Frequency dependence of the penetration of electromagnetic fields through an elliptical hole in a thin metallic wall

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The static approximation suggests that, for a given hole area, the use of a long narrow slot in a beam pipe gives a reduced coupling impedance. But for a long slot the slot length may be comparable with the wavelength, making the static approximation a poor one. In this paper we derive expressions for the generalized polarizability and susceptibility [Cheng, Fedotov, and Gluckstern, Phys. Rev. E **52**, 3127 (1995)] of an elliptical hole in a thin plane metallic screen, as a function of hole dimensions and wavelength. In particular, we construct a variational form that allows us to obtain an approximate analytic result for the resonant frequency of a cavity with such a hole. In the calculations we include the effects of finite wavelength, but still confine our attention to reduced wavelengths no smaller than the primary hole dimensions. We then use these results to estimate the coupling impedance of a long narrow elliptical slot in a beam pipe, and show that the effect of finite wavelength is important. [S1063-651X(96)10508-0]

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I. INTRODUCTION

The penetration of electric and magnetic fields through a hole in a metallic wall plays an important role in many devices. In an accelerator, such holes in the beam pipe serve to allow access for pumping, devices for beam current and beam position measurement, coupling between cavities, etc. As a consequence, the beam generates wakefields in the beam pipe when it passes by such holes and these wakefields are capable of affecting beam quality and stability. In all these and other similar situations, the quantities of importance are the polarizability and susceptibilities of such holes.

When we consider the coupling impedance of a hole in the wall of a beam pipe [2] of rectangular cross section, we must evaluate the following integral over the hole on the inner surface of the beam pipe:

$$I=\int\int_{\text{hole}} ds\ dt[E_s^{(2)}H_t^{(1)}-E_t^{(2)}H_s^{(1)}]. \tag{1.1}$$

Here the normal to the wall is in the n direction, the azimuthal direction is denoted by s, and the direction t is parallel to the axis of the beam pipe. The superscript (1) refers to the fields with no hole and the superscript (2) refers to the fields in the presence of the hole.

The integral in Eq. (1.1) is exactly the same as the integral used to describe the coupling between waveguides and/or cavites [3]. In fact it is also the integral which describes the detuning of a cavity by a hole in a plane cavity wall, with the superscripts (1) and (2) having the same meaning. It is therefore reasonable to relate the frequency dependence of the coupling impedance in Eq. (1.1) to the detuning of the cavity by a hole whose dimensions may be as large as the reduced wavelength.

The conventional treatment of Eq. (1.1) proceeds by way of the polarizability and susceptibilities of the hole in the wall [4,3]. In a previous paper [1] we presented the method of calculating the polarizability and susceptibility for a circular hole in a thick metallic plate as a function of hole dimensions and wavelength. We redefined polarizabilities in terms of the cavity detuning and constructed a generalized polarizability and susceptibility. In this way, we included the frequency dependence of the polarizability and susceptibility as well as the contributions of higher multipole moments of the hole. We should note that this generalized polarizability and these susceptibilities should only be seen as intermediate vehicles to relate the coupling integrals of interest to the detuning of the cavity by the hole. In our earlier work [1] we showed that the variational approach also allows us to derive analytically the low frequency corrections for the polarizability and susceptibilities of a hole in a thin wall, correct to the first order in k^2.

In the present paper we use the previous approach [1] to obtain expressions for the generalized polarizability and susceptibilities of an elliptical hole in a thin plane metallic screen, as a function of hole dimensions and reduced wavelength. These expressions are derived in Sec. III and appear in Eqs. (4.1) and (4.4), correct to the first order in k^2. The free parameter α in Eq. (4.4) is treated as a variational parameter and chosen in Eq. (4.11) to minimize the susceptibility.

Our interest in an elliptical hole is guided by the fact that narrow slots are considered to be the best choice for pumping holes in the design of modern high-intensity accelerators. It is clear that the image currents on the wall will flow more easily around a long thin slot than around a circular hole of the same area. For this reason a long thin slot gives a lower coupling impedance. But the length of the slot may then become comparable with the wavelength and the static values of polarizability and susceptibilities may no longer be adequate. Therefore, it becomes important to estimate the frequency correction for the coupling impedance of a narrow slot. Our results allow us to obtain a more accurate expression for the coupling impedance of an elliptical hole at low frequencies. Specifically, for a long narrow slot, the frequency correction of the impedance turns out to be much larger than the static value.

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II. GENERALIZED POLARIZABILITY OF AN ELLIPTICAL HOLE

In an analysis for the circular hole [1] we showed how to construct a variational form for the resonant frequency of a cavity with such a hole. For our present purposes we use the same general form of the equation for the calculation of the frequency of the detuned cavity,

$$\frac{\tan\beta_N L}{\lambda_N}=\sum_{\nu}\sum_{\mu} K_{N\nu}(\mathrm{M}^{-1})_{\nu\mu}K_{N\mu}. \tag{2.1}$$

Here $\beta_n^2=k^2-\gamma_n^2$ where the γ_n^2 are the eigenvalues of the two-dimensional scalar Helmholtz equation in the cavity region with the appropriate boundary conditions, and $kc/2\pi$ is the frequency. The quantities $M_{\nu\mu}$ comprise a symmetric matrix defined by

$$M_{\nu\mu}=-\sum_{n\neq N}\lambda_n\cot\beta_n L K_{n\mu}K_{n\nu}+\sum_{\sigma}\lambda_\sigma\tan\beta_\sigma g/2K_{\sigma\mu}K_{\sigma\nu}. \tag{2.2}$$

Here

$$K_{n\nu}\equiv\int_{S_1}dS\mathbf{e}_n\cdot\mathbf{f}_\nu,\quad K_{\sigma\nu}\equiv\int dS\,\mathbf{e}_\sigma\cdot\mathbf{f}_\nu, \tag{2.3}$$

where $\mathbf{e}_n$, $\mathbf{e}_\sigma$ are the complete sets for the field expansion in the cavity and iris regions, and the $\mathbf{f}_\nu$ is the complete set for the expansion of the electric field at the interface between the cavity of length L and the iris of length g. As will be seen in Appendix A, we take a cavity of rectangular cross section with dimensions A and B, with the iris hole located at the center of the rectangle.

Note that we have separated the term $n=N$ and moved it to the left side in Eq. (2.1) since we shall be looking at the $\mathrm{TM}_{0N\ell}$ cavity mode corresponding to $\beta_N L=\ell\pi$ or

$$k_{N\ell}^2=\frac{\ell^2\pi^2}{L^2}+\kappa_N^2, \tag{2.4}$$

where ℓ is an integer, and κ_N^2 is the eigenvalue for the cavity waveguide. Here we use notation

$$\lambda_n=\frac{Z_0}{Z_n}=\begin{cases}k/\beta_n, & \mathrm{TM}\\ \beta_n/k, & \mathrm{TE},\end{cases}$$

with Z_n being the impedance of the "cavity" wave guide and $Z_0=\sqrt{\mu_0/\epsilon_0}$ being the impedance of free space. We use latin subscripts $(n,m,N,\ldots)$ for the cavity region and greek subscripts $(\nu,\mu,\sigma,\ldots)$ for the iris region.

