Fast penetration of a magnetic field into a collisionless plasma

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The Hall-effect-driven fast penetration of the magnetic field into a collisionless plasma is studied in a two-dimensional geometry. The magnetic field penetrates in the form of a shock wave, leaving behind the shock an electron vortex. If the plasma density varies by a large factor, the magnetic field penetrates only in a narrow stripe at a certain value of the density. The specific location of that stripe depends on the degree of resistivity of the cathode. The magnetic field can be nonlinearly enhanced during the penetration. [S1063-651X(96)01008-2]

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It is widely accepted that the early evolution of a magnetic field in high energy plasma devices, such as the plasma opening switches and the z pinches, is governed by the Hall effect. The penetration of the magnetic field depends on the local profile of the ion density n and on the curvature of the magnetic field lines. A shocklike penetration is typical, the velocity of the shock being of the order of the mean current velocity, multiplied by a geometry-dependent factor $u\approx I/\lambda^2 ne$. Here I denotes the total current, e the electron charge, n the ion density, and λ the characteristic size of the plasma. The respective energy dissipation is to be related to the so-called electron magnetohydrodynamics (MHD) resistance: the integrals of motion prohibit the electron flow without losing a part of its energy, associated with the magnetic field [1]. The phenomenon is quite well understood in the case of a nonzero resistivity, $\omega_{ce}>\tau_e^{-1}>v_{Ae}/\lambda$, where $\omega_{ce}=eB_0/mc$ is the electron cyclotron frequency, τ_e the electron collision time, $v_{Ae}=B_0/\sqrt{4\pi mn}$ the electron Alfven velocity, m the electron mass, c the speed of light, and B_0 the magnetic field at the plasma boundary cf [1,2].

The integrals of motion of the electron MHD predict a fast penetration in the collisionless limit, as well (assuming that the cathode is resistive); however, despite several attempts [3–6], the existing mathematical models have failed to provide a consistent scenario of the process in this limit. In review [1] it was argued that the excess of the magnetic energy is transferred to the electron vortices, but there was a complete uncertainty about the characteristics of these vortices. The main difficulty is that the process cannot be understood in the framework of a one-dimensional model; the one-dimensional solutions do exist [3], but cannot be tailored with realistic boundary conditions.

In the present paper we use a quasi-one-dimensional description of the shock penetration of magnetic fields. Unlike the quasi-one-dimensional models used in Refs. [4–6], we secure the conservation of the vorticity, which is conserved by the exact two-dimensional equations. It is shown that the collisionless fast penetration takes place in the form of a shock wave with the width of the front of order of the collisionless skin depth. The value of the magnetic field behind the shock B_s is different from its value at the boundary B_0 and varies in space, so that the region behind the shock can be considered as an electron vortex. The detailed shape of the front of the shock, as well as the value of the magnetic field behind the front and the region where the penetration occurs, are determined by the boundary conditions at the cathode.

We restrict ourselves to the two-dimensional geometry where all quantities depend on the spatial coordinates x and y, the magnetic field is parallel to the z-axis $\mathbf{B}=B\hat{\mathbf{z}}$, and the plasma fills the region $x>0$. However, this should not be a very limiting assumption. Indeed, it has been shown (c.f. [2]) that the case of cylindrical geometry with circular field lines can be reduced to the two-dimensional (2D) geometry with straight field lines and modified, effective ion density. Furthermore, it has been argued [7] that in the case of an arbitrary 3D geometry the penetration of the magnetic field into plasma is qualitatively the same as in the 2D case. Also, in order to avoid the delicate problem of boundary conditions at the electrodes, they are considered sufficiently remote. Let the plasma density profile be of the form

$$n=\begin{cases} n_C, & y<0 \\ n(y), & 0<y<a, \\ n_A, & y>a \end{cases}$$

where $n(y)$ is a monotonically increasing function whose characteristic space scale is much longer than the collisionless skin depth c/ω_{pe}. We also assume the following inequalities for the space scales λ and a, the time scale τ, and the penetration speed u of the magnetic field:

$$\omega_{ce}^{-1}\ll\tau\ll\omega_{ci}^{-1},\quad \lambda,a\ll c/\omega_{pi},\quad u\gg V_A. \tag{1}$$

