Driving and synchronizing by chaotic impulses

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For a class of dynamical systems driven by chaotic impulses we give conditions for the occurrence of chaos locking. It is shown how the concept of time-discontinuous coupling of two chaotic systems may lead to generalized synchronization. The method of synchronization is interpreted as a nonlinear analog of the sampling theorem. Furthermore, we examine the effect of amplitude quantization of the driving signal on their synchronization. Even though two time-discontinuously coupled dynamical systems are no more exactly synchronized when the driving signal is digitized, their trajectories are close enough to allow correct transmission of digital information signals between them. [S1063-651X(96)07708-2]

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Over the past few decades, there has been considerable interest in the studies of chaos and its ubiquitous nature. This is due to two facts. First, the study of chaotic behavior in almost all fields of science is essential for an appropriate description and modeling of various phenomena in nature [1]. Second, nonlinear phenomena may lead to new applications in engineering. For example, recently there has been considerable interest in potential applications of synchronized chaotic systems in the area of *analog* communication [2–4]. However, almost exclusively today communications are *digital*. Motivated by this challenge—digital communications—we discuss in this paper the question of the exchange of digital information signals between two synchronized *continuous* chaotic systems.

The paper is organized as follows. First we develop a general theory of driving (chaotic) systems by chaotic impulses. As a consequence, a criterion for the occurrence of generalized synchronization in unidirectionally systems coupled at discrete times is given. Then we address some questions related to synchronizing two identical systems by chaotic impulses and finally we discuss the relevance of our results to digital communication using chaos synchronization.

Consider a driven N-dimensional chaotic dynamical system whose behavior is governed by

$$\dot{\mathbf{x}}=\mathbf{F}(\mathbf{x},\mathbf{s}_T), \tag{1}$$

where $\mathbf{x}$ is an N-dimensional vector and $\mathbf{s}_T$ is a driving signal. We denote with $\mathbf{x}(t,\mathbf{x}_0)$ the trajectory of (1) based on the initial condition $\mathbf{x}_0$ at $t=0$. The driving signal $\mathbf{s}_T$ is a time sequence of m-dimensional chaotic impulses: $\ldots,\mathbf{s}(-2T),\mathbf{s}(-T),\mathbf{s}(0),\mathbf{s}(2T),\ldots$. The impulses $\mathbf{s}(nT)\in\mathbf{R}^m$ are produced through equidistant sampling of a chaotic trajectory $\mathbf{s}(t)$. The dynamics of (1) is influenced by the sequence of impulses $\mathbf{s}_T$ in the following way. Let $\mathbf{x}(T,\mathbf{x}_0)$ be the position of the trajectory $\mathbf{x}(t,\mathbf{x}_0)$ at time $t=T$. At this moment the system is subject to an impulse $\mathbf{s}(T)$ such that the first m components $x_1,x_2,\ldots,x_m$ of the state vector $\mathbf{x}$ are replaced by $x_i(T,\mathbf{x}_0)=s_i(T)$. Put it another way, the dynamical system (1) oscillates freely and independently from the driving signal $\mathbf{s}_T$ except for the equidistant moments $t_n=nT$ when a part $[x_1,x_2,\ldots,x_m]$ of its state vector $\mathbf{x}$ is forced to a new value $\mathbf{s}(nT)$. Throughout the paper this kind of driving will be called *sporadic driving*. The concept of sporadic driving can be easily applied to iterated maps. Here we focus only on time-continuous dynamical systems due to the fact that circuit implementations of continuous systems are much easier than those of iterated maps, which is important when it comes to applications of sporadic driving in engineering.

In order to put the description of (1) and the influence of chaotic impulses $\mathbf{s}_T$ in a more mathematical frame, we show that (1) can be described as follows. Let us decompose the state vector of (1) into two parts $\mathbf{u}=[x_1,x_2,\ldots,x_m]$ and $\mathbf{w}=[x_{m+1},x_{m+2},\ldots,x_N]$ and the vector field $\mathbf{F}$ into $\mathbf{F}_u=[f_1,f_2,\ldots,f_m]$ and $\mathbf{F}_w=[f_{m+1},f_{m+2},\ldots,f_N]$. Then we can rewrite (1) as

$$\dot{\mathbf{u}}=\mathbf{F}_u(\mathbf{u},\mathbf{w})+\delta_T(t)(\mathbf{s}-\mathbf{u}), \tag{2}$$

$$\dot{\mathbf{w}}=\mathbf{F}_w(\mathbf{u},\mathbf{w}), \tag{3}$$

where $\delta_T(t)$ denotes a periodic sequence of Dirac pulses with period T, i.e., $\delta_T(t)=\Sigma_{n=-\infty}^{+\infty}\delta(t-nT)$. Integrating (1) from time $t=nT-\varepsilon$ to $nT+\varepsilon$, we find in the limit $\varepsilon\to 0$ the following description of the dynamics of (2) and (3). The dynamical system (2) and (3) oscillates unforcedly and freely except for the equidistant moments $t_n=nT$ when $\mathbf{u}(t_n)$ is forced to a new value $\mathbf{s}(t_n)$. Using the concept of asymptotic stability [5], the following theorem determines conditions for the occurrence of predictable oscillations of (2) and (3).

