} + S_{a_1} g(a_1,x_2) = [G_{a_1}], \tag{59}$$

with $D_{a_1} = 2\pi i \phi(a_1,x_2,\hat{z}_1,\hat{z}_2)$ and $S_{a_1} = 2\pi i \{[\phi(a_1,x_2,z_1,z_2)]_{x_2}\}|_{z_1=\hat{z}_1, z_2=\hat{z}_2}$. The solution of (59) reads

$$g(a_1,x_2) = g(a_1,a_2)\exp\left(-\int_{a_2}^{x_2} dx_2 \frac{S_{a_1}}{D_{a_1}}\right) + \exp\left(-\int_{a_2}^{x_2} dx_2 \frac{S_{a_1}}{D_{a_1}}\right)\int_{a_2}^{x_2} dx_2 \frac{[G_{a_1}]}{D_{a_1}} \times \exp\left(\int_{a_2}^{x_2} dx_2 \frac{S_{a_1}}{D_{a_1}}\right). \tag{60}$$

Since $g(a_1,a_2)$ and G_{a_1} are known from (56) and (57), the solution of $g(a_1,x_2)$ is explicit. In the same way we can specify $g(x_1,a_2)$, $g(b_1,x_2)$, and $g(x_1,b_2)$. Inserting all these boundary values and corner values into Eqs. (51) and (52), all functions of f^2 and f^1 are fixed and then the function $m(x_1,x_2)$ in (50) follows. With $m(x_1,x_2)$ and the boundary conditions $g(a_1,x_2)$ and $g(x_1,a_2)$ known, Eq. (54) is completely defined and can be solved in various practical ways, though the explicit solution seems to be unavailable.

In seeking the unique solution of (54) we do not need $g(b_1,x_2)$ and $g(x_1,b_2)$ for the boundary conditions. These two known functions again serve as constraints for the function $n(y_1,y_2)$ as we discussed for (23), (29), and (49).

Applying similar procedures, we can transform the more general two-dimensional inverse problem with arbitrary ν_1,ν_2 in (32) to partial differential equations in a systematical way. With larger ν_1 and ν_2, the order of the partial differential equation becomes larger and there are more constraints for function $n(y_1,y_2)$ to be satisfied. We will not go into further details in this respect.

The procedure can also be extended to q-dimensional problems with $q>2$. Here we give only some results of Eq. (1) when the continuous dilatation of the kernel $K(z_1,z_2,\ldots,z_q|x_1,x_2,\ldots,x_q)$ has simple singularity

$$K(z_1,z_2,\ldots,z_q|x_1,x_2,\ldots,x_q) = \frac{\phi(z_1,z_2,\ldots,z_q|x_1,x_2,\ldots,x_q)}{\Pi_{i=1}^{i=q}[x_i - h_i(z_1,z_2,\ldots,z_q)]}. \tag{61}$$

In the spirit same as (35) and (36) we can define 2^q functions $K^{[\pm,\pm,\ldots,\pm]}$ and the corresponding 2^q functions $n^{[\pm,\pm,\ldots,\pm]}$. The final solution of (1) can be written as

$$g(x_1,\ldots,x_q) = \frac{[n]}{(2\pi i)^q u(\hat{z}_1,\ldots,\hat{z}_q)}, \tag{62}$$

where the notation $[n]$ indicates an algebraic summation of 2^q terms of all $n^{[\pm\cdots\pm]}$. Each term takes a sign $(-1)^j$ with j being the number of minus signs in the superscript, namely,

$$[n] = n^{[+\cdots+]} - n^{[-+\cdots+]} - n^{[+-+\cdots+]} + n^{[--+\cdots+]} - n^{[++-+\cdots+]} + n^{[-+-+\cdots+]} + n^{[+--+\cdots+]} - n^{[---+\cdots+]} + \cdots + (-1)^q n^{[-\cdots-]}. \tag{63}$$

The development from (30) to (60) can be fully applied to q-dimensional inverse problem with kernel having high-order singularities

$$K(z_1,z_2,\ldots,z_q|x_1,x_2,\ldots,x_q) = \frac{\phi(z_1,z_2,\ldots,z_q|x_1,x_2,\ldots,x_q)}{\prod_{i=1}^{i=q}[x_i - h_i(z_1,z_2,\ldots,z_q)]^{\nu_i+1}}. \tag{64}$$

Here we will not repeat the similar formalism.

V. CONCLUSION

In conclusion, we have generalized our method of solution to the inverse problem proposed in Ref. [10] to the more general cases. This generalization will enable us to deal with a wider range of practical inverse problems that may occur in many fields of research.

The essential results in our approach are the following. We successfully reduced the inverse problem to strikingly simple algebraic equations [for $\Sigma_{i=1}^{i=q}\nu_i = 0$ in Eq. (64)], explicitly solvable ordinary or partial differential equations (for $\Sigma_{i=1}^{i=q}\nu_i = 1$), or other differential equations that can be handled with essentially different methods from those used conventionally for solving the integral equations. All these reductions are based on the understanding of the properties of the singular sets of the kernel and the discontinuities or singularities of the integral associated to these sets.

As $K(z|x)$ contains high-order poles, the integral $n(y)$ should be subject to certain constraints. These constraints are not due to the integrand $g(x)$. They represent the intrinsic links between the singularities of the kernel and the integral. The understanding of these intrinsic relations may be of interest in predicting the universal behaviors of measurable quantities directly from the structure of transformation operators.

We emphasize again that both the original K and n are analytical and well behaved on the physical supports. The key point is to extend the physical support to the complex plane to analyzing the singularities and discontinuities in the nonphysical regions. Moreover, we also analytically dilate the integral $n(y)$ and find that the discontinuities or singularities occur exactly at the locations of the singularities of the kernel.

Our approach relies on the fact that we need the explicit functional forms for $K(y|x)$ and $n(y)$. In many practical cases, however, we do not have explicit expressions for these functions, but only limited amount of data for $K(y|x)$ and $n(y)$. The analytical continuations cannot be directly made in these cases. To make analytical continuation in these cases, some proper expansion (or successive-expansion) approaches have to be invoked. We plan to discuss this more practical aspect in a future work.

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