We now consider only the case of $g=0$ (zero thickness of the wall) and by separating out the $\nu=1$, $\mu=1$ term we are able to write Eq. (2.1) approximately as

$$\beta_N\tan\beta_N L\cong\frac{(K_{N1}^\chi)^2}{M_{11}^\chi/k}, \tag{2.5}$$

with

$$\frac{M_{11}^\chi}{k}=-\sum_{n\neq N}\frac{\cot\beta_n L}{\beta_n}(K_{n1}^\chi)^2=\sum_{n\neq N}\frac{\coth\rho_n L}{\rho_n}(K_{n1}^\chi)^2, \tag{2.6}$$

where

$$\rho_n\equiv\sqrt{\kappa_n^2-k^2}, \tag{2.7}$$

$$K_{n1}\equiv\int_{S_1}dS\mathbf{e}_n\cdot\mathbf{f}_1. \tag{2.8}$$

In writing Eq. (2.6) we showed earlier [1] that by using the exact static expression for $\mathbf{f}_1(t,s)$ (where the azimuthal direction is denoted by s and the direction t is parallel to the axis of the beam pipe) we can obtain a result for the generalized polarizability which is accurate through terms proportional to k^2. For the trial function of an elliptical hole in a plane cavity wall we choose

$$\mathbf{f}_1(t,s)=-\boldsymbol{\nabla}\left(1-\frac{t^2}{a^2}-\frac{s^2}{b^2}\right)^{1/2}=-\nabla\phi_0, \tag{2.9}$$

where a and b are the semimajor axes in the axial and azimuthal directions of the beam pipe, respectively. Then Eq. (2.8) becomes

$$K_{n1}\equiv\int_{S_1}dS\boldsymbol{\nabla}\phi_n\cdot\boldsymbol{\nabla}\phi_0=\kappa_n^2\int_{S_1}dS\phi_n\phi_0, \tag{2.10}$$

where we used $\mathbf{e}_n=-\boldsymbol{\nabla}\phi_n$ and $\nabla^2\phi_n+\kappa_n^2\phi_n=0$. For large N the left hand side (LHS) of Eq. (2.5) can be rewritten as

$$\beta_N\tan\beta_N L\cong\frac{\beta_N^2L-\ell^2\pi^2}{2L}=\frac{(k^2-k_{N\ell}^2)L}{2}. \tag{2.11}$$

Then from Eqs. (2.5), (2.10), and (2.11) we have

$$\frac{L(k^2-k_{N\ell}^2)}{2}=\kappa_N^4\left(\int dS\phi_N\phi_0\right)^2\Bigg/\left[\sum_{n\neq N}\frac{\kappa_n^4\left(\int dS\phi_n\phi_0\right)^2}{\sqrt{\kappa_n^2-k^2}}\right]. \tag{2.12}$$

Using the definition of generalized polarizability introduced earlier [1],

$$\chi\equiv\frac{k^2-k_{N\ell}^2}{k_{N\ell}^2E_{N\ell}^2(0)}, \tag{2.13}$$

we obtain

$$\chi=2\kappa_N^4I_{N0}^2\Bigg/\left[L\left(\sum_{n\neq N}\frac{\kappa_n^4I_{n0}}{\sqrt{\kappa_n^2-k^2}}\right)k_{N\ell}^2E_{N\ell}^2(0)\right], \tag{2.14}$$

where we define $I_{n0}=\int dS\phi_n\phi_0$, and where $E_{N\ell}(0)$ is the normal component of the electric field for mode N, ℓ at the hole location when the hole is absent. The field $\vec{E}_{N\ell}$ is normalized so that its square, integrated over the cavity volume, is unity.

The various terms in Eq. (2.14) can be evaluated explicitly, as shown in Appendix A. Our final result for χ is

$$\chi=\chi^0\left[1-\frac{\widetilde{P}^2a^2}{5}-\frac{\widetilde{Q}^2b^2}{5}-\frac{k^2b^2K(m)}{5E(m)}\right], \tag{2.15}$$

where $\widetilde{P}=P\pi/A$, $\widetilde{Q}=Q\pi/B$, and χ^0 is the static polarizability for an elliptical hole

$$\frac{1}{\chi^0}=\frac{3}{2\pi ab^2}E(m). \tag{2.16}$$

Here $K(m)$ and $E(m)$ are complete elliptic integrals of the 1st and 2nd kind, and $m=1-b^2/a^2$. The quantities $-\widetilde{P}^2$, $-\widetilde{Q}^2$ are the normalized second derivatives of E_n with respect to t and s.

III. GENERALIZED SUSCEPTIBILITY OF AN ELLIPTICAL HOLE

For the frequency calculation we again use Eq. (2.1). For the $\mathrm{TM}_{1N\ell}$ mode we have

$$\beta_N\tan\beta_N L\cong\frac{(K_{N1}^{\mathrm{TM}})^2}{M_{11}^{\psi}/k}, \tag{3.1}$$

and for the $\mathrm{TE}_{1N\ell}$ mode

$$\frac{\tan\beta_N L}{\beta_N}\cong\frac{(K_{N1}^{\mathrm{TE}})^2}{kM_{11}^{\psi}}, \tag{3.2}$$

where M_{11}^{ψ} is given by

$$M_{11}^{\psi}=-\sum_{n\neq N}\frac{k}{\beta_n}\cot\beta_n L K_{n1}^2-\sum_{n\neq N}\frac{\beta_n}{k}\cot\beta_n L K_{n1}^2, \tag{3.3}$$

$$M_{11}^{\psi}=k\sum_{n\neq N}\frac{\coth\rho_n L}{\rho_n}(K_{n1}^{\mathrm{TM}})^2-\frac{1}{k}\sum_{n\neq N}\rho_n'\coth\rho_n' L(K_{n1}^{\mathrm{TE}})^2. \tag{3.4}$$

Here $\rho_n=\sqrt{x_n^2-k^2}$ and $\rho_n'=\sqrt{y_n^2-k^2}$, where x_n^2 and y_n^2 are the eigenvalues in the Helmholtz equation for TM and TE modes, respectively. In the limiting case of large n we have $\coth\rho_n L\rightarrow 1$, and the equation for M_{11}^{ψ} becomes

$$M_{11}^{\psi}=k\sum_{n\neq N}\frac{(K_{n1}^{\mathrm{TM}})^2}{\sqrt{x_n^2-k^2}}-\frac{1}{k}\sum_{n\neq N}\sqrt{y_n^2-k^2}(K_{n1}^{\mathrm{TE}})^2. \tag{3.5}$$