Theorem 1. Consider the system (2) and (3). Assume that $\dot{\mathbf{w}}=\mathbf{F}_w(\mathbf{s},\mathbf{w})$ is asymptotically stable when driven continuously by $\mathbf{s}(t)$ and for initial conditions $\mathbf{w}_0\in\mathbf{B}\subset eq\mathbf{R}^{N-m}$. Then for the given drive signal $\mathbf{s}_T$ there exists a value T_H such that for all $T<T_H$ system (2) is asymptotically stable.

Asymptotic stability of (2) means the following: the driven system (1) ''forgets'' its history (initial conditions

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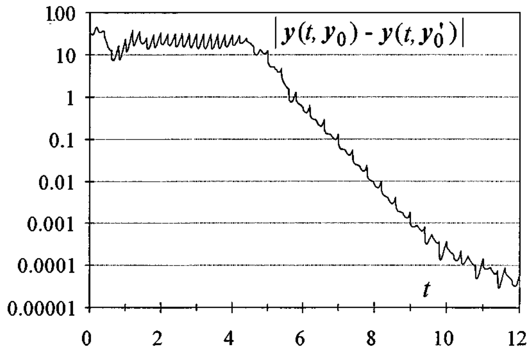

FIG. 1. Difference between two trajectories of (7) based on two different initial conditions y_0 and y_0'. The driving signal s_T is identical for both trajectories. $T=0.2$.

$\mathbf{x}_0$) as time goes on and in the limit $t\rightarrow\infty$ its dynamics is completely determined by the driving signal $\mathbf{s}_T$. In this case we say that the behavior of (1) is *locked* to the chaotic signal $\mathbf{s}_T$. In the special case when a limit cycle behavior is forced by a periodic sequence of impulses, phase locking occurs. Clearly, when (1) exhibits *chaos locking* then its behavior is predictable: driving with the same signal $\mathbf{s}_T$ always results in the same response of (1). The proof of the theorem, which follows from the analytical arguments of [9], and the examples of chaos locking in other systems are planned to be addressed in an extended version of this work [6].

We illustrate the theorem through an example. In this example a driving sequence s_T of one-dimensional chaotic impulses is produced through equidistant sampling of the x_2 variable of a chaotic trajectory of the Rössler system

$$\dot{\mathbf{x}}=\mathbf{R}(\mathbf{x}), \tag{4}$$

where $\mathbf{R}(\mathbf{x})=[2+x_1(x_2-4),-x_1-x_3,x_2+0.45x_3]$.

Sequence s_T drives the Lorenz system

$$\dot{\mathbf{y}}=\mathbf{L}(\mathbf{y})+[0,\delta_T(t)(x_2-y_2),0], \tag{5}$$

where $\mathbf{L}(\mathbf{y})=[\sigma(y_2-y_1),y_1y_3+\rho y_1-y_2,y_1y_2-\beta y_3]$ with $\sigma=10.0$, $\rho=28$, and $\beta=2.66$. In this example we perform one directional *time-discontinuous* coupling between two different *time-continuous* dynamic systems (4) and (5). The term $\delta_T(t)(x_2-y_2)$ in (5) leads to $y_2(t_n)$ being forced to a new value $x_2(t_n)$. In the time intervals between two successive kicks, the Lorenz system (5) behaves chaotically and independently from (4). Denoting $\mathbf{w}=[y_1,y_3]$ and $\mathbf{u}=[y_2]$, one can readily see the compatibility of (4) and (5) with (2) and (3). It is well known that the $\mathbf{w}=[y_1,y_3]$ subsystem of the Lorenz system is asymptotically stable. Therefore Theorem 1 ensures the existence of T_H. Numerically we have found out that (5) is asymptotically stable for $T<T_H=0.31$. The asymptotic stability of (5) is illustrated in Fig. 1, where the difference $|\mathbf{y}(t,\mathbf{y}_0)-\mathbf{y}(t,\mathbf{y}_0')|$ is shown for $T=0.2$. Here $\mathbf{y}(t,\mathbf{y}_0)$ and $\mathbf{y}(t,\mathbf{y}_0')$ denote two trajectories of (5) based on two different initial conditions $\mathbf{y}_0$ and $\mathbf{y}_0'$, while the driving term s_T is identical for both trajectories. The difference between the two trajectories vanishes as time goes to infinity, which means that (5) forgets its initial conditions and follows s_T.