Here ω_{pi} denotes the ion plasma frequency, V_A the Alfven velocity, and λ the characteristic scale of the density variations in the x direction. Then the ions can be considered motionless and the current is created by a quasineutral flow of electrons $\mathbf{j}=-ne\mathbf{v}$. Hence Ampere's law allows us to express the velocity of electrons as

$$\mathbf{v}=-\frac{\nabla b\times\hat{\mathbf{z}}}{n}. \tag{2}$$

Here we have introduced the dimensionless quantities, where the velocities are measured in the units of the electron Alfven velocity v_{Ae}, the magnetic field b in the units of the magnetic field B_0 at the plasma boundary, the ion density n in the units of the density n_C in the vicinity of the cathode, the time t in the units of the reciprocal electron gyrofrequency ω_{ce}^{-1}, and the space lengths in the units of the electron skin depth $c/\omega_{pe}=\sqrt{mc^2/4\pi n_C e^2}$. The electron equation of motion,

$$\frac{\partial \mathbf{v}}{\partial t}+(\mathbf{v}\nabla)\mathbf{v}=-\mathbf{E}-\mathbf{v}\times\mathbf{b}-\frac{\nabla P_e}{n},$$

upon applying curl to both sides and substituting the electric field from Faraday's law, $-\partial\mathbf{b}/\partial t=\nabla\times\mathbf{E}$, can be rewritten as

$$\left(\frac{\partial}{\partial t}+(\mathbf{v}\nabla)\right)\Omega=0,\quad \Omega=\frac{1}{n}\left[b-\nabla\left(\frac{\nabla b}{n}\right)\right]. \tag{3}$$

Here we have neglected the electron-ion collisions, which is legitimate under the condition

$$v_{Ae}\tau_e \gg \lambda, \tag{4}$$

and used the "polytropic" approximation for the electron pressure $P_e(x,y)=P_e[n(x,y)]$. Equation (3) should be supplemented with boundary conditions. The first two boundary conditions are rather natural,

$$b|_{x=0}=1,\quad b|_{x=\infty}=0, \tag{5}$$

the third one needs a detailed derivation. First of all we note that in the regions outside the varying density profile one can expect a stationary one-dimensional distribution of the magnetic field, defined by the simple collisionless skin effect. In the case of a stationary solution, according to Eq. (3), the vorticity Ω is constant along the streamlines $b(x,y)=$ const, hence

$$b-\frac{1}{n}\frac{\partial^2 b}{\partial x^2}=n(y)\Omega(x,y), \tag{6}$$

where

$$\Omega(x,y)=\Omega_0[b(x,y)].$$

The particular form of the functional dependence $\Omega_0(b)$ is defined by the processes at the cathode, which are not considered here. Instead, we assume this function to be specified.

As was indicated in paper [7], in the case of a collisionless cathode and a collisionless plasma, the fast penetration will be suppressed. Indeed, the low collisionality of the cathode means that $\Omega_0(b)\ll b$. Then the system (5), (6) has a quasi-one-dimensional solution at all values of y and system (3), (5) has a stationary solution.

Now we assume that the cathode is resistive so that in its vicinity $v_{Ae}\tau_e\lesssim\lambda$, a condition opposite to (4). Then, at least in some regions of the plasma, we have $\Omega_0(b)\approx b/n$. Equation (6) has a solution, consistent with the boundary conditions (5) if, for all values of $b\in(0,1)$, the following inequality is satisfied:

$$b^2>2n\Pi(b),\quad \Pi(b)\equiv\int_0^b\Omega(b')db'. \tag{7}$$

Thus, starting from some certain value $y>y_0$, there is no solution to Eq. (6). A more detailed definition of y_0 is given later, see Eq. (8). So we can expect that at larger plasma densities $n>n_0\equiv n(y_0)$ the magnetic field penetrates into the bulk, and the line $y=y_0$ presents the lower boundary of the shock (see Fig. 1). Quite naturally, the magnetic field does not penetrate into the region $y>a$ with constant plasma density. In fact, the upper boundary of the shock $y=y_1$ can be located at even smaller values of y, $y_1\leqslant a$ (see below).

Thus, the lower boundary of the shock can be found as the smallest coordinate y_0, such that the equation

$$b_S^2=2n(y_0)\Pi_0(b_S),\quad \Pi_0(b)=\int_0^b\Omega_0(b')db' \tag{8}$$

has a real solution b_S. Equivalently, the critical values n_0 and b_S can be defined by the system

$$b_S^2=2n_0\Pi_0(b_S),\quad b_S=n_0\Omega(b_S), \tag{9}$$

where it is assumed that $b_S<1$ (the opposite situation will be discussed below). Then the boundary condition closing the system (3), (5), can be written as

$$\Omega(x,y)|_{y=y_0}=\Omega_0[b(x,y_0)]. \tag{10}$$

We wish to find a shocklike solution of the system (3), (5), and (10).

