Three-dimensional perturbations in conformal turbulence

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The effects of three-dimensional perturbations in two-dimensional turbulence are investigated through a conformal field theory approach. We compute scaling exponents for the energy spectra of enstrophy and energy cascades, in a strong-coupling limit, and compare them with the values found in recent experiments. The extension of unperturbed conformal turbulence to the present situation is performed by means of a simple physical picture in which the existence of small-scale random forces is closely related to deviations of the exact two-dimensional fluid motion. [S1063-651X(96)09607-9]
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I. INTRODUCTION

It has been recognized that turbulence, with its manifold experimental realizations, is one of the challenging problems for which very different approaches, ranging from pure mathematics to engineering applications, have been developed in an interesting complementary way. One of the most important methods to study turbulence is, in fact, the formulation via field theory, based on its relationship with stochastic partial differential equations [1,2]. However, such a technique is far from being well established and complete, so that new ideas and important improvements are constantly appearing on the subject.

Recently, Polyakov suggested that nonunitary minimal models of conformal field theory could be used to describe the physics of two-dimensional turbulence [3]. The advantage of this proposal is that one can deal in a controllable way with a set of anomalous dimensions and short-distance products. An infinite number of inertial range exponents follows from this approach [4–6] and one of the still open problems is how to find ''selection rules,'' which would define the experimentally relevant minimal models or the connection between them and statistical characterizations of the random forces acting on the system. These ideas have attracted the attention of many authors and generalizations have been investigated, such as, for instance, possible boundary effects [7], alternative physical pictures for the enstrophy and energy cascades[8], and magnetohydrodynamic turbulence [9].

We will consider, in this paper, the problem of conformal turbulence including in its formalism the influence of three-dimensional effects. Our motivation comes from a number of experimental studies, in which approximately two-dimensional fluids were observed, together with the unavoidable presence of three-dimensional perturbations [10–13]. It was verified that a quasi-two-dimensional fluid is perturbed by small-scale forces originated from the the degrees of freedom related to the direction perpendicular to the plane of motion. We will take this fact into account, noting that there are also compressibility effects that cannot be neglected in an effective two-dimensional theory of the perturbed system. A generalization of the conformal approach will be devised and inertial range exponents will be obtained here in reasonable agreement with the experimental data.

This paper is organized as follows. In the next section we briefly review the most important and practical aspects of conformal turbulence in order to make the paper as self-contained as possible. In Sec. III we discuss some of the experiments carried out to investigate two-dimensional turbulence. This will motivate us to define an effective (and infinite) set of stochastic partial differential equations that represents a quasi-two-dimensional fluid under the influence of three-dimensional perturbations. The conformal approach is then introduced in order to solve the Hopf equations for the turbulence problem. Furthermore, the constant enstrophy and energy flux conditions are also studied. Explicit solutions are found and described in Sec. IV and, in Sec. V, the problem of boundary effects is discussed. Finally, in Sec. VI we comment on our results and on possible directions for future investigations.

II. CONFORMAL TURBULENCE

The minimal models of conformal field theory [14] are generically defined by a pair of relatively prime numbers (p,q) with $p<q$. These models contain a subset of $(p-1)(q-1)/2$ scalar primary operators $\psi_{(m,n)}$, labeled by $1\leqslant m<p$ and $1\leqslant n\leqslant(q-1)/2$ if p is even or $1\leqslant m\leqslant(p-1)/2$ and $1\leqslant n<q$ otherwise, having dimensions $\Delta_{(m,n)}=[(pn-qm)^2-(p-q)^2]/4pq$. The reason for the choice of scalar operators is that we will be dealing with isotropic correlation functions in the turbulence problem. The operator product expansion (OPE) of two primary operators $\psi_{(r_1,s_1)}(z)$ and $\psi_{(r_2,s_2)}(z')$, with $|z-z'|\rightarrow 0$, is written as

$$\psi_{(r_1,s_1)}(z)\psi_{(r_2,s_2)}(z') = \sum_{(r_3,s_3)} (a\overline{a})^{(\Delta_{(r_3,s_3)}-\Delta_{(r_1,s_1)}-\Delta_{(r_2,s_2)})} \times\sum_{(n,m)} C^{(r_3,s_3)}_{\{(n_1,\ldots,n_k);(m_1,\ldots,m_l)\}} L_{-n_1}\ldots L_{-n_k} \times\overline{L}_{-m_1}\ldots\overline{L}_{-m_l} a^{\Sigma n}\overline{a}^{\Sigma m}\psi_{(r_3,s_3)}(z), \quad (2.1)$$

