Inverse Faraday effect and propagation of circularly polarized intense laser beams in plasmas

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The magnetic field generation through inverse Faraday effect and its effects on the propagation of a circularly polarized light wave are studied in a self-consistent way for relativistic intensities. The following results are presented. (i) The magnetic field is produced by two sources, the circular motion of single electrons which produces plasma magnetization, and the inhomogeneity of both the electron density and light intensity which produces nonzero currents in the azimuthal direction. The magnetic field is calculated for various profiles of electron density and light intensity. (ii) For the case of a plane wave in a homogeneous plasma, the cutoff frequency is calculated as a function of light intensity, which is different from that without consideration of magnetic field generation. An ultra-intense magnetic field as large as hundreds of MG is obtainable in an overdense plasma where the wave can propagate owing to the induced transparency. (iii) The evolution equations for a laser beam of finite width are derived. Due to magnetic field generation, the critical power for self-focusing of the laser beam is reduced by a factor of $(1+\omega_p^2/\omega^2)^{-1}$; the magnetic field tends to reduce the effect of the electron cavitation resulting from the transverse ponderomotive force. [S1063-651X(96)04908-2]

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I. INTRODUCTION

Relativistic interaction of electromagnetic (em) waves with plasma was first investigated about 40 years ago by Akhiezer and Polovin [1], who showed the complicated nonlinear coupling even between homogeneous plasma and plane em waves. Subsequent studies on the interactions between optical beams of finite width and inhomogeneous plasmas, which are concerned with the concept of laser-ignited inertial confinement fusion, have discovered a variety of more complicated structures of laser-plasma coupling. One of them is the self-focusing (or self-trapping) of intense laser beams in plasmas [2–4]. In the powerful laser field, the electron mass increases due to the relativistic motion. This effect modifies the plasma refractive index in such a way that the refractive index is larger in high intensity regions than that in low intensity regions, and hence results in the self-focusing of the beams. Then, Sun *et al.* noted the fact that the ponderomotive force of the laser beam tends to expel plasma [5], and the expulsive force results in a lowered electron density or even electron cavitation channel in the high intensity region. This channel is beneficial for the focusing of the laser beam. In general, both the relativistic effect and the displacement of electrons due to the ponderomotive force should be taken into account. An analytical treatment of similar model equations was given by Kurki-Suonio, Morrison, and Tajima for the case when the electron cavitation does not occur [6]. Borisov *et al.* further extended the same problem by including inhomogeneity of plasma density [7]. These investigations are expected to be valid for beam length (or pulse duration) much longer than the width and at times before significant motion of ions. Recently, two-dimensional self-focusing of short intense laser pulse was studied [8–10]. It has been supposed that this kind of self-focusing is useful for the plasma based particle accelerators [11] and the recently proposed concept of the fast ignitor [12].

However, the studies on self-focusing of laser beams are still not complete, even for the case of a circularly polarized light wave. As early as the 1970s, the excitation of a magnetic field by a circularly polarized em wave in plasma, known as the inverse Faraday effect [13], was found experimentally. It was shown that in the low intensity limit, the magnetic field produced is proportional to the intensity of the incident wave. Therefore the magnetic field produced can be as large as tens of MG or even larger when the incident light wave is at relativistic intensities. It can largely modify the propagation of the light wave. To our knowledge, this problem has not received sufficient attention up to now.

In this work, we will study the following problems. First, we calculate the magnetic field generation in a self-consistent way, clarifying that the magnetic field has two sources. (1) One source is related to the circular motion of single electrons in the wave which is equivalent to a magnetic dipole. The superposition of all the magnetic dipoles constitutes the magnetization of the plasma [14]. (2) The other source is related to the inhomogeneity of both the electron density and the intensity of the laser beam. If there is no such inhomogeneity, the latter will contribute nothing. In earlier studies, the magnetic field was not calculated self-consistently, i.e., the magnetic field was not taken into account in the equation of motion [15–18]. Meanwhile, either the first or the second source was ignored.

Secondly, the dispersion relation of a circularly polarized relativistic wave is reconsidered, now taking magnetic field generation into account. The cutoff frequency is studied as a function of the intensity of the wave, which is shown more clearly than that given in [14].

Thirdly, self-focusing of laser beams is studied, including the self-generated magnetic field. This improves earlier results in several ways, for example, the critical power for self-focusing is modified, and the electron cavitation channel is reduced. In our study we assume that the laser beam is

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short enough to neglect ion motion, while long enough as compared to the beam width to neglect the longitudinal variation. The latter allows us to treat the problem of beam self-focusing in one dimension, a method adopted in some earlier studies [2–7].

In Sec. II the calculation of the magnetic field produced through inverse Faraday effect is given. Section III presents the dispersion relation of a plane circularly polarized wave. Section IV gives the evolution equation of a laser beam with self-generated magnetic fields taken into account. The stationary eigenmode solutions of the evolution equation are found numerically in Sec. V. A summary of the results is given in Sec. VI.

II. MAGNETIC FIELD GENERATION

For a cold relativistic electron fluid,

$$\frac{d\mathbf{P}}{dt}=-e\left(\mathbf{E}_L-\nabla\cdot\Phi+\frac{1}{c}\,\mathbf{v}\times(\mathbf{B}_L+\mathbf{B}_s)\right), \tag{1}$$

where $\mathbf{E}_L$ and $\mathbf{B}_L$ are the electric and magnetic field of the laser beam, respectively, and Φ and $\mathbf{B}_s$ are the slowly varying electric potential and the longitudinal magnetic field along the direction of laser propagation either produced through inverse Faraday effect or externally applied. Taking the electric field $\mathbf{E}_L$ in the form

$$\mathbf{E}_L=\tfrac{1}{2}E_0(\hat{\mathbf{e}}_x+i\lambda\hat{\mathbf{e}}_y)\exp(ikz-i\omega t+i\psi_0)+\text{c.c.}, \tag{2}$$

where $E_0(x,y,z,t)$ is the real amplitude, slowly varying in time and space, λ is either equal to 1 or -1, corresponding to right- and left-circular polarization, respectively, $\hat{\mathbf{e}}_x$ and $\hat{\mathbf{e}}_y$ are the unit vectors in x and y direction, ψ_0 is a phase slowly varying in time and space, used to enforce $\nabla\cdot\mathbf{E}_L=0$ for pure transverse waves, and c.c. denotes the complex conjugate terms. From this, we find the oscillation velocity at fundamental frequency is

$$\mathbf{v}_\perp=-\frac{i\eta}{2}\,\frac{eE_0}{m\omega\gamma}\,(\hat{\mathbf{e}}_x+i\lambda\hat{\mathbf{e}}_y)\exp(i\psi)+\text{c.c.}, \tag{3}$$

where $\eta=(1-\lambda\omega_c/\omega\gamma)^{-1}$, $\gamma=1/\sqrt{1-|\mathbf{v}|^2/c^2}$ is the relativistic factor, $\omega_c=eB_s/mc$ is the cyclotron frequency, and $\psi=kz-\omega t+\psi_0$. From this equation, we find the relativistic factor in the form

$$\gamma=[1+(\eta eE_0/m\omega c)^2]^{1/2}, \tag{4}$$

which differs from that without a constant magnetic field by a factor of η. This factor has been ignored in some earlier studies, thereby losing the self-consistency [15–18]. Here the longitudinal velocity of electrons is ignored on the assumption that the intensity of the laser beam changes slowly in the longitudinal direction. From the continuity equation, we find the density perturbation of electrons at fundamental frequency

$$\begin{aligned} n&=-\frac{i}{\omega}\,\nabla\cdot(n_0\mathbf{v}_\perp)\\ &=-\frac{eE_0}{2m\omega\gamma}\,(\hat{\mathbf{e}}_x+i\lambda\hat{\mathbf{e}}_y)\exp(i\psi)\cdot\nabla\left(\frac{\eta n_0}{\gamma}\right)+\text{c.c.}, \end{aligned} \tag{5}$$