Once again, for large N the LHS of Eq. (3.1) can be rewritten as

$$\beta_N\tan\beta_N L\cong\frac{L}{2}(k^2-k_{N\ell}^2), \tag{3.6}$$

and the LHS of Eq. (3.2) can be rewritten as

$$\frac{\tan\beta_N L}{\beta_N}\cong\frac{L}{2}\left(\frac{L}{\pi\ell}\right)^2(k^2-k_{N\ell}^2). \tag{3.7}$$

Using Eqs. (3.1), (3.2), and (3.5), Eqs. (3.6) and (3.7) become

$$\frac{L}{2}(k^2-k_{N\ell}^2)=k^2(K_{N1}^{\mathrm{TM}})^2/\Sigma \quad \text{for } \mathrm{TM}_{1N\ell}, \tag{3.8}$$

$$\frac{L}{2}(k^2-k_{N\ell}^2)=\pi^2\ell^2(K_{N1}^{\mathrm{TE}})/L^2\Sigma \quad \text{for } \mathrm{TE}_{1N\ell}, \tag{3.9}$$

where

$$\sum \equiv k^2\sum_{n\neq N}(K_{n1}^{\mathrm{TM}})^2/\sqrt{x_n^2-k^2}-\sum_{n\neq N}\sqrt{y_n^2-k^2}(K_{n1}^{\mathrm{TE}})^2. \tag{3.10}$$

In order to obtain the expressions for K_{n1}^{TM}, K_{n1}^{TE} we are guided by our experience for the circular hole [1] and use the trial function whose components are

$$f_{1t}=\sqrt{1-u^2-v^2}-\frac{\partial\phi}{\partial t},\quad f_{1s}=-\frac{\partial\phi}{\partial s}, \tag{3.11}$$

with $\phi=\alpha a u\sqrt{1-u^2-v^2}$, where we define $t=ua$ and $s=vb$. Then for (n, TM)

$$\begin{aligned}K_{n1}^{\mathrm{TM}}&=\int_{S_1}dS\,\mathbf{e}_n\cdot\mathbf{f}_1=-\int dS\,\boldsymbol{\nabla}\phi_n\cdot\mathbf{f}_1\\&=-\int dS\frac{\partial\phi_n}{\partial t}\phi_0+\int dS\,\boldsymbol{\nabla}\phi_n\cdot\boldsymbol{\nabla}\phi\\&=I_{n0}'+I_{n0}'',\end{aligned} \tag{3.12}$$

where $\phi_0=\sqrt{1-u^2-v^2}$,

$$I_{n0}'=b\int\int du\,dv\,\phi_n\left(\frac{\partial\phi_0}{\partial u}\right), \tag{3.13}$$

$$I_{n0}''=x_n^2ab\int\int du\,dv\,\phi_n\phi; \tag{3.14}$$

for (n, TE)

$$\begin{aligned}K_{n1}^{TE}&=\int dS(\hat{\mathbf{n}}\times\nabla\psi_n)\cdot\mathbf{f}_1\\&=a\int\int du\,dv\,\psi_n\frac{\partial\phi_0}{\partial v}-\int dS\left(\frac{\partial\psi_n}{\partial t}\frac{\partial\phi}{\partial s}-\frac{\partial\psi_n}{\partial s}\frac{\partial\phi}{\partial t}\right)\\&=Y_{n0},\end{aligned} \tag{3.15}$$

with

$$Y_{n0}=a\int\int du\,dv\,\psi_n\frac{\partial\phi_0}{\partial v}. \tag{3.16}$$

Using the definition of generalized susceptibility introduced earlier [1],

$$\psi=-\frac{k^2-k_M^2}{k^2H_M^2(0)}, \tag{3.17}$$

Eqs. (3.8) and (3.9) become the variational forms for the susceptibilities, which can be evaluated explicitly, as shown in Appendix B. Here $H_M(0)$ is the normalized tangential component of the magnetic field for mode M at the hole location when the hole is absent.

The final expressions for the susceptibilities are

$$\psi^{\rm TE}=\psi_{ss}^0\Bigg\{1-\frac{\lambda_0^2}{5}+\frac{k^2b^2}{5}\Bigg[2+\frac{K(m)-E(m)}{(m-1)K(m)+E(m)} -2\alpha\frac{K(m)-E(m)}{(m-1)K(m)+E(m)} +\frac{\alpha^2}{1-m}\Bigg(\frac{K(m)-E(m)}{(m-1)K(m)+E(m)} +\frac{m[2E(m)-K(m)]}{(m-1)K(m)+E(m)}\Bigg)\Bigg]\Bigg\}. \tag{3.18}$$

Here ψ_{ss}^0 is the static susceptibilty for an elliptical hole (in the azimuthal direction),

$$\frac{1}{\psi_{ss}^0}=\frac{3[(m-1)K(m)+E(m)]}{2\pi b^2am}, \tag{3.19}$$

and $\lambda_0^2=\widetilde{P}^2a^2+\widetilde{Q}^2b^2$. The quantities $-\widetilde{P}^2,-\widetilde{Q}^2$ are the normalized second derivatives of H_s with respect to t,s and

$$\psi^{\rm TM}=\psi_{ss}^0\Bigg\{1-\frac{\lambda_0^2}{5}\Bigg(1+2\alpha\frac{a^2x_N^2}{\lambda_0^2}\Bigg)+\frac{k^2b^2}{5}\Bigg[2 +\frac{K(m)-E(m)}{(m-1)K(m)+E(m)}-2\alpha\frac{K(m)-E(m)}{(m-1)K(m)+E(m)} +\frac{\alpha^2}{1-m}\Bigg(\frac{K(m)-E(m)}{(m-1)K(m)+E(m)} +\frac{m[2E(m)-K(m)]}{(m-1)K(m)+E(m)}\Bigg)\Bigg]\Bigg\}, \tag{3.20}$$

where $x_N^2=\widetilde{P}^2+\widetilde{Q}^2$. Equations (3.18) and (3.20) are variational forms with respect to the parameter α. The minimum values of ψ occur when $\alpha^{\rm TE}$ and $\alpha^{\rm TM}$ are given by Eqs. (B35) and (B36).

IV. DISCUSSION

In order to obtain a clear physical picture we represent our results in terms of field derivatives at the hole, since the fields must satisfy the Helmholtz equation for finite k.