*Electronic address: moriconi@if.ufrj.br where $|r_1-r_2|+1\leqslant r_3\leqslant \min(r_1+r_2-1,2p-r_1-r_2-1)$, $|s_1-s_2|+1\leqslant s_3\leqslant \min(s_1+s_2-1,2q-s_1-s_2-1)$, and we have introduced, in (2.1), the Virasoro generators of conformal transformations L_{-n} and $\overline{L}_{-n}$. The interest in these models is related not only to their finite number of primary operators but also to the fact that their dimensions and the form of short-distance products are completely known.

Let us look now at the problem of turbulence in two dimensions and show how it may be matched [3] with the above operator structures. The motion of an incompressible fluid is assumed, even in the turbulent regime, to be described by the Navier-Stokes equations for the velocity field

$$\partial_t v_\alpha+\left(\delta_{\alpha\gamma}-\frac{\partial_\alpha\partial_\gamma}{\partial^2}\right)v_\beta\partial_\beta v_\gamma=f_\alpha+\nu\partial^2 v_\alpha, \tag{2.2}$$

where f_α represents a random force acting at large scales, determined by a characteristic length L, and $\nu\to 0$ is the viscosity, associated with the small scale where dissipation effects come into play, yielding a natural uv cutoff for the system. In terms of the stream function ψ, related to the velocity field by $v_\alpha=\epsilon_{\beta\alpha}\partial_\beta\psi$, we may write the following equation for the vorticity field $\omega=\partial^2\psi$:

$$\partial_t\omega+\epsilon_{\alpha\beta}\partial_\alpha\psi\partial^2\partial_\beta\psi=\epsilon_{\alpha\beta}\partial_\alpha f_\beta+\nu\partial^2\omega. \tag{2.3}$$

One of the fundamental problems of turbulence theory is to find solutions of the Hopf equations, for statistical averages over realizations of the velocity field,

$$\partial_t[\langle\omega(x_1,t)\omega(x_2,t)\cdots\omega(x_n,t)\rangle]=0, \tag{2.4}$$

where the time derivative is expressed through the use of Eq. (2.3). In the inertial range, the standard view of the problem is that both forcing and viscosity terms may be neglected in order to formulate an effective set of Hopf equations. Considering, furthermore, the convection term in (2.3) as a vanishing point-split product of fields, that is, $\oint_{|z-z'|=|a|}(dz'/a)\epsilon_{\alpha\beta}\partial_\alpha\psi(z)\partial^2\partial_\beta\psi(z')\to 0$, when $|z-z'|\to 0$, we would have, then, an exact solution of (2.4). A concrete realization of this possibility may be achieved if we regard the stream function ψ as a primary operator of some conformal minimal model. In this case we may use all the available information on operator dimensions and OPE's to extract physical results from the analysis of the problem. According to this assumption, let ϕ be the primary operator that has the lowest dimension $\Delta\phi$ appearing in the OPE $\psi\psi$, between fields with the same dimension $\Delta\psi$. Taking $a\equiv|a|\exp(i\theta)$, we will thus have

$$\begin{aligned}\lim_{|a|\to 0}\oint_{|z-z'|=|a|}&\frac{dz'}{a}\epsilon_{\alpha\beta}\partial_\alpha\psi(z)\partial^2\partial_\beta\psi(z')\\ &\sim\int d\theta[\partial_a^2\partial_a\partial_z-\partial_a^2\partial_{\overline{a}}\partial_{\overline{z}}](a\overline{a})^{(\Delta\phi-2\Delta\psi)}\\ &\quad\times\sum C_{\{n;m\}}L_{-n_1}\cdots L_{-n_k}\overline{L}_{-m_1}\cdots\overline{L}_{-m_l}a^{\Sigma n}\overline{a}^{\Sigma m}\phi(z,\overline{z})\\ &\sim(a\overline{a})^{(\Delta\phi-2\Delta\psi)}[L_{-2}\overline{L}^2_{-1}-\overline{L}_{-2}L^2_{-1}]\phi,\end{aligned} \tag{2.5}$$