Equation (3) can be simplified by neglecting the curvature of the current lines. This approximation is justified in the current layer at the front of the shock, under the assumption that the width of the shock is much smaller than its curvature. Switching from the independent variables x and y to the magnetic field b and the logarithm of the density $N=\ln(n/n_0)$ we have

$$\left(w\frac{\partial}{\partial b}+\frac{\partial}{\partial N}\right)\Omega=0,\quad \Omega=e^{-N}\left(b-\frac{e^{-N}}{2}\frac{\partial j^2}{\partial b}\right). \tag{11}$$

Here $w=w(N)=u/(\partial n^{-1}/\partial y)$, $u=u(y)$ is the velocity of the shock in the direction of the x axis, $j=\partial b/\partial\xi$ is the current density, ξ the coordinate across the front, and the time derivative $\partial/\partial t$ is substituted by $-u\partial/\partial x=uj(d\xi/dx)\partial/\partial b$. The boundary conditions in the new variables are

$$\Omega|_{N=0}=\Omega_0(b),\quad \Omega|_{b=0}=0, \tag{12a}$$

$$\Omega|_{b=b_s(N)}=b_s e^{-N}, \tag{12b}$$

$$\Pi|_{b=b_s(y)}=e^{-N}b_s^2/2, \tag{12c}$$

where $b_s(N)$ denotes the magnetic field behind the front [so that $b_S=b_s(0)$], and the potential $\Pi(b,N)$ is given by Eq. (7). Conditions (12b) and (12c) express the continuity of the vorticity and the current density, respectively. However, depending on the behavior of the characteristics of Eq. (11), the above boundary conditions may be insufficient. If the electrons behind the front move faster than the front [see Fig. 1(a) and Fig. 2(a)], the following condition should be added:

$$\Omega[b_s(N),N]=\Omega_0[b_s(N)],\quad b_s(N)<1, \tag{12d}$$

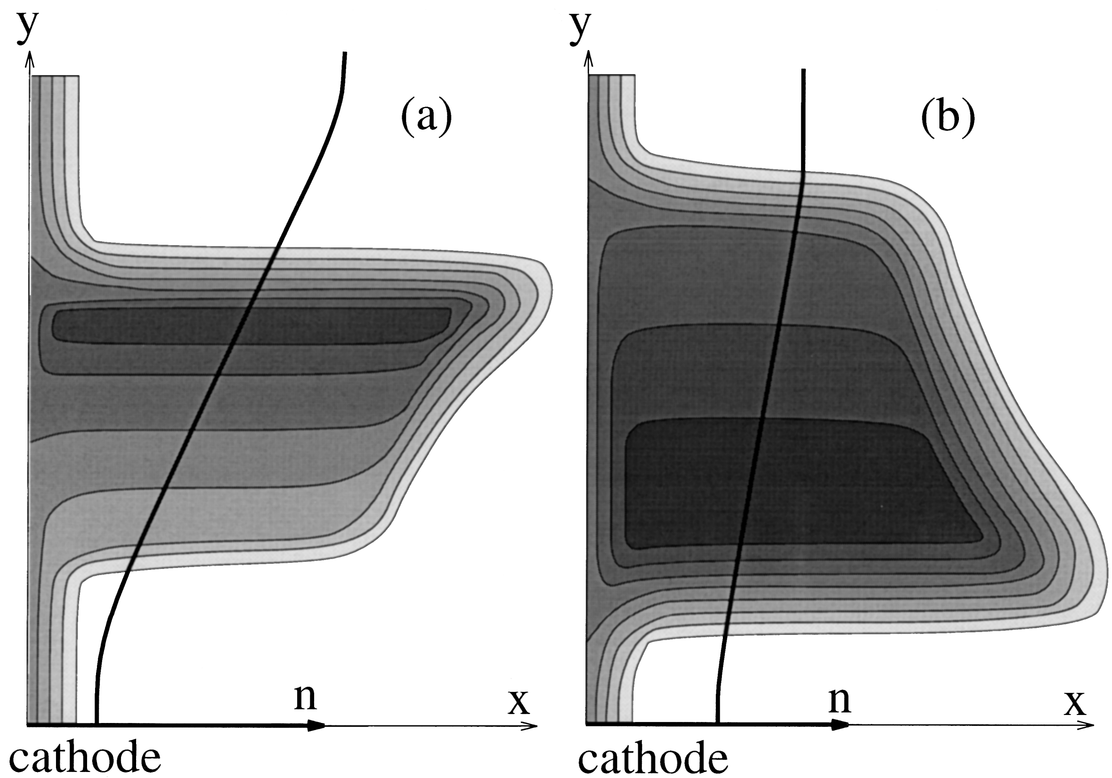

FIG. 1. Qualitative picture of the penetration. Darker regions correspond to stronger magnetic fields. The current created behind the shock can be faster (a) or slower (b) than the shock. Besides, if the plasma density varies strongly (a), the magnetic field penetrates only in a thin stripe at certain values of the density.

$$\Omega[b_s(N),N]=\Omega[b_s(N),N_1], \quad b_s(N)>1, \tag{12e}$$

where we have used the notation $N_1=N(y_1)$.