where n_0 is the slowly varying electron density. In deriving this equation, it is important to use the condition $\nabla\cdot\mathbf{E}_L=0$; Otherwise, incorrect results are obtained [17,18]. The slowly varying current is calculated by time averaging

$$\mathbf{J}_s=-\langle en\mathbf{v}\rangle=\frac{\lambda\,\eta e^3E_0^2}{2\gamma m^2\omega^3}\left(\hat{\mathbf{e}}_y\,\frac{\partial}{\partial x}-\hat{\mathbf{e}}_x\,\frac{\partial}{\partial y}\right)\left(\frac{\eta n_0}{\gamma}\right), \tag{6}$$

where $\langle\,\rangle$ denotes time average over ω^{-1}. We notice that this current depends on the inhomogeneity of both the plasma density and light intensity. If the beam is cylindrically symmetric, it reduces to

$$\mathbf{J}_s=\frac{\lambda\,\eta e^3E_0^2}{2\gamma m^2\omega^3}\,\frac{\partial}{\partial r}\left(\frac{\eta n_0}{\gamma}\right)\hat{\mathbf{e}}_\theta. \tag{7}$$

On the other hand, the motion of a single electron in the circularly polarized laser field produces a magnetic dipole moment $\mu=-e/2c\langle\mathbf{r}_0\times\mathbf{v}_\perp\rangle$ with $\mathbf{r}_0$ the orbit radius, which produces magnetization of the plasma when it is summed for all electrons,

$$\mathbf{M}=-\frac{\lambda\,\eta^2e^3n_0E_0^2}{2c\gamma^2m^2\omega^3}\,\hat{\mathbf{e}}_z. \tag{8}$$

The total magnetic field is calculated from [19]

$$\nabla\times\mathbf{B}_s=\frac{4\pi}{c}\,\mathbf{J}_s+4\pi\nabla\times\mathbf{M}. \tag{9}$$

In cylindrical geometry, with no externally applied constant magnetic field and $B_s=0$ for $r\to\infty$, we find from Eq. (9), by using Stokes's theorem,

$$\mathbf{B}_s=-\frac{2\pi\lambda ec}{\omega}\left[\frac{\eta^2n_0|a|^2}{\gamma^2}-\int_r^{+\infty}\frac{\eta|a|^2}{\gamma}\,\frac{d}{dr}\left(\frac{\eta n_0}{\gamma}\right)dr\right]\hat{\mathbf{e}}_z, \tag{10}$$

or

$$\frac{\omega_c}{\omega}=-\frac{\lambda}{2}\,\frac{\omega_p^2}{\omega^2}\left[\frac{\eta^2|a|^2}{\gamma^2}-\frac{1}{n_0}\int_r^{+\infty}\frac{\eta|a|^2}{\gamma}\,\frac{d}{dr}\left(\frac{\eta n_0}{\gamma}\right)dr\right], \tag{11}$$

where $a=eE_0\exp(i\psi_0)/m\omega c$ is the normalized complex amplitude and $\omega_p^2=4\pi n_0e^2/m$. Obviously, for a linearly polarized laser beam with $\lambda=0$, there are no current $\mathbf{J}_s$ and magnetization $\mathbf{M}$, and therefore no longitudinal magnetic field generation.

In the non-self-consistent way, η and γ in the above equation are simply substituted with 1 and $\gamma_0=\sqrt{1+|a|^2}$, where γ_0 is the normal relativistic factor without the magnetic field. In this case, the magnetic field can be calculated directly from these equations. Here we can give some examples for the purpose of comparison with the self-consistent result. When the electron density n_0 is homogeneous, and the amplitude of the laser field approaches zero as $r\to+\infty$,

$$B_s=-\frac{\lambda}{2}\left(\frac{m\omega c}{e}\right)\left(\frac{\omega_p^2}{\omega^2}\right)\left(\frac{3|a|^2}{2\gamma_0^2}-\frac{1}{2}\,\ln\gamma_0^2\right). \tag{12}$$

If both the laser field and the density of electrons are inhomogeneous, for example, when both have Gaussian shape $|a|=a_0\exp(-r^2/2L_a^2)$, $a_0^2\ll 1$, and $n_0(r)=N_0\exp(-r^2/L_n^2)\equiv N_0 f_1(r)$, then

$$B_s=-\frac{\lambda a_0^2}{2}\left(\frac{m\omega c}{e}\right)\left(\frac{\omega_p'^2}{\omega^2}\right)\left(1+\frac{L_a^2}{L_a^2+L_n^2}\right)\times\exp\left[-\left(\frac{r^2}{L_a^2}+\frac{r^2}{L_n^2}\right)\right], \tag{13}$$

where $\omega_p'^2=4\pi N_0 e^2/m$; otherwise for the same Gaussian $|a|$, but density cavitation $n_0(r)=N_0[1-\exp(-r^2/L_n^2)]\equiv N_0 f_2(r)$, we have

$$B_s=-\frac{\lambda a_0^2}{2}\left(\frac{m\omega c}{e}\right)\left(\frac{\omega_p'^2}{\omega^2}\right)\left[1-\left(1+\frac{L_a^2}{L_a^2+L_n^2}\right)\times\exp(-r^2/L_n^2)\right]\exp(-r^2/L_a^2). \tag{14}$$

Self-consistently, the magnetic field produced can be calculated numerically. To do this, it is easier to solve a differential equation than to solve an integration equation. Assuming $n_0(r)=N_0 f(r)$, and taking the derivative of Eq. (11), or directly from Eq. (9), we find

$$\frac{d(\omega_c/\omega)}{dr}=-\frac{\lambda}{2}\frac{\omega_p'^2}{\omega^2}\left(1+\frac{3}{2}\frac{\beta_0\eta^3|a|^2 f}{\gamma^5}\frac{\omega_p'^2}{\omega^2}\right)^{-1}\times\left[2\frac{\eta^2|a|^2}{\gamma^2}\frac{df}{dr}+\frac{\eta^2 f}{\gamma^2}\times\left(1-\frac{3}{2}\frac{\beta_0\eta^3|a|^2}{\gamma^2}\right)\frac{d|a|^2}{dr}\right], \tag{15}$$

where $\beta_0=[1+\lambda(\omega_c/\omega)|a|^2(\eta/\gamma)^3]^{-1}$. This equation is easily solved for given functions $f(r)$ and $|a(r)|^2$. Figures 1 and 2(a) display a comparison between the non-self-consistent analytical expressions [Eqs. (12)–(14)] and the self-consistent result obtained from Eq. (15). It shows that only for the case $\omega_p^2/\omega^2\ll 1$ or $|a|^2\ll 1$ when the generated magnetic field is small, the non-self-consistent results approach the self-consistent results. The plotted results are for $\lambda=1$. If one takes $\lambda=-1$, the magnetic fields just change sign.