as the dominant contribution in this short distance product. It is important to note that in order to get (2.5) it was necessary to set $C_{\{1;2\}}=C_{\{2;1\}}$ and $C_{\{1;(1,1)\}}=C_{\{(1,1);1\}}$, as it follows from the pseudoscalar nature of the ϵ factor above. We then see that (2.5) vanishes with $|a|\to 0$ if

$$\Delta\phi>2\Delta\psi, \tag{2.6}$$

which is one of the constraints that the chosen minimal model has to satisfy. An additional constraint comes from the condition of a constant enstrophy or energy flux through the inertial range, meaning that $\langle\dot{\omega}(x)\omega(0)\rangle\sim x^0$ or $\langle\dot{v}_\alpha(x)v_\alpha(0)\rangle\sim x^0$, respectively. In the case of a constant enstrophy flux, we have

$$\begin{aligned}\langle\dot{\omega}(x)\omega(0)\rangle\sim(a\overline{a})^{(\Delta\phi-2\Delta\psi)}\langle[(L_{-2}\overline{L}^2_{-1}\\ -\overline{L}_{-2}L^2_{-1})\phi(x)]\partial^2\psi(0)\rangle.\end{aligned} \tag{2.7}$$

The correlation function on the right-hand side of (2.7) is now evaluated by means of a purely dimensional argument, as $L^{-2(\Delta\phi+\Delta\psi+3)}$, which makes sense if one thinks that there is an effective ir cutoff in the theory at the scales where the forcing terms act. Imposing (2.7) to be independent of L, we get

$$\Delta\phi+\Delta\psi+3=0. \tag{2.8}$$

In the case of an energy cascade, the argument is the same and the constraint turns out to be $\Delta\phi+\Delta\psi+2=0$. It is known that there is an infinite number of minimal models compatible with (2.6) and (2.8) [4]. The general belief, and still an open problem, is that there may be universality classes, associated with the statistical properties of the forcing terms, that would single out one or another of the possible solutions.

An alternative analysis of conformal turbulence regards the existence of boundary effects on the vacuum expecation values (VEV's) of single operators in nonunitary theories [15]. In this case, one has to consider the OPE between $\phi(x)$ and $\psi(0)$ in (2.7), picking up the most relevant operator, let us say χ. Now (2.8) is modified to $\Delta\phi+\Delta\psi-\Delta\chi+3=0$, with an analogous change for the constant energy flux condition. Some of these further solutions (in the enstrophy cascade picture) were obtained in Ref. [7].

The connection of the conformal approach with real experiments or numerical simulations is made through the computation of inertial range exponents, which describe the decrease of energy in the region of higher Fourier modes. In the situation where VEV's of single operators vanish, the inertial range exponents are given by $4\Delta\psi+1$ and, in the opposite case, by $4\Delta\psi-2\Delta\phi+1$. Good agreement has been reached between the former possibility, for the the direct enstrophy cascade case, and numerical simulations [16,17] of the two-dimensional Navier-Stokes equations.

III. THREE-DIMENSIONAL EFFECTS

In a series of interesting experiments, Hopfinger *et al.* [10–12] studied the turbulence phenomenon as it happens in a rotating tank, where at its bottom there was an oscillating grid responsible for perturbations of the fluid motion. According to the Taylor-Proudman theorem [18–20], a rotating fluid tends to behave as if it were two dimensional and in fact this was observed in the form of coherent structures (vortices) organized in the direction parallel to the rotation axis of the tank. However, ''defects'' in the vortices were seen to propagate from the very turbulent region at the bottom of the tank up to the effectively two-dimensional system. The essential picture extracted from these observations is that the fluid should be best described in terms of two-dimensional equations containing not only large-scale forcing terms but also small-scale random perturbations, originated from either vortex breakdown or soliton pulses propagating along vorticity filaments. The experimental data suggested then the existence of an inertial range, likely to be related to a direct enstrophy cascade and well approximated by $E(k)\sim k^{-2.5}$, which represents a less steep energy spectrum than the one obtained by Kraichnan [21], $E(k)\sim k^{-3}$, or even other proposals [22,23], not excluding conformal turbulence [4]. This puzzling result is presently understood to be due only to the measurement techniques used in the experiments, based on the analysis of the dispersion of suspended particles in the fluid [12]. More recently, similar experiments were conduced by Narimousa *et al*. [13] and direct measurements of the turbulent velocity field were recorded. The results pointed out the existence of a possible energy spectra $E(k)\sim k^{-5/3}$ at lower wave numbers, in agreement with the conjecture of an inverse energy cascade [21], and a range at higher wave numbers, where $E(k)\sim k^{-5.5\pm 0.5}$. In this region, the spectral slope was seen to depend on the controlling external conditions, with results varying from $E(k)\sim k^{-5.0}$ up to $E(k)\sim k^{-6.0}$. It is worth noting that a spectral law $E(k)\sim k^{-5}$ follows from Rhines's theory of β-plane turbulence [24] and, alternatively, is closely approximated by some solutions of the constant enstrophy flux condition in the conformal approach, such as the minimal models (9,71) or (11,87).