The theorem can also be related to the notion of generalized synchronization (GS) [7]. Two systems are said to have the property of GS if a functional relation exists between the states of both systems and the synchronized manifold defined with this relation is an attractor. Consider the class of unidirectionally coupled systems

$$\dot{\mathbf{x}}=\mathbf{f}(\mathbf{x}), \quad \dot{\mathbf{y}}=\mathbf{g}(\mathbf{y},\mathbf{s}), \tag{6}$$

where $\mathbf{s}=h(\mathbf{x})$ is an m-dimensional vector function of $\mathbf{x}$. Recently, a general criterion for the occurrence of GS in (6) has been proposed [8]. This criterion can be generalized as follows.

Corollary 1. Assume that the second system in (6) is asymptotically stable when driven by $\mathbf{s}(t)$. Then there exists a value T_H such that for all $T<T_H$ two systems

$$\dot{\mathbf{x}}=\mathbf{f}(\mathbf{x}), \quad \dot{\mathbf{y}}=\mathbf{g}(\mathbf{y},\mathbf{s}_T), \tag{7}$$

where s_T is produced through equidistant sampling of $\mathbf{s}$, have the property of GS.

In other words, if GS occurs in (6), then sporadically coupled systems (7) have also the property of GS.

In the remaining part of this paper we address the case when $\mathbf{s}_T$ is produced by sampling $\mathbf{u}$ projections of solutions of an autonomous copy of (2) and (3)

$$\dot{\mathbf{u}}=\mathbf{F}_u(\mathbf{u},\mathbf{w}), \quad \dot{\mathbf{w}}=\mathbf{F}_w(\mathbf{u},\mathbf{w}). \tag{8}$$

In this case, the theorem is a generalization of the synchronization method of Pecora and Carroll [2] and may be rewritten as follows.

Corollary 2. Consider system (8). Assume that $\dot{\mathbf{w}}=\mathbf{F}_w(\mathbf{u},\mathbf{w})$ is asymptotically stable when driven by $\mathbf{u}(t)$ and for initial conditions $\mathbf{w}_0\in\mathbf{B}\subseteq\mathbf{R}^{N-m}$. Then there exists a value T_H such that for all $T<T_H$ sporadic driving of a copy of (8)

$$\dot{\mathbf{u}}'=\mathbf{F}_u(\mathbf{w}',\mathbf{u}')+\delta_T(t)(\mathbf{u}-\mathbf{u}'), \quad \dot{\mathbf{w}}'=\mathbf{F}_w(\mathbf{w}',\mathbf{u}') \tag{9}$$

results in synchronization between (8) and (9), i.e., $|\mathbf{x}-\mathbf{x}'|\rightarrow 0$ when $t\rightarrow\infty$.

In other words, if two dynamical systems synchronize for a particular Pecora-Carroll decomposition [2] then there exists a nonzero value T_H such that sporadic coupling between the two systems leads to their synchronization for all $T<T_H$.

The synchronization in the systems (8) and (9) was considered by Amritkar and Gupte [9]. Corollary 2 might show its usefulness in various scientific disciplines. A promising area of application is communications because all proposed communications systems based on synchronized chaos so far consisted of pairs of *identical* dynamical systems. Now we will illustrate Corollary 2 through an example based on the Lorenz system. The driven system is again defined by (5), while the driving chain of one-dimensional chaotic impulses $\delta_T(t)x_2(t)$ is obtained from a copy of the Lorenz system identical to (5) [without the term $\delta_T(t)(x_2-y_2)$]. From the stability of the (y_1,y_3) subsystem of the Lorenz system and from Corollary 2 it follows that there exists a maximal period T_H of $\delta_{T_H}(t)$ that still allows the two Lorenz systems to synchronize. Figure 2 shows the largest conditional

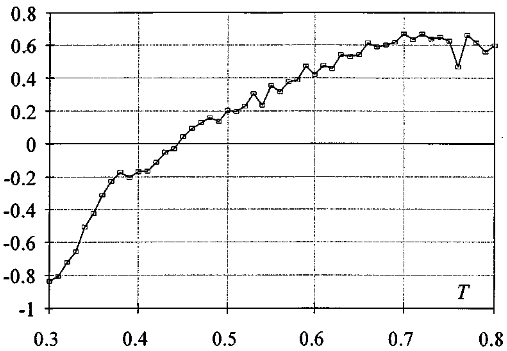

FIG. 2. Dependence of the largest CLE of (7) on the coupling period T. The driving signal s_T is obtained from an identical copy of the Lorenz system.