In the case of the polarizability we rewrite Eq. (2.15) in the following form:

$$\chi=\chi^0\Bigg[1+e_{tt}\frac{a^2}{5}+e_{ss}\frac{b^2}{5}+\frac{k^2b^2}{5}\frac{K(m)}{E(m)}\Bigg], \tag{4.1}$$

with

$$e_{tt}=\frac{\partial^2E_n}{\partial t^2}\Bigg/E_n(0), \tag{4.2}$$

$$e_{ss}=\frac{\partial^2E_n}{\partial s^2}\Bigg/E_n(0). \tag{4.3}$$

It is at first surprising that our results for the susceptibility are different for the TE and TM modes, since the cavity walls have been removed to infinity. But the TE and TM modes have different higher derivatives at the hole. Therefore, we can write a single expression which covers both cases. Specifically, we rewrite Eqs. (3.18) and (3.20) in the following form:

$$\psi=\psi_{ss}^0\Bigg[1+\frac{a^2}{5}h_{tt}^s+\frac{b^2}{5}h_{ss}^s+\frac{k^2b^2}{5}(u+2v\alpha+\alpha^2w) -\frac{2}{5}\alpha a^2(h_{ts}^t-h_{tt}^s)\Bigg], \tag{4.4}$$

which is valid for either a TM or TE mode. Here

$$-h_{tt}^s=-\frac{\partial^2H_s}{\partial t^2}\Bigg/H_s(0)=\widetilde{P}^2=\frac{P^2\pi^2}{A^2}, \tag{4.5}$$

$$-h_{ss}^s=-\frac{\partial^2H_s}{\partial s^2}\Bigg/H_s(0)=\widetilde{Q}^2=\frac{Q^2\pi^2}{B^2}, \tag{4.6}$$

$$h_{ts}^t-h_{tt}^s=\frac{\partial^2H_t}{\partial t\partial s}\Bigg/H_s(0)-\frac{\partial^2H_s}{\partial t^2}/H_s(0)=\begin{cases}\widetilde{P}^2+\widetilde{Q}^2, & \text{TM}\\ 0, & \text{TE}\end{cases}, \tag{4.7}$$

with $H_s(0)$ corresponding to the value of H_s that would exist at the location of the center of the hole in the absence of a hole. The quantities u, v, and w are defined by the relations

$$u=2+\frac{K(m)-E(m)}{(m-1)K(m)+E(m)}, \tag{4.8}$$

$$v=-\frac{K(m)-E(m)}{(m-1)K(m)+E(m)}, \tag{4.9}$$

$$w=\frac{1}{1-m}\Bigg[\frac{(1-m)K(m)+(2m-1)E(m)}{(m-1)K(m)+E(m)}\Bigg]. \tag{4.10}$$

The minimum value of ψ occurs when

$$\alpha=-\frac{v}{w}+\frac{1}{w}\frac{a^2}{b^2k^2}(h_{ts}^t-h_{tt}^s). \tag{4.11}$$

Then we can write

$$\psi=\psi_{ss}^0\Bigg[1+\frac{a^2}{5}h_{tt}^s+\frac{b^2}{5}h_{ss}^s+\frac{k^2b^2}{5}\Bigg(u-\frac{v^2}{w}\Bigg) +\frac{2}{5}\frac{a^2v}{w}(h_{ts}^t-h_{tt}^s)-\frac{1}{5}\frac{a^4}{w}\frac{(h_{ts}^t-h_{tt}^s)^2}{k^2b^2}\Bigg] \tag{4.12}$$

or

$$\psi=\psi_{ss}^{0}\left[1+\frac{a^{2}}{5}h_{tt}^{s}+\frac{b^{2}}{5}h_{ss}^{s}+\frac{k^{2}b^{2}}{5}A+\frac{2}{5}a^{2}B(h_{ts}^{t}-h_{tt}^{s})-\frac{1}{5}\frac{a^{4}}{k^{2}b^{2}}C(h_{ts}^{t}-h_{tt}^{s})^{2}\right],\tag{4.13}$$

with

$$A=2+\frac{mE(m)[K(m)-E(m)]}{[(m-1)K(m)+E(m)]\Delta(m)},\tag{4.14}$$

$$B=\frac{(m-1)[K(m)-E(m)]}{\Delta(m)},\tag{4.15}$$

$$C=\frac{(1-m)[(m-1)K(m)+E(m)]}{\Delta(m)}.\tag{4.16}$$

Here

$$\Delta(m)\equiv(2m-1)E(m)+K(m)(1-m).\tag{4.17}$$

The results obtained above can be used to estimate the frequency corrections for the polarizability and susceptibilities of a hole for different values of b/a with the corresponding field configuration.

In the present paper our main problem of interest is the case when the magnetic field (in the azimuthal direction) is perpendicular to the longest dimension of the elliptical hole. In Appendix C we consider the case where the magnetic field is parallel to the long dimension of the hole.

V. THE COUPLING IMPEDANCE OF A LONG NARROW ELLIPTICAL SLOT

In the design of modern high-intensity accelerators, narrow slots oriented along the chamber axis are the best choice for pumping holes, giving low impedance for a finite hole area. Therefore, in this section we apply our results to estimate the coupling impedance of a long narrow elliptical slot. For this we consider a beam traveling along the axis of a beam pipe. We also consider a long and narrow elliptical slot in the pipe wall. Therefore, we can write approximately $m=1-(b^2/a^2)\rightarrow 1$, $E(m)\rightarrow 1$, and $K(m)\rightarrow \ln(4a/b)$. Denoting the t direction by z and using Eq. (4.1), with $E_n\rightarrow e^{-jkz}$, $e_{tt}=-k^2$, and $e_{ss}=0$, for the polarizability of such a slot we obtain

$$\chi=\chi^{0}\left(1-\frac{k^{2}a^{2}}{5}\right).\tag{5.1}$$

In the same limit, using Eq. (4.4), the expression for the susceptibility becomes

$$\psi=\psi_{ss}^{0}\left(1-2\frac{k^{2}a^{2}}{5}+(\alpha-1)^{2}\frac{k^{2}a^{2}}{5}\right).\tag{5.2}$$

The minimum of this expression occurs when $\alpha=1$, giving

$$\psi=\psi_{ss}^{0}\left(1-\frac{2}{5}k^{2}a^{2}\right).\tag{5.3}$$

The definition of the longitudinal coupling impedance is

$$Z_{\parallel}(k)=-\frac{1}{I_0}\int_{-\infty}^{\infty}dzE_z(0,z;k)e^{jkz},\tag{5.4}$$

where the z direction is along the pipe axis. The integral can be rewritten in the form

$$|I_0^2|Z_{\parallel}(k)=-\int_{\text{hole}}dSE_zH_{1\theta},\tag{5.5}$$

where the subscript 1 denotes the fields in the lossless pipe without the obstacle.