The variation of exponents obtained in the experiments may have a theoretical counterpart in the existence of a set of operator anomalous dimensions, making it interesting to analyze the problem from the conformal field theory point of view. It is clear, however, that the inertial range exponents, found in Ref. [4], cannot reproduce the experimental situation. We believe that the important ingredient, missing in the previous conformal approach, is precisely the existence of three-dimensional perturbations, which must be taken into account in any realistic model of a quasi-two-dimensional fluid.

In view of the above considerations, let us write the two-dimensional Navier-Stokes equations as

$$\partial_t v_\alpha + v_\beta \partial_\beta v_\alpha = \nu \partial^2 v_\alpha + f_\alpha^{(1)} + g f_\alpha^{(2)} - \partial_\alpha P, \tag{3.1}$$

where $f_\alpha^{(1)}$ and $f_\alpha^{(2)}$ are stirring forces defined at large (L) and small ($\mu \ll L$) scales, respectively. The dimensionless constant g represents, roughly, a coupling with the three-dimensional modes of the fluid. We assume that the dissipation scale a is related, in principle, to the other scales of the problem as $a \ll \mu \ll L$. This means that even though the perturbations act at very small scales, when compared to the macroscopic size of the system, they are still much larger than the scale where dissipation occurs.

An important point here is that the condition of incompressibility, when formulated in three dimensions, reads $\partial_1 v_1 + \partial_2 v_2 + \partial_3 v_3 = 0$, suggesting that the ''projection'' of this constraint to the two-dimensional world has to be given by $\partial_\alpha v_\alpha = O(g)$, in the framework of Eq. (3.1). The velocity field may be described, then, by means of a stream function ψ and a velocity potential ϕ as

$$v_\alpha = \epsilon_{\beta\alpha} \partial_\beta \psi + g \partial_\alpha \phi. \tag{3.2}$$

It is of further interest to study, besides the vorticity ω, the divergence of v_α, given by $\rho = g \partial^2 \phi$. An exact, although infinite, chain of equations may be generated if we expand ψ and ϕ in powers of g, substituing them into (3.1) and collecting the coefficients of the obtained series. Defining, in this way,

$$\psi = \sum_{n=0}^{\infty} g^n \psi^{(n)}, \quad \omega = \sum_{n=0}^{\infty} g^n \omega^{(n)},$$

$$\phi = \sum_{n=0}^{\infty} g^n \phi^{(n)}, \quad \rho = \sum_{n=0}^{\infty} g^{n+1} \rho^{(n)}, \tag{3.3}$$

we get the set of coupled equations

$$\partial_t \omega^{(n)} + \sum_{p=0}^{n} \epsilon_{\alpha\beta} \partial_\alpha \psi^{(p)} \partial_\beta \partial^2 \psi^{(n-p)} + \sum_{p=0}^{n-1} [\partial_\beta \phi^{(p)} \partial_\beta \partial^2 \psi^{(n-p-1)} + \partial^2 \phi^{(p)} \partial^2 \psi^{(n-p-1)}] = \nu \partial^2 \omega^{(n)} + \epsilon_{\alpha\beta} \partial_\alpha f_\beta^{(2)} \delta_{n,1}, \tag{3.4a}$$

$$\partial_t \omega^{(0)} + \epsilon_{\alpha\beta} \partial_\alpha \psi^{(0)} \partial_\beta \partial^2 \psi^{(0)} = \nu \partial^2 \omega^{(0)} + \epsilon_{\alpha\beta} \partial_\alpha f_\beta^{(1)}, \tag{3.4b}$$