Meanwhile, we note according to Eq. (11) that $\omega_c/\omega\sim(\omega_p/\omega)^2(|a|^2/\gamma^2)$, therefore either for $|a|\ll 1$ or $|a|\gg 1$, one has $\omega_c/\gamma\omega\ll 1$. We then obtain for η and γ in the order of $O(\omega_c/\omega)$

$$\eta\approx 1+\lambda\frac{\omega_c}{\gamma_0\omega},$$

$$\gamma=\gamma_0+\lambda\frac{\omega_c}{\omega}\frac{|a|^2}{\gamma_0^2}, \tag{16}$$

$$\frac{\eta}{\gamma}=\frac{1}{\gamma_0}\left(1+\lambda\frac{\omega_c}{\omega}\gamma_0^{-3}\right),$$

and

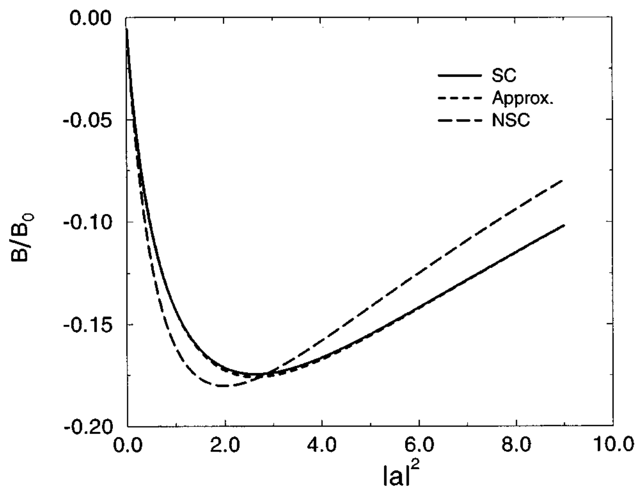

FIG. 1. Magnetic field produced in homogeneous plasma as a function of the light intensity calculated in self-consistent (SC) and non-self-consistent [NSC, Eq. (12)] ways, or self-consistently but with approximation to order $O(\omega_c/\omega)$ for $\omega_p^2/\omega^2=0.8$ and right-circular polarization ($\lambda=1$). The magnetic field is normalized to $B_0=m\omega c/e$.

$$\frac{d(\omega_c/\omega)}{dr}=-\frac{\lambda}{2}\frac{\omega_p'^2}{\omega^2}\left(1+\frac{3}{2}\frac{f|a|^2}{\gamma_0^5}\frac{\omega_p'^2}{\omega^2}\right)^{-1}\times\left[\left(1+2\lambda\frac{\omega_c}{\omega}\gamma_0^{-3}\right)\frac{2|a|^2}{\gamma_0^2}\frac{df}{dr}+\left(1-\frac{|a|^2}{2}+\lambda\frac{\omega_c}{\omega}\frac{4-11|a|^2}{2\gamma_0^3}\right)\frac{f}{\gamma_0^4}\frac{d|a|^2}{dr}\right]. \tag{17}$$

Figures 1 and 2(b) show a comparison between Eqs. (15) and (17). It shows that the approximation to order $O(\omega_c/\omega)$ is generally reasonable.

It should be pointed out that Eqs. (15) and (17) are valid for inhomogeneous distributions of the light intensity and electron density. When both the light intensity and plasma density are homogeneous, the total volume current density vanishes, but there is still surface contribution. The magnetic field is simply

$$B_s=-\frac{\lambda}{2}\left(\frac{m\omega c}{e}\right)\left(\frac{\omega_p^2}{\omega^2}\right)\frac{\eta^2|a|^2}{\gamma^2}, \tag{18}$$

and the corresponding differential equation is

$$\frac{d(\omega_c/\omega)}{d|a|^2}=-\frac{\lambda}{2}\frac{\omega_p^2}{\omega^2}\frac{\eta^2}{\gamma^2}\left(1+\frac{\beta_0\eta^3|a|^2}{\gamma^5}\frac{\omega_p^2}{\omega^2}\right)^{-1}\times\left(1-\frac{\beta_0\eta^3|a|^2}{\gamma^2}\right). \tag{19}$$

Figure 3 displays ω_c/ω as a function of $|a|^2$ for some value of ω_p^2/ω^2. The non-self-consistent result (taking $\eta=1$, $\gamma=\gamma_0$) is also given for comparison. The latter overestimates the magnetic field. Compared to Fig. 1, one can see that the

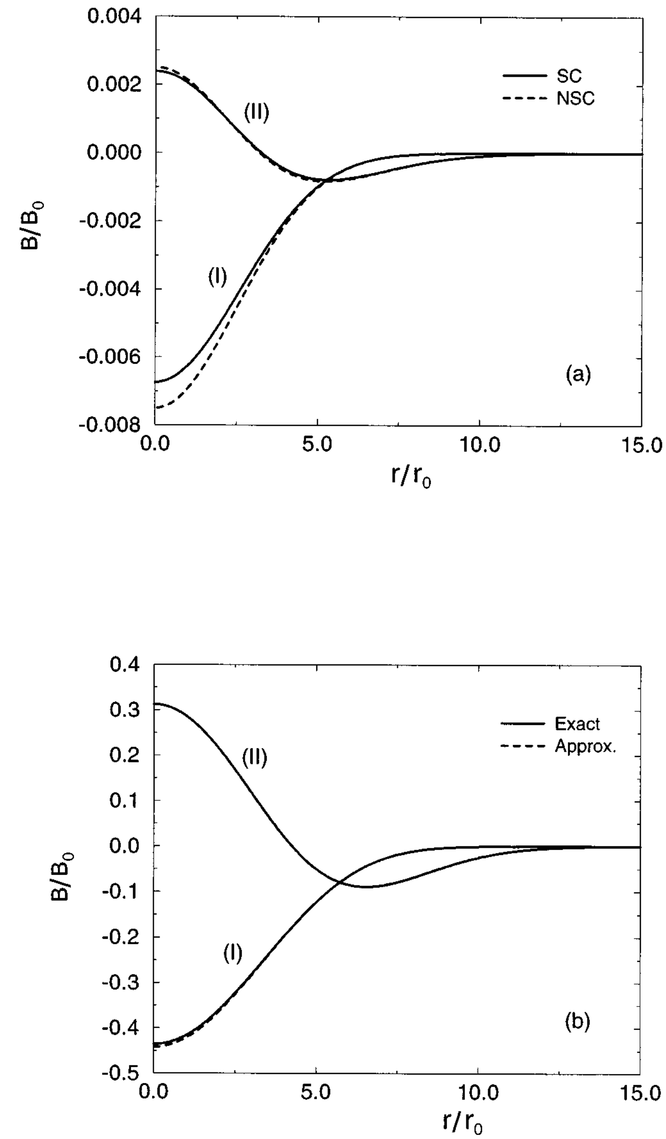

FIG. 2. (a) Magnetic field distribution calculated in a self-consistent (SC) or non-self-consistent (NSC) way for two different profiles of plasma density given in the text [curve (I) for $f_1(r)$ and curve (II) for $f_2(r)$] and for a Gaussian beam for $|a|=0.1$, $\omega_p'^2/\omega^2=0.1$, $L_n=L_a=5.0r_0$, and $\lambda=1$. (b) Magnetic field distribution calculated exactly or with approximation to order $O(\omega_c/\omega)$ for the same density profiles and Gaussian beam for $|a|=2.0$, $\omega_p'^2/\omega^2=0.8$, $L_n=L_a=5.0r_0$, and $\lambda=1$. Magnetic field in units $B_0=m\omega c/e$; radius in arbitrary units r_0.

volume current density induced by the intensity inhomogeneity tends to cancel the magnetic field generation in homogeneous plasma.