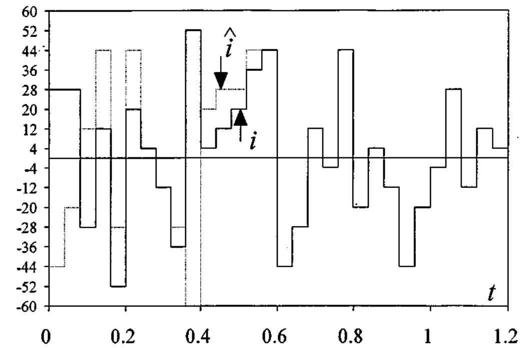

FIG. 4. Information signal i (solid line) and recovered information signal $\hat{i}$ (dashed line) for systems (11) and (12) when $q=128$, $T=0.04$, and $\Omega_i=\{4\Delta(2k+1)|k=-8,\ldots,7\}$.

Lyapunov exponent (CLE) [7] of (5) versus T. All CLEs of (5) are negative for $T<T_H=0.45$. The different T_H values for the sporadically coupled Rössler-Lorenz and Lorenz-Lorenz systems are due to the fact that T_H depends on the driving signal s_T. Through numerical simulations we have checked the synchronization between the two Lorenz systems for different values of parameters σ,ρ,β. For every set of parameters σ,ρ,β there exists a maximal distance between samples of $x_2(t)$ that still allows synchronized motion of the two Lorenz systems.

There is another possible interpretation of the synchronization between the two Lorenz systems. The responding Lorenz system interpolates the samples s_T and produces an interpolated signal $y_2(t)$. If $T<T_H$ then the interpolation is successful and $y_2(t)=x_2(t)$, which means that $x_2(t)$ is completely described by its samples $\ldots,x_2(-2T),x_2(-T),x_2(0),x_2(T),x_2(2T),\ldots$. According to the *sampling theorem*, any function of time $f(t)$ that is *band limited* to B (cycles/sec) is completely described by its sample values every $\frac{1}{2B}$ sec, the samples extending throughout time domain [11]. One might wrongly draw the conclusion that the power spectrum $P_{x_2}(f)$ of $x_2(t)$ is band limited to $f_{T_H}=1/2T_H$. However, power spectra of chaotic signals are exponentially decreasing [10] and thus with *infinite* width. Figure 3 shows the power spectrum of $x_2(t)$. Frequency $f_{T_H}=1.11$ is denoted in Fig. 3. A significant percentage (30%) of the power of $x_2(t)$ is contained out of the frequency range $f>1.11$. Despite the infinite width of

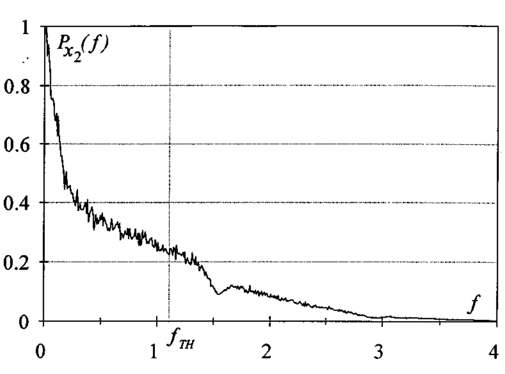

FIG. 3. Power spectrum of $x_2(t)$ normalized with its maximum value.

$P_{x_2}(f)$, it is possible to interpolate the samples of $x_2(t)$ because they are generated by deterministic ordinary differential equations (ODEs) that are *known* at the response. The sampling theorem proves that interpolation of the sampled signal can be performed by passing it through an ideal *linear* low-pass filter with limit frequency $1/2T$. Therefore, Corollary 2 is a counterpart of sampling theorem for chaotic signals: when one applies sporadic coupling then the sampled signal s_T should be processed *nonlinearly* with (5) in order to interpolate it between the samples. There is another interesting consequence of the previous discussion. The value T_H determines the minimum frequency bandwidth of the communication channel modeled as an ideal low-pass filter f_{T_H} that connects two chaotic systems and synchronizes them through the concept of sporadic coupling.