$$\begin{aligned} \partial_t \rho^{(n)} &+ \sum_{p=0}^{n-1} [\partial_\alpha \partial_\beta \phi^{(p)} \partial_\alpha \partial_\beta \phi^{(n-p-1)} + \partial_\alpha \phi^{(p)} \partial_\alpha \partial^2 \phi^{(n-p-1)}] + \sum_{p=0}^{n} [2 \epsilon_{\alpha\beta} \partial_\beta \partial_\sigma \phi^{(p)} \partial_\alpha \partial_\sigma \psi^{(n-p)} + \epsilon_{\alpha\beta} \partial_\alpha \psi^{(n-p)} \partial_\beta \partial^2 \phi^{(p)}] \\ &+ \sum_{p=0}^{n+1} [\partial_\alpha \partial_\beta \psi^{(p)} \partial_\alpha \partial_\beta \psi^{(n-p+1)} - \partial^2 \psi^{(p)} \partial^2 \psi^{(n-p+1)}] = \nu \partial^2 \rho^{(n)}, \end{aligned} \tag{3.4c}$$

$$\partial_t \rho^{(0)} + 2 \partial_\alpha \partial_\beta \psi^{(0)} \partial_\alpha \partial_\beta \psi^{(1)} + 2 \epsilon_{\alpha\beta} \partial_\beta \partial_\sigma \phi^{(0)} \partial_\alpha \partial_\sigma \psi^{(0)} + \epsilon_{\alpha\beta} \partial_\alpha \psi^{(0)} \partial_\beta \partial^2 \phi^{(0)} - 2 \partial^2 \psi^{(0)} \partial^2 \psi^{(1)} = \nu \partial^2 \rho^{(0)} + \partial_\alpha f_\alpha^{(2)}, \tag{3.4d}$$

and, finally, the constraint of incompressibility for the g-independent part of the velocity field, which defines the pressure term,

$$(\partial_\alpha\partial_\beta\psi^{(0)})(\partial_\alpha\partial_\beta\psi^{(0)})-\partial^2\psi^{(0)}\partial^2\psi^{(0)}=\partial_\alpha f_\alpha^{(1)}-\partial^2 P. \quad (3.5)$$

In the above expressions, $n\geqslant 1$. We have obtained, therefore, a set of stochastic partial differential equations. In a statistical description, reflecting a stable asymptotic limit for the correlation functions of ω and ρ, Hopf equations may be straightforwardly written as

$$\partial_t\left\langle \prod_{i=1}^{N} \omega^{(n_i)}(x_i,t)\prod_{j=N+1}^{M} \rho^{(n_j)}(x_j,t)\right\rangle=0. \quad (3.6)$$

We observe now that Eq. (3.4b) is identical to the one that corresponds to an unperturbed ($g=0$) two-dimensional fluid. This means that the field $\psi^{(0)}$ will be related to an enstrophy or energy cascade, even in the presence of three-dimensional effects. This field plays the role of an external random variable in the other equations, since its dynamics is independent of the other components $\psi^{(n)}$ or to the field ϕ (in general, the subset $\{\psi^{(0)},\psi^{(1)},\ldots,\psi^{(n)},\phi^{(0)},\phi^{(1)},\ldots,\phi^{(n-1)}\}$ contains fields that act like external random perturbations in the equations for $\psi^{(p)}$ and $\phi^{(p-1)}$, with $p\geqslant n+1$). Considering that (3.4) gives relatively complex equations, the analysis of the problem might seem hopeless, perhaps being addressed only to a numerical treatment. However, we can extend the conformal approach, applied previously to the unperturbed case, to find here solutions of the Hopf equations. Our basic assumption is that not only $\psi^{(0)}$ but also the other components in the power expansions of ψ and ϕ are primary operators that belong to some minimal model in a conformal field theory. It is necessary, then, to define a scale l, possibly associated with intermittency effects, which allows us to write the dimensionally correct expansion

$$\psi=\sum_{n=0}^{\infty} f_n l^{2(\Delta\psi^{(n)}-\Delta\psi^{(0)})}g^n\psi^{(n)},$$

$$\phi=\sum_{n=0}^{\infty} f'_n l^{2(\Delta\phi^{(n)}-\Delta\psi^{(0)})}g^n\phi^{(n)}, \quad (3.7)$$

where $\psi^{(n)}$ and $\phi^{(n)}$ have dimensions $\Delta\psi^{(n)}$ and $\Delta\phi^{(n)}$, respectively.