For holes whose dimensions are small compared to the wavelength, the integral can be expressed in terms of the fields $E_{1r},H_{1\theta}$ near the hole and the electric polarizability and magnetic susceptibility of the hole. In our case, we have

$$\frac{Z_{\parallel}(k)}{Z_0}=\frac{jk}{8\pi^{2}R^{2}}(\psi-\chi),\tag{5.6}$$

where R is the radius of the pipe and ψ, χ are the generalized susceptibility and polarizability of a hole, respectively [1]. Then, using Eqs. (5.1) and (5.3), we have

$$\frac{Z_{\parallel}(k)}{Z_0}=\frac{jk}{8\pi^{2}R^{2}}\frac{\psi_{ss}^{0}\chi^{0}}{\psi_{tt}^{0}}\left(1-\frac{k^{2}a^{2}}{5}\frac{\psi_{tt}^{0}}{\psi_{ss}^{0}}\right),\tag{5.7}$$

where we used the relation $1/\psi_{tt}^0+1/\psi_{ss}^0=1/\chi^0$ for an elliptical hole and the fact that $\psi_{tt}^0\gg\psi_{ss}^0$. Using the same approximation of a narrow slot, the expressions for the static polarizabilty and susceptibilities become

$$\chi^{0}=\frac{2\pi ab^{2}}{3E(m)}\rightarrow\frac{2\pi}{3}ab^{2},\tag{5.8}$$

$$\psi_{ss}^{0}=\frac{2\pi ab^{2}m}{3[(m-1)K(m)+E(m)]}\rightarrow\frac{2\pi}{3}ab^{2},\tag{5.9}$$

$$\psi_{tt}^{0}=\frac{2\pi a^{3}m}{3[K(m)-E(m)]}\rightarrow\frac{2\pi}{3}\frac{a^{3}}{[\ln(4a/b)-1]}.\tag{5.10}$$

Finally we obtain

$$\frac{Z_{\parallel}(k)}{Z_0}=\frac{jkab^{2}}{12\pi R^{2}}\left\{\frac{b^{2}}{a^{2}}\left[\ln\left(\frac{4a}{b}\right)-1\right]-\frac{k^{2}a^{2}}{5}\right\}.\tag{5.11}$$

In Eq. (5.11) the first term is the static approximation and the second term is the new correction obtained by considering the frequency dependence of the polarizability and susceptibilities. Note that the frequency dependent correction reduces the inductive impedance obtained in the static approximation.

For commonly used values of $b/a\approx 1/5$, the frequency correction becomes important even for $ka\approx 0.3$. And in the case where $b\ll a$ the impedance behavior will be essentially that of the correction term. Therefore, the frequency correction in such cases is very important.

For the rectangular narrow slot, in the static approximation the numerical result contains terms proportional to w^4/ℓ and w^3, with terms proportional to $w^2\ell$ being canceled, where w and ℓ are the width and the length of the slot, respectively [5]. By considering the frequency dependence, in analogy with an elliptical slot, one would now expect to have similar terms proportional to $w^2\ell(k^2\ell^2)$, which will play an essential role for small values of w/ℓ.

VI. SUMMARY

We used the method developed earlier [1] to obtain the frequency dependence of the polarizability and susceptibilities of an elliptical hole in a thin plane metallic screen. We then used the results to estimate the coupling impedance of a long narrow elliptical slot in a beam pipe.

The frequency correction of the impedance turns out to be much larger than the static value for long narrow slots. Therefore it is important to take this correction into account, when one works with a long narrow slot. We should note that our results are for a thin metallic screen where the thickness is less than the slot width. We also note that our results for an elliptical hole in a plane wall are a good approximation for a slot in a real beam pipe, where the slot is parallel to the beam. But when a long narrow slot is perpendicular to the beam one has to take into account the curvature of the beam pipe.

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APPENDIX A: EXPLICIT DERIVATION OF THE POLARIZABILITY

Here we present the detailed calculation of the polarizability from the variational form of Eq. (2.14). We first find the electric field normal to the hole at the hole location in the absence of the hole. We write

$$\mathbf{E}_\perp = C\mathbf{e}_N \sin\frac{\ell\pi n}{L}, \quad \mathbf{e}_N = -\nabla\phi_N, \tag{A1}$$

$$E_n = C\frac{L}{\ell\pi}\phi_N\kappa_N^2\cos\frac{\ell\pi n}{L}. \tag{A2}$$

Then from $\int E^2 dv = 1$ we get

$$C^2 = \frac{2\ell^2\pi^2}{L^3 k_{N\ell}^2} \tag{A3}$$

and

$$k_{N\ell}^2 E_{N\ell}^2 = \frac{2\kappa_N^4\phi_N^2(0)}{L}. \tag{A4}$$

Therefore for the generalized polarizability we have

$$\chi = [I_{N0}^2] \Bigg/ \left[\phi_N^2(0)\sum_{n\neq N}\frac{\kappa_n^4 I_{n0}^2}{\sqrt{\kappa_n^2 - k^2}}\right]. \tag{A5}$$

We obtain the expressions for ϕ_n and ϕ_N by considering the rectangular form of the cavity waveguide

$$\phi_n = \tilde{C}\cos\frac{p\pi t}{A}\cos\frac{q\pi s}{B},$$

$$\phi_N = \tilde{C}\cos\frac{P\pi t}{A}\cos\frac{Q\pi s}{B}, \tag{A6}$$

where p, q, P, and Q are odd numbers, in order to satisfy the boundary conditions at $t = \pm A/2$ and $s = \pm B/2$. We note that P and Q are introduced in order to distinguish the specific mode N. Here

$$\kappa_n^2 = \left(\frac{p\pi}{A}\right)^2 + \left(\frac{q\pi}{B}\right)^2, \tag{A7}$$

$$\int \phi_n^2 dS = \frac{1}{\kappa_n^2} = \tilde{C}^2\frac{AB}{4}, \tag{A8}$$

which gives $\tilde{C} = 2/\kappa_n\sqrt{AB}$. Then

$$I_{n0} = \int dS\,\phi_n\phi_0$$
$$= \frac{2}{\sqrt{AB}}\int\int\frac{dt\,ds}{\kappa_n}\sqrt{1 - \frac{t^2}{a^2} - \frac{s^2}{b^2}}\cos\frac{p\pi t}{A}\cos\frac{q\pi s}{B}, \tag{A9}$$

which becomes

$$\kappa_n I_{n0} = \frac{4\pi ab}{\sqrt{AB}}\int_0^1 \sigma d\sigma\sqrt{1-\sigma^2}J_0(\sigma\sqrt{\mu^2+\nu^2})$$
$$= \frac{4\pi ab}{\sqrt{AB}}\frac{j_1(\sqrt{\mu^2+\nu^2})}{\sqrt{\mu^2+\nu^2}}, \tag{A10}$$

where $\mu = p\pi a/A$, $\nu = q\pi b/B$, and $j_1(\sqrt{\mu^2+\nu^2})$ is the spherical Bessel function of order 1.