The velocity of the front w can be found from the boundary condition (12c) by differentiating it with respect to the coordinate y and taking into account conditions (12b) and (7),

$$w=b_s/2, \quad u=(b_s/2)\partial n^{-1}/\partial y. \tag{13}$$

It is worth emphasizing that the boundary-value problem (11) and (12) does not depend on the density profile $n(y)$.

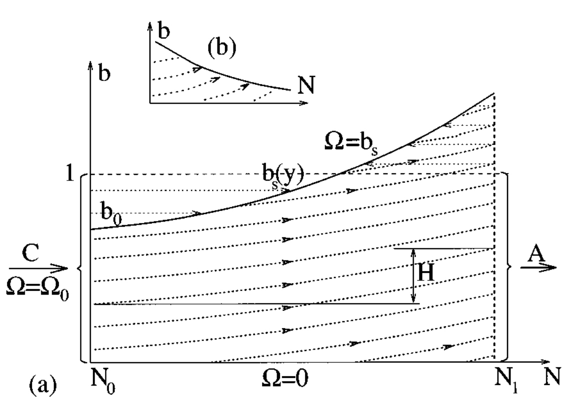

FIG. 2. A sketch of characteristics (bold dotted lines) of Eq. (11). The exact Eq. (3) enables us to continue the characteristics beyond the shock front (thin dotted lines). The characteristics originating at the cathode are designated by C, and those terminating at the anode, by A. H is the integral (19).

Equation (11) is integrated along the characteristics, as shown in Fig. 2

$$\Omega(b,N)=\Omega_0[b_0(b,N)],$$

$$b_0=b-\frac{1}{2}\int_0^N b_s(N')dN', \text{ if } b_0(b,N)<b_S, \tag{14a}$$

and

$$\Omega\left(b_s+\int_{N(b_s)}^N \frac{b_s(N')}{2}dN'\right)=\Omega(b_s), \text{ if } b_0(b,N)>b_S. \tag{14b}$$

Here $N(b_s)$ is the inverse of the function $b_s(N)$. In the expression (14a) the function $\Omega_0(b)$ is implied to be extrapolated to the negative values of $b<0$ to a constant $\Omega_0(b)=0$.

Still, our solution (14) includes the unknown function $b_s(N)$. The method to determine this function depends on the behavior of the characteristics (Fig. 2). Let us consider some particular cases. If the front moves faster than the electrons behind the front [$db_s/dN<w$; Fig. 2(b)], one can introduce the auxiliary function $b_{s0}(N)=b_0[b_s,N(b_s)]$, so that

$$b_s(N)=b_{s0}(N)+\int_0^N\frac{b_s(N')}{2}dN'. \tag{15}$$

Formula (15) can be rewritten in a differential form as

$$(d/dN)[b_s\exp(-N/2)]=\exp(-N/2)db_{s0}/dN.$$

Solution (14a) can be used to find $\Pi(b_s)=\Pi_0(b_{s0})$, so that the boundary condition (12c) yields

$$\exp(-N/2)=F(b_{s0}), \quad F(b)=d\sqrt{2\Pi_0(b)}/db. \tag{16}$$

Equation (16) defines implicitly the function $b_{s0}(N)$. Finally, the function $b_s(N)$ is found from the boundary condition (12b):

$$b_s=\exp(N)\Omega_0(b_{s0}). \tag{17}$$

If the front is slower than the electrons behind it $db_s/dN>w$ the unknown function can be found from the boundary conditions (12b), (12d), (12e). The case of $b_s(N)<1$ immediately results in