Next we address the effect of amplitude quantization of the driving signal on the synchronization of two chaotic systems. Here we will restrict our investigations to the simplest quantizer (certainly not the best according to many criteria): a uniform scalar quantizer that divides the amplitude range of the input signal into q equal amplitude quanta each of length Δ. The quantizer output signal is equal to the medium value of the amplitude quantum to which the input signal belongs. The amplitude range of the quantizer is $(-A_Q,+A_Q)$. The quantized signal will be denoted as $x_{2q}(t)=Q(x_2(t))$, where $Q(\,)$ denotes the operation of the quantizer. In what follows, once again we time-discontinuously couple two identical Lorenz systems. The driving sequence of chaotic impulses is produced by equidistant sampling of the quantized signal $x_{2q}(t)=Q(x_2(t))$. The response system is

$$\dot{\mathbf{y}}=\mathbf{L}(\mathbf{y})+[0,\delta_T(t)(x_{2q}-y_2),0]. \tag{10}$$

Infinite couplings at times $t_n=nT$ force $y_2(t_n)$ to the value $x_{2q}(t_n)$, which is almost never equal to $x_2(t_n)$. Thus, at the coupling moments t_n, x_2 and y_2 are not forced to be equal to each other, but rather their difference is kept within the range $(-\Delta/2,+\Delta/2)$. As a consequence, the two Lorenz systems never synchronize exactly, but their difference $|\mathbf{x}-\mathbf{y}|$ is not significant and is small enough not to preclude digital communication applications. We have numerically solved ODEs defining the two Lorenz systems and we have computed the difference between $x_{2q}(t_n)$ and $y_{2q}(t_n)=Q(y_2(t_n))$ in

100 000 sampling intervals for $A_Q=25$. An extended version of the results obtained is planned to be presented somewhere else [6]. As an example, numerical integrations have shown that the difference $|x_{2q}(t_n)-y_{2q}(t_n)|$ is always smaller than 4Δ when $q=128$ and $T=0.04$. Such a closeness between $x_{2q}(t_n)$ and $y_{2q}(t_n)$ enables us to construct the digital communication system

$$\dot{\mathbf{x}}=\mathbf{L}(\mathbf{x})+[0,\delta_T(t)(s-x_2),0],$$

$$s=x_2+i(\mathrm{mod}A_Q), \tag{11}$$

$$s_Q=Q(s) \quad \text{(transmitter)}$$

$$\dot{\mathbf{y}}=\mathbf{L}(\mathbf{y})+[0,\delta_T(t)(s_Q-y_2),0],$$

$$y_{2q}=Q(y_2), \tag{12}$$

$$\hat{i}=S_E[s_Q-y_{2q}(\mathrm{mod}A_Q)] \quad \text{(receiver)},$$

where s_Q is the transmitted signal and i is the information signal. We emphasize that both s_Q and i are discrete-time digital signals taking values from finite alphabets $\Omega_s=\{(\Delta/2)(2k+1-q)|k=0,1,\ldots,q-1\}$ and Ω_i, respectively. A new digit of the information signal is generated at times nT. The last equation in (12) denotes the operation of a slicer $S_E(\,)$ that chooses a digit from the alphabet Ω_i that is closest to $s_q-y_{2q}(\mathrm{mod}A_Q)$.

The result $|x_{2q}(t_n)-y_{2q}(t_n)|<4\Delta$ when $q=128$ and $T=0.04$ is still valid even after the insertion of the information signal i. Let the information signal i take values from the finite alphabet $\Omega_i=\{4\Delta(2k+1)|k=-8,\ldots,7\}$, that is, distance between the symbols from the alphabet Ω_i is at least $2(4\Delta)$. If $q=128$ and $T=0.04$, then $\hat{i}=i$ after a *finite* period during which (12) sufficiently approaches the state of the driving Lorenz system (11). Figure 4 illustrates the *exact recovery* of the information signal after a finite period $t=0.56$. We stress here that alternatively one could also use s_Q in the first equation of the transmitter instead of s. This leads to perfect synchronization, but it may also turn the chaotic dynamics of the transmitter into a periodic motion. In such cases a new set of parameter values has to be found with chaotic dynamics. We will discuss this equation also in [6].

Let us briefly summarize the results presented in this paper. We have proposed the concept of sporadic driving and have given conditions for the occurrence of chaos locking and generalized synchronization. We have shown that *time discretization* of the driving signal (sporadic coupling) does not destroy the synchronized motion of two chaotic systems. To a certain extent the synchronization is also robust with respect to the *amplitude discretization* (quantization) of the driving signal. Through the concept of sporadic coupling one can synchronize chaotic systems connecting them with a band-limited channel. The robustness of the synchronization to the quantization of the driving signal is large enough to allow digital transmission of a digital information signal between two chaotic systems.

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