For large A, B, p, and q we can convert the sum over p and q in Eq. (A5) to an integral by writing

$$\frac{2a\pi}{A}\sum_p = \int d\mu, \quad \frac{2b\pi}{B}\sum_q = \int d\nu. \tag{A11}$$

Equation (A5) then becomes

$$\chi = \frac{4\pi^2 ab j_1^2(\lambda_0)/\lambda_0^2}{\int\int d\mu d\nu\frac{(\mu^2/a^2+\nu^2/b^2)j_1^2(\lambda)}{\lambda^2\sqrt{\mu^2/a^2+\nu^2/b^2-k^2}}}, \tag{A12}$$

where $\lambda_0^2 = P^2\pi^2a^2/A^2 + Q^2\pi^2b^2/B^2$, $\lambda^2 = \mu^2+\nu^2$, and where the limits on μ and ν are extended to $-\infty$ to ∞. By making the change of variables $\mu = \lambda\cos\alpha$, $\nu = \lambda\sin\alpha$, the integral in the denominator of Eq. (A12) can be rewritten as

$$\int_0^\infty d\lambda\, j_1^2(\lambda)\int_0^{2\pi}d\alpha\left(\sqrt{\cos^2\alpha/a^2+\sin^2\alpha/b^2} + \frac{k^2}{2\lambda^2\sqrt{\cos^2\alpha/a^2+\sin^2\alpha/b^2}}\right). \tag{A13}$$

We will need the results

$$\int_0^{2\pi} d\alpha \sqrt{\cos^2\alpha/a^2+\sin^2\alpha/b^2}=4E(m)/b, \quad \text{(A14)}$$

where

$$E(m)\equiv\int_0^{\pi/2} d\phi\sqrt{1-(1-b^2/a^2)\sin^2\phi} \quad \text{(A15)}$$

is an elliptic integral of the second kind, with $m\equiv 1-b^2/a^2$,

$$\int_0^{2\pi}\frac{d\alpha}{\sqrt{\cos^2\alpha/a^2+\sin^2\alpha/b^2}}=4bK(m), \quad \text{(A16)}$$

where

$$K(m)\equiv\int_0^{\pi/2}\frac{d\phi}{\sqrt{1-m\sin^2\phi}} \quad \text{(A17)}$$

is an elliptic integral of the first kind, and

$$\int_0^{\infty} j_1^2(\lambda)d\lambda=\pi/6, \quad \text{(A18)}$$

$$\int_0^{\infty}\frac{j_1^2(\lambda)d\lambda}{\lambda^2}=\pi/15. \quad \text{(A19)}$$

With the aid of these results, Eq. (A12) can be written in the form

$$\chi=\left[\frac{\pi ab j_1^2(\lambda_0)}{\lambda_0^2}\right]\Bigg/\left[\frac{E(m)}{6b}\left(1+\frac{k^2b^2}{5}\frac{K(m)}{E(m)}\right)\right]. \quad \text{(A20)}$$

By expanding the numerator for small λ_0, $j_1^2(\lambda_0)/\lambda_0^2=(1-\lambda_0^2/5)/9$, we have

$$\chi=\chi^0\left[1-\frac{P^2\pi^2a^2}{5A^2}-\frac{Q^2\pi^2b^2}{5B^2}-\frac{k^2b^2K(m)}{5E(m)}\right] \quad \text{(A21)}$$

or

$$\chi=\chi^0\left[1-\frac{\tilde{P}^2a^2}{5}-\frac{\tilde{Q}^2b^2}{5}-\frac{k^2b^2K(m)}{5E(m)}\right], \quad \text{(A22)}$$

where $\tilde{P}=P\pi/A$, $\tilde{Q}=Q\pi/B$, and χ^0 is the static polarizability for an elliptical hole [3],

$$\frac{1}{\chi^0}=\frac{3}{2\pi ab^2}E(m). \quad \text{(A23)}$$

APPENDIX B: EXPLICIT DERIVATION OF THE SUSCEPTIBILITY

Here we present the detailed calculation of the susceptibilities from the variational form of Eqs. (3.8) and (3.9). We first find the magnetic field tangent to the hole at the hole location in the absence of the hole.

For TM modes we write

$$H_n=0, \quad \text{(B1)}$$

$$\mathbf{H}_\perp=C\hat{\mathbf{n}}\times\boldsymbol{\nabla}\phi\cos\frac{\ell\pi n}{L}. \quad \text{(B2)}$$

Then from

$$\int dvH^2=1 \quad \text{(B3)}$$

we have $C^2=2/L$ and

$$H_M^2(0)=\frac{2}{L}(\nabla\phi_N(0))^2. \quad \text{(B4)}$$

For TE modes we write

$$\mathbf{H}_\perp=\tilde{C}\boldsymbol{\nabla}\psi\cos\frac{\ell\pi n}{L}, \quad \text{(B5)}$$

$$H_n=-\tilde{C}\frac{L}{\ell\pi}y_N^2\psi\sin\frac{\ell\pi n}{L}. \quad \text{(B6)}$$

Then from Eq. (B3) we obtain $\tilde{C}^2=2\ell^2\pi^2/L^3k_{N\ell}^2$ and

$$H_M^2(0)=\frac{2\ell^2\pi^2}{L^3k_{N\ell}^2}(\nabla\psi_N(0))^2. \quad \text{(B7)}$$

Using Eqs. (3.8), (3.9), and (3.17) we have

$$\psi^{\rm TM}=\frac{-(I'_{N0}+I''_{N0})^2}{(\nabla\phi_N(0))^2\Sigma}, \quad \text{(B8)}$$

$$\psi^{\rm TE}=\frac{-Y_{N0}^2}{(\nabla\psi_N(0))^2\Sigma}, \quad \text{(B9)}$$

where Σ, defined in Eq. (3.10), is now written as

$$\sum=k^2\sum_{n\neq N}(I'_{n0}+I''_{n0})^2/\sqrt{x_n^2-k^2}-\sum_{n\neq N}\sqrt{y_n^2-k^2}Y_{n0}^2. \quad \text{(B10)}$$