$$\Omega_0[b_s(N)]\exp(N)=b_s(N). \tag{18}$$

If the opposite inequality holds, there are two different regimes of penetration, depending on the extent of the plasma density variation. The electrons caught by the shock have never been at the resistive cathode and thus carry zero vorticity. In Fig. 2, these electrons correspond to the characteristics, originating from the bottom. Moving along the front towards the abode, the electrons ''climb up'' towards higher values of b (Fig. 2). In the case of a strong density variation, there is a point $N=N_1$ where they attain the value $b=1$. Thus, at that point, the electrons of zero vorticity occupy the whole interval ($b\in[0,1]$); further, they continue their motion towards the anode at the plasma boundary. Consequently in the region $y>y_1$ there will be the collisionless skin layer $b=\exp(-x)$. The other electrons ($b>1$) turn back towards the cathode, and start rotating in the electron vortex [Fig. 1(a)]. The upper boundary of the shock is then found from the condition

$$H=\int_0^{N_1}\frac{b_s}{2}dN=1. \tag{19}$$

In other words, the points $N=0$, $b=0$ and $N=N_1$, $b=1$ lie on the same characteristic (Fig. 2).

In the case of weak density variations, Eq. (19) cannot be satisfied. Indeed, upon substituting $N_1=\ln(n_A)$, we obtain $H<1$. Then the shock penetrates in the whole region of varying density, so that $y_1=a$.

In both cases, the desired function $b_S(N)$ can be found from the equation $b_s=\exp(N-N_1)\Omega(b_s,N_1)$ [which follows from Eqs. (12b) and (12e)], where the function $\Omega(b_s,N_1)$ is to be found from the expressions (14) and (18). Thus the function $b_s(N)$ can be constructed step by step towards higher values of the magnetic field, as is shown in Fig. 1(b). The procedure is rather sophisticated and is not presented here. Still, Eq. (19) provides a simple and useful estimate of the upper boundary of the shock N_1. For $b_s\approx 1$ we obtain $H\approx N_1=\ln[n(y_1)/n(y_0)]$. In the case of strong density variations, the upper boundary is given by the condition $H=1$, hence

$$n(y_1)\approx en(y_0). \tag{20}$$

In other words, the front of the shock will terminate before reaching the line $y=a$ if $\ln[n(a)/n(y_0)]\gtrsim 1$. In the plasma opening switches, the overall density variation can be more than an order of magnitude. In that case, the magnetic field will penetrate only into a narrow layer of the plasma in the vicinity of the cathode. The width of the layer is given by the estimate (20).

The problem is more complicated, if Eq. (8) formally yields $b_S=1$. In this case, the formally obtained value of the lower boundary of the shock y_{0*} would contradict the boundary conditions (12b) and (12c). Here the real lower boundary of the shock is somewhat closer to the cathode, and the penetrating magnetic field b_S is larger than the magnetic field at the plasma boundary $b_S>1$. The lower boundary of the shock is to be found from the system (9), where the extrapolated (to the region $b>1$) function $\Omega_0(b)$ should satisfy the following functional equation:

$$\Omega_0(b)=\Omega_0(b-H), \text{ if } db_s/dN>0,$$

$$\Omega_0(b_s)=b_s, \text{ if } db_s/dN<0.$$

In practice it may be difficult to solve this equation. However, it is possible to show that the shocklike solution does exist in that case as well. A possible scheme of currents is shown in Fig. 1(b).

As an illustration, we present the explicit solution for the model vorticity function

$$\Omega=b_1\Theta(b-b_2), \quad b_1,b_2<1,$$

and the density profile

$$n(y)=\exp(\alpha y).$$

Then in the region of the shock we have

$$y_0\lesssim y\lesssim y_1, \quad y_0=\alpha^{-1}\ln(2b_2/b_1), \quad y_1=y_0-\ln(b_2),$$

and the magnetic field is given by

$$b=b_1 n(y)\begin{cases} f[(2x-tb_1\alpha)/2\sqrt{n(y)}], & y_0\lesssim y\lesssim y_1, \\ f[(y_0-y)/\sqrt{n(y_0)}], & y\approx y_0, \\ f[(y-y_1)/\sqrt{n(y)}], & y\approx y_1, \end{cases}$$

where the notation is introduced,

$$f(x)=\begin{cases} 1-\exp(x), & x<0 \\ \exp(-x), & x>0 \end{cases}.$$

At the plasma boundary (i.e., in the region $0<x\lesssim 1$), there is a simple collisionless skin layer.

Thus we have proposed a simple model capturing essential features of 2D nonlinear collisionless penetration of the magnetic field into electron MHD plasmas. The magnetic field penetrates in the form of a growing electron vortex with a self-consistent profile of the magnetic field inside the vortex . The location and detailed structure of the vortex depend on the boundary conditions on the cathode. In the case of strong ion density variations, the vortex is localized over an octave of the density variation. So, the magnetic field penetrates only in a narrow stripe in the vicinity of the cathode. The theory also predicts a nonlinear enhancement of the magnetic field.

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