III. THE CUTOFF FREQUENCY OF A CIRCULARLY POLARIZED RELATIVISTIC LIGHT WAVE

The general wave equation is

$$\nabla^2 \mathbf{E}_L - \frac{1}{c^2}\frac{\partial^2 \mathbf{E}_L}{\partial t^2} = \frac{4\pi}{c^2}\frac{\partial \mathbf{J}}{\partial t}, \tag{20}$$

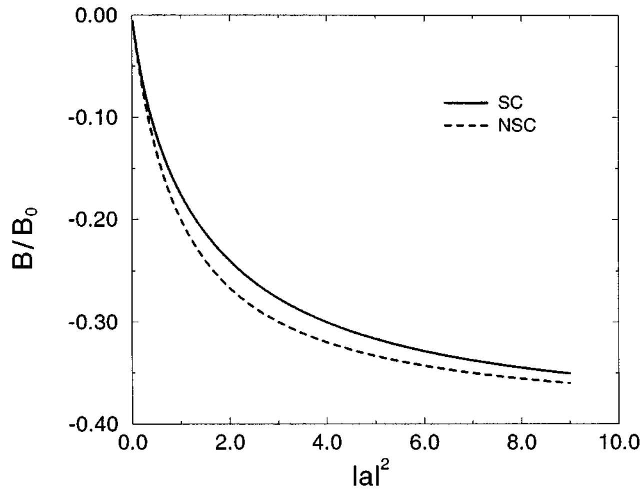

FIG. 3. Magnetic field produced in homogeneous plasma by a right-circularly polarized plane wave ($\lambda=1$) for $\omega_p^2/\omega^2=0.8$, normalized to $B_0=m\omega c/e$.

with

$$\mathbf{J} = -e n_s \mathbf{v}_\perp = \frac{i\eta e^2 n_s}{m\omega\gamma}\mathbf{E}_L, \tag{21}$$

and the density of electrons n_s is determined by Poisson's equation

$$\nabla^2 \Phi = 4\pi e(n_s - n_0). \tag{22}$$

Here n_0 is the unperturbed electron density or the ion density with effective charge number equal to one. For plane waves, $n_s=n_0$, we then have the dispersion relation of circularly polarized relativistic light,

$$\omega^2 = k^2c^2 + \frac{\omega_p^2}{\gamma}\left(1-\lambda\,\frac{\omega_c}{\omega\gamma}\right)^{-1}. \tag{23}$$

This dispersion relation was also given by Akhiezer and Polovin [1], where they considered the propagation of a circularly polarized relativistic wave with an externally applied magnetic field along the propagation direction. Here, ω_c/ω in the dispersion relation is calculated from the self-generated magnetic field [Eq. (18)] and γ is given by Eq. (4).

The cutoff frequency above which the laser beam can propagate in the plasma is

$$\omega_{\rm CF}^2 = \frac{\omega_p^2}{\gamma}\left(1-\lambda\,\frac{\omega_c}{\gamma\omega_{\rm CF}}\right)^{-1}. \tag{24}$$

If the magnetic field is produced uniquely by inverse Faraday effect, Eq. (18) gives

$$\frac{\omega_c}{\omega_{\rm CF}} = -\frac{\lambda}{2}\frac{\omega_p^2}{\omega_{\rm CF}^2}\frac{|a|^2}{\gamma^2}\left(1-\lambda\,\frac{\omega_c}{\gamma\omega_{\rm CF}}\right)^{-2}, \tag{25}$$

therefore

$$\frac{\omega_{\rm CF}^2}{\omega_p^2} = \frac{1}{\gamma}\left(1+\frac{|a|^2}{2\gamma}\frac{\omega_{\rm CF}^2}{\omega_p^2}\right)^{-1}, \tag{26}$$

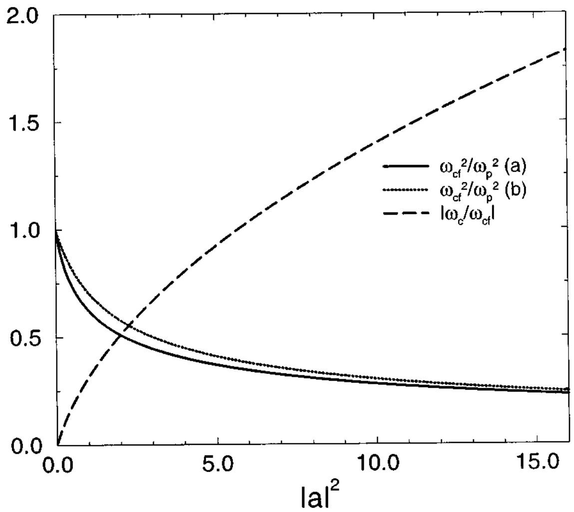

FIG. 4. The cutoff frequency of an intense circularly polarized light wave in homogeneous plasma and the magnetic field produced. Curve (a) takes into account the self-generated magnetic field and curve (b) does not account for it.

$$\frac{\omega_c}{\omega_{\mathrm{CF}}}=-\frac{\lambda}{2}\frac{\omega_{\mathrm{CF}}^2}{\omega_p^2}|a|^2, \tag{27}$$

with $\lambda=(1+\eta_{\mathrm{CF}}^2|a|^2)^{1/2}$ and $\eta_{\mathrm{CF}}=(1-\lambda\omega_c/\gamma\omega_{\mathrm{CF}})^{-1}$. Figure 4 shows $\omega_{\mathrm{CF}}^2/\omega_p^2$ and $|\omega_c/\omega_{\mathrm{CF}}|$ as functions of $|a|^2$. The cutoff frequency for intense circularly polarized light is slightly smaller than $\omega_p^2/\sqrt{1+|a|^2}$, the well-known result without considering the magnetic field generation. With the approximation (16), we have

$$\frac{\omega_{\mathrm{CF}}^2}{\omega_p^2}=\frac{\gamma_0^3}{|a|^2}\left[-1+\left(1+\frac{2|a|^2}{\gamma_0^4}\right)^{1/2}\right]. \tag{28}$$

When $|a|\gg 1$, we find

$$\omega_{\mathrm{CF}}^2/\omega_p^2=1/|a|, \tag{29}$$

$$\omega_c/\omega_{\mathrm{CF}}=-(\lambda/2)|a|. \tag{30}$$

We note that the magnetic field generated increases linearly with the amplitude of the light wave. It shows that one can obtain an ultra-intense magnetic field in an overdense plasma where the light wave can propagate through induced transparency. As an example, if the incident wave has a wavelength $\lambda_0=1\ \mu$m in vacuum and $|a|=3.0$, which corresponds to an intensity of about 1.23×10^{19} W/cm^2, the plasma has a cutoff density $n_c=|a|n_c^{(0)}=|a|\omega_0^2 m/(4\pi e^2)=3.35\times10^{21}$/cm^3, and the magnetic field produced is around 160 MG. This value is comparable with these produced through other mechanisms [20,21].

IV. THE EVOLUTION EQUATIONS OF CIRCULARLY POLARIZED LASER BEAMS

For a laser beam with finite transverse size, $\nabla\Phi$ is not zero. Assuming the amplitude envelope of the laser beam changes slowly in longitudinal space as compared to that in transverse direction, taking the time average over ω^{-1} in Eq. (1), after some tedious algebra calculation, we obtain

$$\nabla_\perp\Phi=\left\langle\frac{1}{c}\mathbf{v}_\perp\times\mathbf{B}_L+\frac{m}{e}(\mathbf{v}_\perp\cdot\nabla)\gamma\mathbf{v}_\perp\right\rangle$$
$$=\frac{\eta e}{2m\omega^2\gamma}\left[\nabla_\perp E_0^2+\frac{\lambda\omega_c}{\omega}\nabla_\perp\left(\frac{\eta}{\gamma}E_0^2\right)\right], \tag{31}$$

where we have ignored the longitudinal component and use the relation $\nabla\cdot\mathbf{E}_L=0$. We emphasize that this expression is valid for a circularly polarized light wave with a slowly varying magnetic field, either produced by the light wave or applied externally, in the propagation direction. Now we put the above equations in dimensionless form by introducing $\mathbf{a}_L=e\mathbf{E}_L/m\omega c$, $a=eE_0\exp(i\psi_0)/m\omega c$, $\phi=e\Phi/mc^2$, $k_p=\omega_p/c$, and $N_s=n_s/n_0$ with $\omega_p^2=4\pi n_0e^2/m$. The resulting wave equation is

$$\nabla^2\mathbf{a}_L-\frac{1}{c^2}\frac{\partial^2\mathbf{a}_L}{\partial t^2}=\frac{\eta k_p^2}{\gamma}N_s\mathbf{a}_L, \tag{32}$$