In order to find expressions for ϕ_N and ψ_N we consider the same rectangular cavity as in the case of the polarizability. We choose the magnetic field to be in the s direction. Therefore, for the TM case we have

$$\phi_n=C\sin\frac{p\pi t}{A}\cos\frac{q\pi s}{B}, \quad \text{(B11)}$$

where p is even and q is odd, in order to satisfy the proper boundary conditions at $t=\pm A/2$, $s=\pm B/2$, respectively, and $C=2/x_n\sqrt{AB}$. Using Eq. (3.13),

$$x_nI'_{n0}=-\frac{2b}{\sqrt{AB}}\int\int udud v\sin\left(\frac{p\pi au}{A}\right)\times\cos\left(\frac{q\pi bv}{B}\right)\Bigg/\sqrt{1-u^2-v^2}, \quad \text{(B12)}$$

which becomes

$$x_n I'_{n0} = -\frac{4\pi b\mu}{\sqrt{AB}\sqrt{\mu^2+\nu^2}} \times \int_0^1 \frac{d\sigma\sigma^2 J_1(\sigma\sqrt{\mu^2+\nu^2})}{\sqrt{1-\sigma^2}}, \tag{B13}$$

where $\mu = p\pi a/A$, $\nu = q\pi b/B$. And using

$$\int_0^1 d\sigma\sigma^2(1-\sigma^2)^{\nu-1/2} J_1(\sigma\lambda) = \frac{\Gamma(\nu+1/2)J_{\nu+3/2}(\lambda)}{2(\lambda/2)^{\nu+1/2}}, \tag{B14}$$

we obtain

$$x_n I'_{n0} = -\frac{4\pi b\mu j_1(\sqrt{\mu^2+\nu^2})}{\sqrt{AB}\sqrt{\mu^2+\nu^2}}. \tag{B15}$$

Using Eq. (3.14),

$$x_n I''_{n0} = \frac{2ba^2x_n^2}{\sqrt{AB}} \int\int u\,du\,dv \times \sin\left(\frac{p\pi au}{A}\right)\cos\left(\frac{q\pi bv}{B}\right)\alpha\sqrt{1-u^2-v^2}, \tag{B16}$$

which becomes

$$x_n I''_{n0} = \frac{4\pi b\mu\alpha a^2 x_n^2 j_2(\sqrt{\mu^2+\nu^2})}{\sqrt{AB}(\mu^2+\nu^2)}. \tag{B17}$$

For the TE case

$$\psi_n = \widetilde{C}\cos\frac{p\pi t}{A}\,\sin\frac{q\pi s}{B}, \tag{B18}$$

where p is even and q is odd, in order to satisfy the proper boundary conditions at $t = \pm A/2$, $s = \pm B/2$, respectively. Using Eq. (3.16), we find

$$y_n Y_{n0} = -\frac{2a}{\sqrt{AB}} \int\int v\,du\,dv \times \cos\left(\frac{p\pi au}{A}\right)\sin\left(\frac{q\pi bv}{B}\right) \Big/ \sqrt{1-u^2-v^2}, \tag{B19}$$

from which we obtain

$$y_n Y_{n0} = -\frac{4\pi a\nu j_1(\sqrt{\mu^2+\nu^2})}{\sqrt{AB}\sqrt{\mu^2+\nu^2}}. \tag{B20}$$

By converting the sum over p and q in Eq. (B9) to an integral, for the TE case we have

$$\psi^{\rm TE} = -\frac{4\pi^2a^3bj_1^2(\lambda_0)}{k^2\lambda_0^2} \Bigg/ \left[\int\int \frac{d\mu\,d\nu(I'_{n0}+I''_{n0})^2}{\sqrt{\mu^2/a^2+\nu^2/b^2-k^2}} - \frac{a^2}{k^2b^2}\int\int \frac{d\mu\,d\nu\,\nu^2 j_1^2(\lambda)\sqrt{\mu^2/a^2+\nu^2/b^2-k^2}}{\lambda^2(\mu^2/a^2+\nu^2/b^2)}\right]. \tag{B21}$$

Using Eqs. (B15) and (B17) we rewrite the first integral in the denominator of Eq. (B21) as $I_1+I_2+I_3$, where

$$I_1 = \int\int \frac{d\mu\,d\nu\,\mu^2 j_1^2(\sqrt{\mu^2+\nu^2})}{\sqrt{\frac{\mu^2}{a^2}+\frac{\nu^2}{b^2}-k^2}\left(\frac{\mu^2}{a^2}+\frac{\nu^2}{b^2}\right)(\mu^2+\nu^2)}, \tag{B22}$$

$$I_2 = \int\int d\mu\,d\nu \frac{-2\alpha a^2\mu^2 j_1(\sqrt{\mu^2+\nu^2})j_2(\sqrt{\mu^2+\nu^2})}{\sqrt{\mu^2/a^2+\nu^2/b^2-k^2}[(\mu^2+\nu^2)^{3/2}]}, \tag{B23}$$

$$I_3 = \int\int d\mu\,d\nu \frac{\alpha^2a^2(\mu^2/a^2+\nu^2/b^2)\mu^2 j_2^2(\sqrt{\mu^2+\nu^2})}{\sqrt{\mu^2/a^2+\nu^2/b^2-k^2}[(\mu^2+\nu^2)^2]}, \tag{B24}$$

for which we obtain

$$I_1 = \frac{4b^3\pi}{15m}\left[\frac{E(m)+(m-1)K(m)}{1-m}\right], \tag{B25}$$

with $m = 1-b^2/a^2$,

$$I_2 = -2\alpha a^2\frac{\pi}{30}\frac{4b}{m}[K(m)-E(m)], \tag{B26}$$

and

$$I_3 = \alpha^2a^4\frac{\pi}{10}\frac{4}{3bm}\Delta(m), \tag{B27}$$

where $\Delta(m)$ is given in Eq. (4.17). The second integral in the denominator of Eq. (B21) can be rewritten as I_0+I_4, where

$$I_0 = \frac{4b}{m}\frac{\pi}{6}[E(m)+(m-1)K(m)] \tag{B28}$$

and

$$I_4 = -\frac{k^2}{2}\frac{\pi}{15}\frac{4b^3}{m}[K(m)-E(m)]. \tag{B29}$$

Putting Eqs. (B25)–(B29) in Eq. (B21) and expanding $j_1^2(\lambda_0)/\lambda_0^2$ for small λ_0, we finally obtain

$$\psi^{\rm TE}=\psi^0_{ss}\left\{1-\frac{\lambda_0^2}{5}+\frac{k^2b^2}{5}\left[2+\frac{K(m)-E(m)}{(m-1)K(m)+E(m)}\right.\right.$$
$$-2\alpha\frac{K(m)-E(m)}{(m-1)K(m)+E(m)}$$
$$+\frac{\alpha^2}{1-m}\left(\frac{K(m)-E(m)}{(m-1)K(m)+E(m)}\right.$$
$$\left.\left.\left.+\frac{m[2E(m)-K(m)]}{(m-1)K(m)+E(m)}\right)\right]\right\}, \tag{B30}$$

where ψ^0_{ss} is the static susceptibility of an elliptical hole [3],

$$\frac{1}{\psi^0_{ss}}=\frac{3[(m-1)K(m)+E(m)]}{2\pi b^2 am}. \tag{B31}$$

Note that for the circular hole limit

$$b\to a,\quad m=1-b^2/a^2\to 0,$$

$$E(m)\to\frac{\pi}{2}(1-m/4),\quad K(m)\to\frac{\pi}{2}(1+m/4),$$

we recover our result for the circular hole [1],

$$\psi^{\rm TE}=\psi^0\left[1-\frac{\lambda_0^2}{5}+\frac{k^2a^2}{5}(3-2\alpha+3\alpha^2)\right]. \tag{B32}$$