$$N_s=\max(0,1+k_p^{-2}\nabla_\perp^2\phi), \tag{33}$$

$$\nabla_\perp\phi=\frac{\eta}{2\gamma}\left[\nabla_\perp|a|^2+\frac{\lambda\omega_c}{\omega}\nabla_\perp\left(\frac{\eta}{\gamma}|a|^2\right)\right], \tag{34}$$

with γ and η given in Sec. II. Here we assume the plasma density is homogeneous. These three equations describe the propagation of optical beams in plasma with a magnetic field in the propagation direction. Substituting $\mathbf{a}_L=a/2(\hat{\mathbf{e}}_x+i\lambda\hat{\mathbf{e}}_y)\exp(ikz-i\omega t)$ into Eq. (32), we have

$$\frac{\partial a}{\partial z}+\frac{1}{v_g}\frac{\partial a}{\partial t}-\frac{i}{2k}\left[\nabla_\perp^2+k_p^2\left(\sigma-\frac{\eta N_s}{\gamma}\right)\right]a=0, \tag{35}$$

where the dispersion relation $\omega^2=k^2c^2+\sigma\omega_p^2$ for laser beams with finite transverse size has been used, $v_g=d\omega/dk$ is the group velocity, and $\sigma\leqslant1$ is a constant eigenvalue meaningful only for the stationary solution of Eq. (35) and depending on the light power of the eigenmode [5,7]. Physically, σ is related to the effect of relativistic electron-mass increase and charge displacement owing to the transverse ponderomotive force, which tends to reduce the plasma frequency in the dispersion relation. The higher the power of the trapped eigenmode, the smaller the σ value. Now making substitution of $z'=z-v_gt$ and normalizing $\xi=k_p^2z'/k$ and $\rho_\perp=k_p\mathbf{r}_\perp$, we obtain the evolution equation for a circularly polarized laser beam in plasma,

$$\left(\nabla_\perp^2+2i\frac{\partial}{\partial\xi}+\sigma\right)a=\frac{\eta N_s}{\gamma}a, \tag{36}$$

where $\nabla_\perp^2=d/d\rho(\rho d/d\rho)/\rho$ for cylindrical geometry. To simplify the problem, we calculate all quantities to order $O(\omega_c/\omega)$ since this is a very good approximation as shown in Sec. II, where η and γ are given by Eq. (16), and $\nabla_\perp\phi$ is reduced to

$$\nabla_\perp \phi = \left(\frac{1}{2\gamma_0} + \lambda \, \frac{\omega_c}{\omega} \, \frac{4+|a|^2}{4\gamma_0^4} \right) \nabla |a|^2$$
$$= \nabla_\perp \gamma_0 + \frac{\lambda}{2} \, \frac{\omega_c}{\omega} \, \nabla_\perp \left(\frac{1}{2} \ln \gamma_0^2 - \frac{3}{2} \, \gamma_0^{-2} \right),$$

and

$$\nabla_\perp^2 \phi = \beta_5 \gamma_0^{-3} \left| \frac{da}{d\rho} \right|^2 + \frac{\beta_4}{2\gamma_0} \, a^* \left(\frac{d^2 a}{d\rho^2} + \frac{1}{\rho} \, \frac{da}{d\rho} \right) + \frac{\lambda}{2} \, \frac{d}{d\rho} \left(\frac{\omega_c}{\omega} \right) \frac{\beta_3}{\gamma_0} \, a^* \, \frac{da}{d\rho} + \text{c.c.}, \tag{37}$$

where

$$\beta_3 = \frac{4+|a|^2}{2\gamma_0^3},$$

$$\beta_4 = 1 + \lambda \, \frac{\omega_c}{\omega} \, \frac{4+|a|^2}{2\gamma_0^3},$$

$$\beta_5 = 1 + \lambda \, \frac{\omega_c}{\omega} \, \frac{4 - 9|a|^2 - |a|^4}{2\gamma_0^3}.$$

If the magnetic field is produced exclusively through the inverse Faraday effect, then the cyclotron frequency is calculated simply by Eq. (17) with d/dr substituted with $d/d\rho$. One finds that the evolution equations for right-and left-circularly polarized laser beams are the same, although the magnetic fields have opposite signs for the two cases.

V. EFFECT OF SELF-GENERATED MAGNETIC FIELD ON THE PROPAGATION OF LASER BEAMS

A. Critical power of self-trapping

The stationary equation is

$$\frac{1}{\rho} \, \frac{d}{d\rho} \left(\rho \, \frac{da}{d\rho} \right) + \sigma a = \frac{\eta N_s}{\gamma} \, a, \tag{38}$$

which is similar to the one given by Sun *et al.* except for the different forms of N_s, γ, and η. Here, we take a to be real. The critical power is calculated as follows. Near the critical power, $|a| \ll 1$, and the beam radius $\rho_0 \gg 1$, therefore $N_s \sim 1 + a^2/\rho_0^2 \approx 1$, $\sigma \approx 1$, and $\omega_c/\omega \approx -\lambda a^2 \omega_p^2/(2\omega^2) \ll 1$. To the order of $O(|a|^2)$, we have $\eta \approx 1 - a^2 \omega_p^2/(2\omega^2)$, $\gamma \approx 1 + a^2/2$, and Eq. (38) reduces to

$$\frac{1}{\rho} \, \frac{d}{d\rho} \left(\rho \, \frac{da}{d\rho} \right) + \left[\sigma - 1 + \frac{1}{2} \left(1 + \frac{\omega_p^2}{\omega^2} \right) a^2 \right] a = 0. \tag{39}$$

Assuming $\kappa^2 = 1 - \sigma$, $\theta = 1/2(1 + \omega_p^2/\omega^2)$, and $a = (\kappa^2/\theta)^{1/2} H(\kappa\rho)$, one has

$$\frac{1}{\rho} \, \frac{d}{d\rho} \left(\rho \, \frac{dH}{d\rho} \right) - H + H^3 = 0, \tag{40}$$

with the boundary conditions $dH/d\rho|_{\rho=0} = 0$ and $H|_{\rho \to +\infty} = 0$. The solution of this equation is well known [22,23]. From this, the critical power is calculated as

$$P_{\text{cr}} = \frac{m^2 c^5}{2e^2} \, \frac{\omega^2}{\omega_p^2} \int_0^\infty |a|^2 \rho \, d\rho$$
$$= \frac{m^2 c^5}{e^2} \, \frac{1}{2\theta} \, \frac{\omega^2}{\omega_p^2} \int_0^\infty H^2(\rho) \rho \, d\rho$$
$$= 1.62 \times 10^{10} \times \left(\frac{\omega^2}{\omega_p^2} \right) \left(1 + \frac{\omega_p^2}{\omega^2} \right)^{-1} W, \tag{41}$$

where, in the last integration, we have made use of the result given in Ref. [7]. Compared with previous studies, we find that the critical power is reduced by a factor of $(1 + \omega_p^2/\omega^2)^{-1}$ due to the magnetic field generation. This result is also confirmed in the following numerical calculations. Physically, it can be understood from the basic dispersion relation Eq. (23): it shows that self-generated magnetic field tends to increase the refractive index in high intensity regions in addition to that caused by the relativistic effect. The critical power is, of course, just a necessary condition for self-focusing of a laser beam. The sufficient condition is related with some globally conserved quantities of the beam [23,24]. Taking into account the magnetic field generation, it should also be modified.

B. Stationary solutions

Assuming that no cavitation occurs, we set $N_s = 1 + \nabla_\perp^2 \phi$ in Eq. (38) and find

$$\frac{d^2 a}{d\rho^2} + \frac{1}{\rho} \, \frac{da}{d\rho} - \lambda \beta_1 \beta_3 \, \frac{d}{d\rho} \left(\frac{\omega_c}{\omega} \right) a^2 \, \frac{da}{d\rho} - \beta_1 \beta_2 \, \frac{a}{\gamma_0^2} \left(\frac{da}{d\rho} \right)^2 + \beta_1 \gamma_0^2 \left(\sigma - \frac{\eta}{\gamma} \right) a = 0, \tag{42}$$

where

$$\beta_1 = \left(1 - \lambda \, \frac{\omega_c}{\omega} \, \frac{(6+a^2) a^2}{2\gamma_0^3} \right)^{-1},$$

$$\beta_2 = 1 + \lambda \, \frac{\omega_c}{\omega} \, \frac{6 - 9a^2 - a^4}{2\gamma_0^3},$$

and β_3 as defined following Eq. (37). To obtain the stationary eigenmode, Eq. (42) should be solved together with the density equation and magnetic field equation. The magnetic field equation is given by Eq. (17), which we rewrite by substituting f with N_s as

$$\frac{d}{d\rho} \left(\frac{\omega_c}{\omega} \right) = -\lambda \epsilon \left(\beta_7' a^2 \, \frac{dN_s}{d\rho} + \beta_8' N_s a \, \frac{da}{d\rho} \right), \tag{43}$$

with

$$\beta_6 = \left(1 + \frac{3}{2}\frac{\epsilon N_s a^2}{\gamma_0^5}\right)^{-1},$$

$$\beta_7 = \left(1 + 2\lambda \frac{\omega_c}{\omega} \gamma_0^{-3}\right)\gamma_0^{-2},$$

$$\beta_8 = \left(1 - \frac{a^2}{2} + \lambda \frac{\omega_c}{\omega}\frac{4-11a^2}{2\gamma_0^3}\right)\gamma_0^{-4},$$

$\beta_7' = \beta_6\beta_7$, $\beta_8' = \beta_6\beta_8$, and $\epsilon = \omega_p^2/\omega^2$ is the factor for the unperturbed plasma density. In the density equation [Eq. (34)], the scalar potential is given by Eq. (37), which can be simplified by substituting with Eq. (42)

$$\nabla_\perp^2 \phi = -\beta_1' \gamma_0(\sigma - \eta/\gamma)a^2 + \frac{\beta_5'}{\gamma_0^3}\left(\frac{da}{d\rho}\right)^2 + \frac{\lambda\beta_3'}{\gamma_0}\frac{d}{d\rho}\left(\frac{\omega_c}{\omega}\right)a\frac{da}{d\rho}, \tag{44}$$

with $\beta_1' = \beta_1\beta_4$, $\beta_3' = \beta_3 + \beta_1\beta_3\beta_4 a^2$, $\beta_5' = \beta_5 + \beta_1\beta_2\beta_4 a^2$. Substituting Eq. (43) into Eq. (44), we finally find the equation for the electron density,

$$\left(\frac{\epsilon\beta_3'\beta_7'}{\gamma_0} a^3 \frac{da}{d\rho}\right)\frac{dN_s}{d\rho} + \left[1 + \frac{\epsilon\beta_3'\beta_8'}{\gamma_0} a^2\left(\frac{da}{d\rho}\right)^2\right]N_s - \frac{\beta_5'}{\gamma_0^3}\left(\frac{da}{d\rho}\right)^2 + \beta_1'\gamma_0\left(\sigma - \frac{\eta}{\gamma}\right)a^2 - 1 = 0. \tag{45}$$

We note that, due to the magnetic field generation through inverse Faraday effect, the electron density is now determined by an ordinary differential equation. This equation and Eq. (42) are valid for $N_s > 0$. Otherwise, when electron cavitation occurs, Eq. (42) has to be replaced by Eq. (38) setting $N_s = 0$. By electron cavitation, we mean that the electron density becomes zero in some region as used in [5]. We emphasize that for $\epsilon \to 0$ (and therefore $\omega_c/\omega \to 0$), all equations in this and the previous sections reduce to those given by Sun *et al.* Equations (42), (43), and (45) are now to be completed with the boundary conditions

$$a(\rho)|_{\rho\to\infty} = 0, \quad \left.\frac{da}{d\rho}\right|_{\rho=0} = 0,$$

$$N_s|_{\rho\to\infty} = 1, \quad \left.\frac{\omega_c}{\omega}\right|_{\rho\to\infty} = 0.$$

They are solved numerically by the shooting method [25]. Similarly to Ref. [5], we use the asymptotic solution of Eq. (42) at some large radius ρ_∞ as a starting point for numerically integrating inward. The asymptotic solution is the modified Bessel function,

$$a(\rho) \sim C_\infty(\kappa\rho)^{-1/2}\exp(-\kappa\rho)$$

for $\rho \to \infty$ and $\kappa^2 = 1 - \sigma$. In our calculations, Eqs. (42) [or Eq. (38)] and (43) are integrated using an explicit scheme, and Eq. (45) is solved implicitly on the consideration that the coefficient of $dN_s/d\rho$ may be a small quantity.

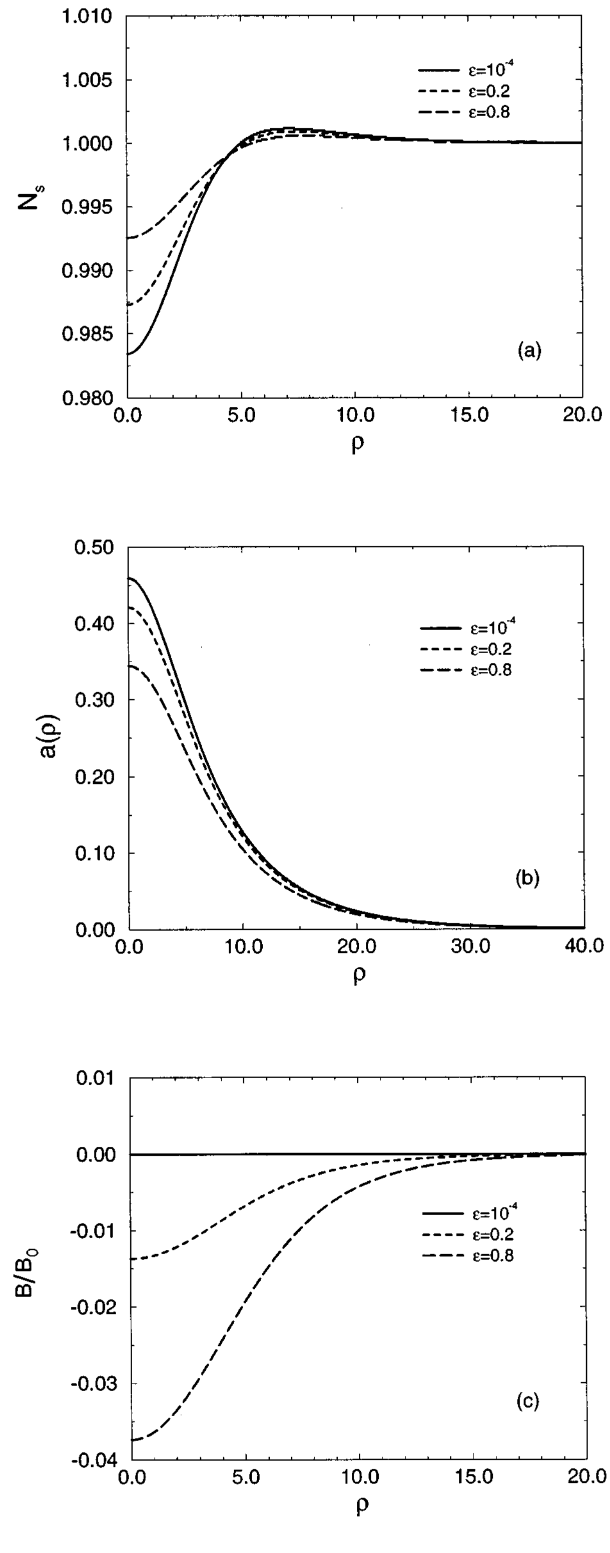

FIG. 5. The self-focusing eigenmode for $\sigma = 0.98$ at various plasma densities. (a) Electron density; (b) amplitude profile; (c) magnetic field ($\lambda = 1$) in units $B_0 = m\omega c/e$.