For the TM modes we use Eq. (B8) and, after the same procedure as for the TE modes, we obtain

$$\psi^{\rm TM}=\psi^0_{ss}\left\{1-\frac{\lambda_0^2}{5}\left(1+2\alpha\frac{a^2x_N^2}{\lambda_0^2}\right)+\frac{k^2b^2}{5}\left[2\right.\right.$$
$$+\frac{K(m)-E(m)}{(m-1)K(m)+E(m)}-2\alpha\frac{K(m)-E(m)}{(m-1)K(m)+E(m)}$$
$$+\frac{\alpha^2}{1-m}\left(\frac{K(m)-E(m)}{(m-1)K(m)+E(m)}\right.$$
$$\left.\left.\left.+\frac{m[2E(m)-K(m)]}{(m-1)K(m)+E(m)}\right)\right]\right\}. \tag{B33}$$

For the circular hole limit we again recover our earlier result [1],

$$\frac{\psi^{\rm TM}}{\psi^0}=1-\frac{\lambda_0^2}{5}(1+2\alpha)+\frac{k^2a^2}{5}(3-2\alpha+3\alpha^2). \tag{B34}$$

We now use the fact that Eqs. (B30) and (B33) are variational forms with respect to the parameter α. The minimum values of ψ occur when

$$\alpha^{\rm TE}=\frac{(m-1)[K(m)-E(m)]}{(m-1)K(m)+E(m)(1-2m)}, \tag{B35}$$

$$\alpha^{\rm TM}=\frac{(m-1)[K(m)-E(m)-x_N^2a^2/k^2b^2]}{(m-1)K(m)+E(m)(1-2m)}, \tag{B36}$$

leading finally to

$$\psi^{\rm TE}=\psi^0_{ss}\left[1+k^2b^2\left(D-\frac{\lambda_0^2}{5k^2b^2}\right)\right], \tag{B37}$$

$$\psi^{\rm TM}=\psi^0_{ss}\left[1+k^2b^2\left(D-\frac{\lambda_0^2}{5k^2b^2}F+\frac{3}{5k^4b^4}G\right)\right], \tag{B38}$$

where

$$D=\frac{1}{5}\left[2+\frac{E(m)m[K(m)-E(m)]}{[(m-1)K(m)+E(m)]\Delta(m)}\right], \tag{B39}$$

$$F=1+\frac{a^2x_N^2}{\lambda_0^2}\frac{(1-m)[K(m)-E(m)]}{\Delta(m)}, \tag{B40}$$

$$G=a^4x_N^4\frac{(1-m)[E(m)+K(m)(m-1)]}{\Delta(m)}. \tag{B41}$$

Here $x_N^2=P^2\pi^2/A^2+Q^2\pi^2/B^2$, $\lambda_0^2=P^2\pi^2a^2/A^2+Q^2\pi^2b^2/B^2$.

APPENDIX C: THE MAGNETIC FIELD ALONG THE LONGEST DIMENSION OF THE HOLE

Here we present results for the case when the magnetic field is aligned along the longest dimension of the hole. We again choose the magnetic field in the azimuthal direction. Then we have $b>a$ and the elliptic integrals in our expressions should be changed according to

$$E(m)\to\frac{b}{a}E(\tilde{m}), \tag{C1}$$

$$K(m)\to\frac{a}{b}K(\tilde{m}), \tag{C2}$$

$$m\to-\frac{b^2}{a^2}\tilde{m}, \tag{C3}$$

with $\tilde{m}=1-a^2/b^2$.

Then χ in Eq. (4.1) should be replaced by

$$\tilde{\chi}=\chi^0\left[1+e_{tt}\frac{a^2}{5}+e_{ss}\frac{b^2}{5}+\frac{k^2a^2}{5}\frac{K(\tilde{m})}{E(\tilde{m})}\right]. \tag{C4}$$

Correspondingly, ψ in Eq. (4.4) should be replaced by

$$\tilde{\psi}=\psi^0_{tt}\left[1+\frac{a^2}{5}h^s_{tt}+\frac{b^2}{5}h^s_{ss}+\frac{k^2b^2}{5}(\tilde{u}+2\tilde{v}\alpha+\alpha^2\tilde{w})\right.$$
$$\left.-\frac{2}{5}\alpha a^2(h^t_{ts}-h^s_{tt})\right], \tag{C5}$$

where

$$\tilde{u}=2+\frac{K(\tilde{m})(\tilde{m}-1)+E(\tilde{m})}{K(\tilde{m})-E(\tilde{m})}, \tag{C6}$$

$$\widetilde{v}=-\frac{K(\widetilde{m})(\widetilde{m}-1)+E(\widetilde{m})}{K(\widetilde{m})-E(\widetilde{m})}, \tag{C7}$$

$$\widetilde{w}=\frac{(\widetilde{m}-1)K(\widetilde{m})+(\widetilde{m}+1)E(\widetilde{m})}{K(\widetilde{m})-E(\widetilde{m})}. \tag{C8}$$

Finally

$$\widetilde{\psi}=\psi_{tt}^{0}\left[1+\frac{a^2}{5}h_{tt}^{s}+\frac{b^2}{5}h_{ss}^{s}+\frac{k^2b^2}{5}\widetilde{A}+\frac{2}{5}a^2\widetilde{B}(h_{ts}^{t}-h_{tt}^{s}) -\frac{1}{5}\frac{a^4}{k^2b^2}\widetilde{C}(h_{ts}^{t}-h_{tt}^{s})^2\right], \tag{C9}$$

with

$$\widetilde{A}=2+\frac{\widetilde{m}E(\widetilde{m})}{K(\widetilde{m})-E(\widetilde{m})}\left[\frac{K(\widetilde{m})(\widetilde{m}-1)+E(\widetilde{m})}{(\widetilde{m}+1)E(\widetilde{m})+K(\widetilde{m})(\widetilde{m}-1)}\right], \tag{C10}$$

$$\widetilde{B}=-\frac{K(\widetilde{m})(\widetilde{m}-1)+E(\widetilde{m})}{(\widetilde{m}+1)E(\widetilde{m})+K(\widetilde{m})(\widetilde{m}-1)}, \tag{C11}$$

$$\widetilde{C}=\frac{K(\widetilde{m})-E(\widetilde{m})}{(\widetilde{m}+1)E(\widetilde{m})+K(\widetilde{m})(\widetilde{m}-1)}. \tag{C12}$$

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