Figure 5 shows a result without electron cavitation with the eigenvalue $\sigma = 0.98$. The line for $\epsilon = 0.0001$ corresponds to a very small plasma density and a small magnetic field. For this case, the result given by Sun *et al.* is recovered [5]. With increasing plasma density ($\epsilon = 0.2$ and 0.8, for ex-

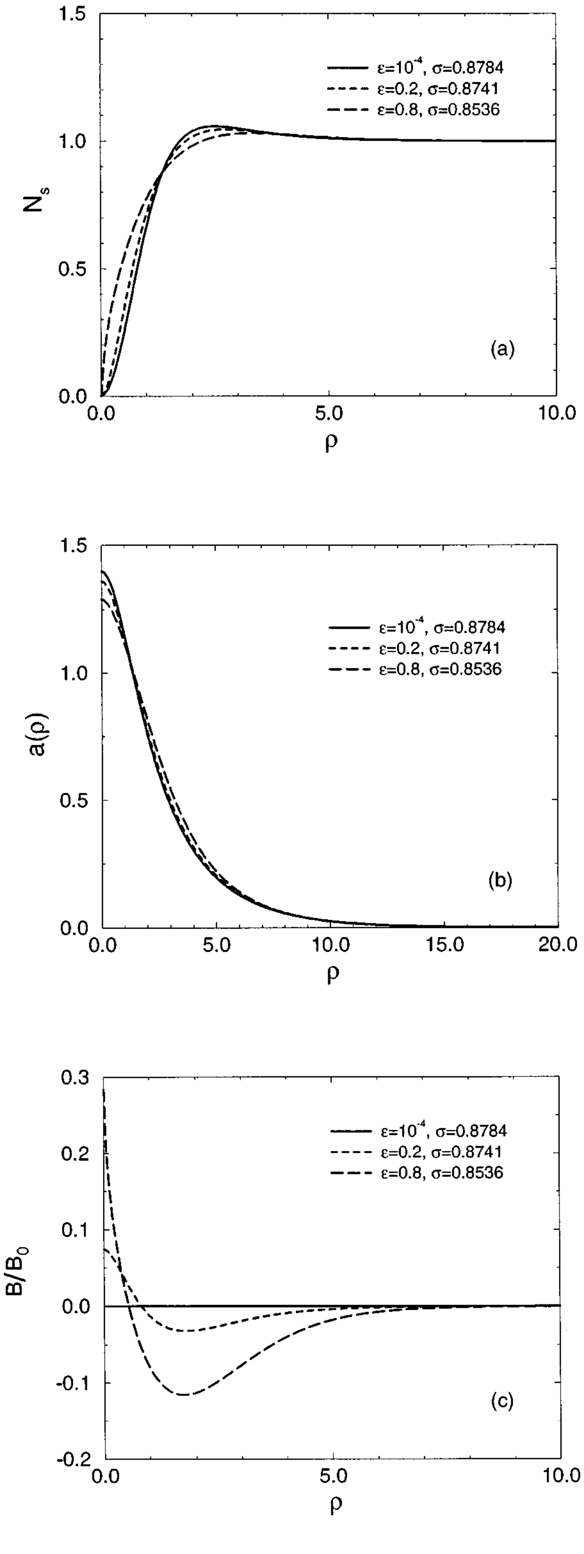

FIG. 6. The self-focusing eigenmode for $\sigma=\sigma_c$ at various plasma densities. (a) Electron density; (b) amplitude profile; (c) magnetic field ($\lambda=1$) in units $B_0=m\omega c/e$.

ample), the magnetic field increases, and its effect becomes apparent: the maximum amplitude of the beam reduces, the beam tends to be focused at low light intensity, and the magnetic field tends to prevent the electrons from being dragged away from the high intensity region by the ponderomotive

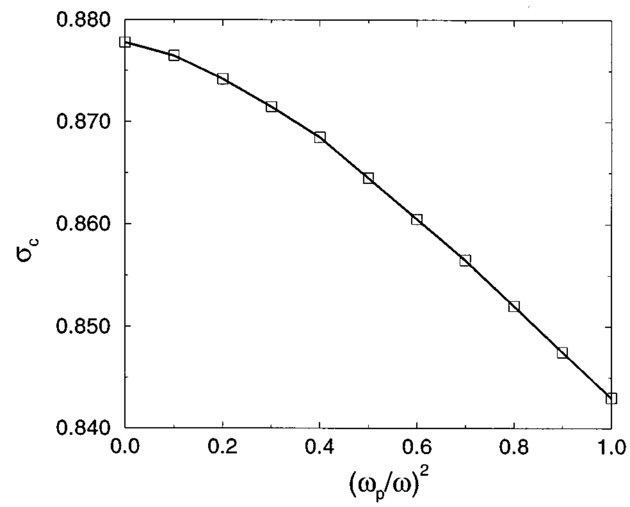

FIG. 7. The eigenvalue σ_c as a function of plasma density $(\omega_p/\omega)^2$ when the electron cavitation begins to occur.

force. The magnetic field distribution in this case is similar to the result in homogeneous plasma. For left-circularly polarized beams ($\lambda=-1$), the distributions for the electron density and amplitude of the laser beam do not change, and the magnetic field changes its sign. For this case, the magnetic field plays a positive role for self-focusing of the laser beam as seen from the dispersion relation Eq. (23).

As the eigenvalue decreases, the peak amplitude of the trapped eigenmode increases, and more electrons are dragged out of the high intensity region. For $\epsilon=0.0001$, when $\sigma=\sigma_c=0.8784$ (slightly different from the value 0.8778 as given by Sun *et al.*), cavitation begins to occur as shown in Fig. 6. Due to the magnetic field, the critical eigenvalue σ_c for electron cavitation to occur is a function of plasma density. When the plasma density increases, σ_c decreases as plotted in Fig. 7. In this case, the magnetic field distribution is somewhat similar to the result for density profile $f_2(r)$ given in Sec. II. Because it changes sign in the radial distribution, its effect on self-focusing is indefinite: near $\rho=0$, $\eta=(1-\lambda\omega_c/\omega\gamma)^{-1}>1$, it plays a negative role for self-focusing; but for $\eta<1$ in the outer region, it becomes positive for self-focusing.

With the further decrease of eigenvalue, a cavitation channel develops. One may note that, due to the magnetic confinement, the normalized size of the cavitation channel reduces. The higher the plasma density, the smaller the normalized radius of the cavitation channel. However, the difference of the amplitude profiles for different plasma densities is not large. Inside the cavitation channel, the magnetic field is a constant like that inside a current coil (see Fig. 8). Outside the channel, the magnetic field changes sign in the radial distribution, therefore its effect on self-focusing is also indefinite, as in the last case.

In the following, we define the normalized trapping power and radius of the self-trapped eigenmode as

$$P=\int_0^\infty |a(\rho)|^2\rho\, d\rho, \tag{46}$$

$$\rho_a=\left(\frac{1}{P}\int_0^\infty |a(\rho)|^2\rho^3 d\rho\right)^{1/2}. \tag{47}$$

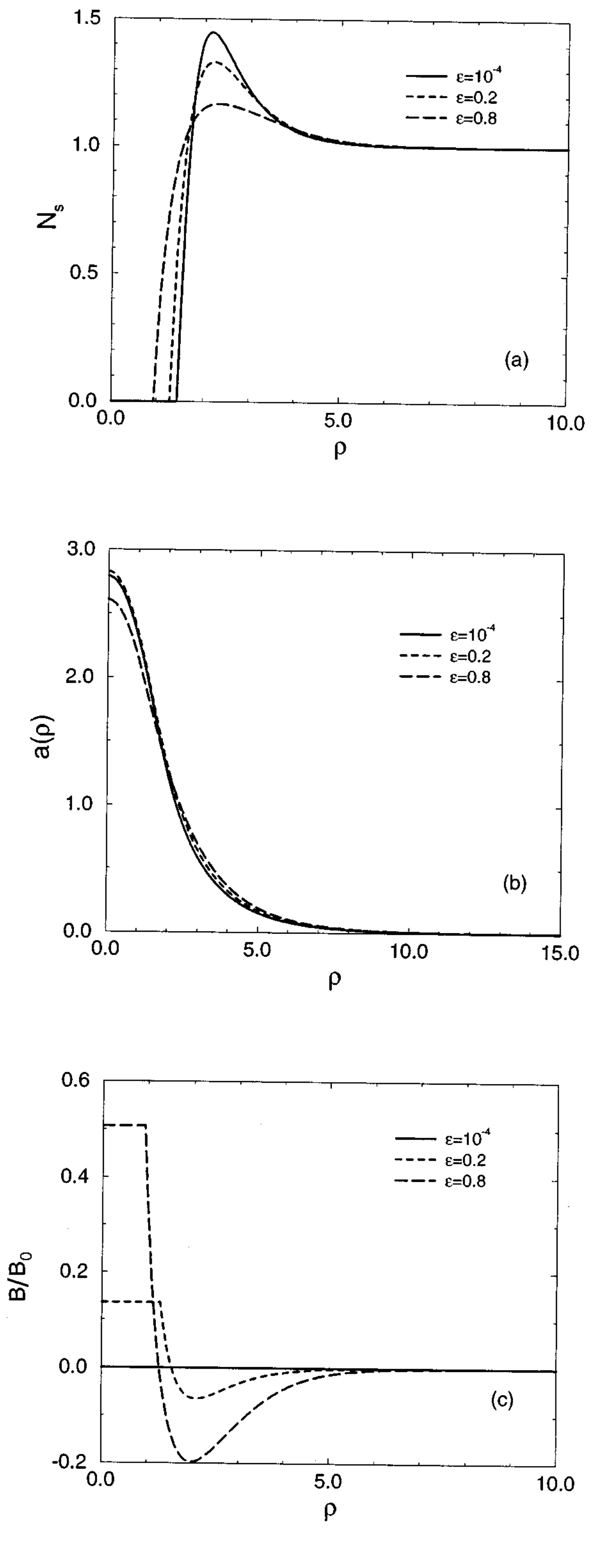

FIG. 8. The self-focusing eigenmode for $\sigma=0.7$ at various plasma densities. (a) Electron density; (b) amplitude profile; (c) magnetic field ($\lambda=1$) in units $B_0=m\omega c/e$.

For given plasma density, a higher trapping power always corresponds to a lower eigenvalue. For given value of $\sigma<0.9$, the laser power seems to be insensitive to the plasma density as shown in Fig. 9. This indicates that when the light intensity is high enough, the cavitation channel plays a fundamental role in self-focusing and traps most of the beam

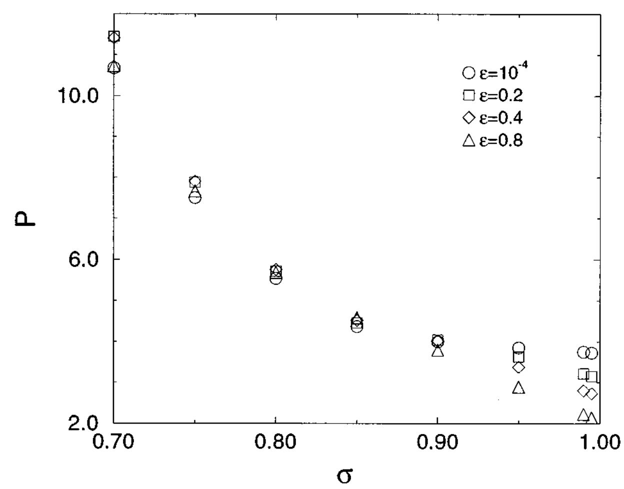

FIG. 9. Normalized self-trapping power P as a function of the eigenvalue σ at various plasma densities.

energy, although both the relativistic and magnetic effects are important for self-focusing outside the channel. For $\sigma>0.9$, the laser power becomes sensitive to the plasma density. When $\sigma\rightarrow 1$, i.e., in the relativistic self-focusing limit, the normalized trapping power scales as $(1+\omega_p^2/\omega^2)^{-1}$, in complete consistency with the analytical calculation as shown above. The normalized focused radius is shown as a function of the trapping power in Fig. 10. One may note that, when $P>5.0$, the radius is almost independent of power and is slightly enhanced when the plasma density increases. Again, this shows that most of the energy is trapped in the cavitation channel when the laser beam is at high intensities.

VI. CONCLUSIONS

The magnetic field generated through inverse Faraday effect by a circularly polarized light wave in plasma is studied in a self-consistent way, which allows us to calculate the magnetic field in plasmas for various density profiles and for light beams at relativistic intensities. Due to the magnetic

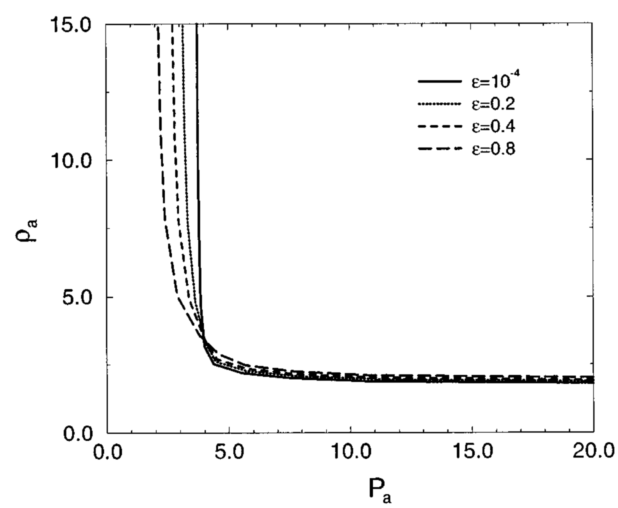

FIG. 10. Normalized radius ρ_a of self-focusing eigenmodes as a function of the trapping power P at various plasma densities.

field generation, the relativistically induced transparency of an intense circularly polarized light wave in overdense plasma is modified in such a way that it can propagate through a slightly denser plasma than a linearly polarized light wave at the same intensity. Meanwhile, an ultra-intense magnetic field is produced when it propagates in overdense plasma through induced transparency.

The magnetic field generation results in a reduced critical power for the self-focusing of a light beam. Generally, at low light intensities, the magnetic field plays a positive role for self-focusing of the beam. In this case, it does not change sign in the radial direction, i.e., either negative for a right-circularly polarized wave or positive for a left-circularly polarized wave. At high light intensities, the effect of the magnetic field on self-focusing is indefinite because it changes sign in the radial direction owing to the electron cavitation. High constant magnetic field is found inside the cavitation channel. In both cases, the effect of the electron displacement and cavitation is reduced to some extent due to the magnetic confinement of the electrons.

Our results are limited by the assumption of long pulse length. For short pulses, the magnetic field structure should be modified [18]. But the magnetic field may not have a direct effect on the wake field generation because the longitudinal oscillation velocity is basically parallel to the magnetic field. The large magnetic field may be useful to guide electron beams in plasma-based particle accelerators.

ACKNOWLEDGMENT

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