The introduction of a scale l in (3.7) means that the perturbed system exhibits a breaking of scale invariance in the inertial range. We will see later that this phenomenon is signaled by the existence of constant enstrophy or energy fluxes that depend on the small scales of three-dimensional perturbations. It is conceptually important to understand the physical origin of l. A clue for this comes from the structure of couplings between $\psi^{(0)}$ and the other fields, as expressed in Eq. (3.4). As we have already observed, $\psi^{(0)}$ is effectively an external field in the equations for $\psi^{(n)}$ (with $n\geqslant 1$) and $\phi^{(n)}$ (for any n). In this way, it is plausible to have a relation between l and the scales involved in the dynamics of $\psi^{(0)}$. Now, if we consider the turbulent limit of the equations for $\psi^{(0)}$, corresponding to $\nu\to 0$ (or, alternatively, $a\to 0$), we are left essentially with the correlation length L of large scale random forces. A simple choice thus is to consider $l=L$. In this respect, one may observe that the small scale μ could also be used in the definition of l. We have, however, physical reasons to believe that this does not happen: μ is related to the forcing terms in the equations for $\psi^{(1)}$ and $\phi^{(1)}$, which we expect to be irrelevant when compared to the nonlinear convection terms in the range of wave numbers given by $|\vec{k}|\ll 1/\mu$.

It is interesting to note that there is an analogy between our problem and the statistical mechanics of second-order phase transitions for a system close to its critical point. In this case, one can study deviations of the critical temperature T_c by means of an expansion in $(T-T_c)$ and through the use of the operator structure of the critical theory [25]. Here, in the turbulence context, the ''critical theory'' is just what we get when $g\to 0$.

In order to simplify the notation we will keep using (3.3), with the above observations in mind. We are interested to obtain possible combinations of primary operators, in Eq. (3.7), that would not affect, in the limit $\mu\to 0$, the constant enstrophy or energy fluxes, obtained from the dynamics of the field $\psi^{(0)}$. Within this point of view, it is important, therefore, to consider short-distance products of a certain number operators, as it follows from (3.4). Taking two generic primary operators $O_1^{(p)}$ and $O_2^{(p')}$ (for example, $\phi^{(p)}$ and $\psi^{(p')}$), with dimensions $\Delta O_1^{(p)}$ and $\Delta O_2^{(p')}$, respectively, we may write

$$O_1^{(p)}(z,\overline{z})O_2^{(p')}(z',\overline{z'})=(a\overline{a})^{(\Delta A_{O_1,O_2}^{(p,p')}-\Delta O_1^{(p)}-\Delta O_2^{(p')})} \times\sum C_{\{(n_1,\ldots,n_k);(m_1,\ldots,m_l)\}}^{O_1^{(p)},O_2^{(p')}} L_{-n_1}\cdots L_{-n_k}\overline{L_{-m_1}}\cdots\overline{L_{-m_l}}a^{\Sigma n}\overline{a}^{\Sigma m}A_{O_1,O_2}^{(p,p')}(z,\overline{z}), \quad (3.8)$$

where $A_{O_1,O_2}^{(p,p')}$ is the primary operator with the lowest dimension in the above OPE. The short-distance products appearing in (3.4) are listed below together with the conformal field theory representation, obtained after straightforward computations:

$$\epsilon_{\alpha\beta}\partial_\alpha\psi^{(p)}\partial_\beta\partial^2\psi^{(p')}\sim(a\overline{a})^{(\Delta A_{\psi\psi}^{(p,p')}-\Delta\psi^{(p)}-\Delta\psi^{(p')})}[L_{-2}\overline{L}_{-1}^2-\overline{L}_{-2}L_{-1}^2]A_{\psi\psi}^{(p,p')},$$

$$\partial_\beta\phi^{(p)}\partial_\beta\partial^2\psi^{(p')}\sim\partial^2\phi^{(p)}\partial^2\psi^{(p')}\sim(a\overline{a})^{(\Delta A_{\phi\psi}^{(p,p')}-\Delta\phi^{(p)}-\Delta\psi^{(p')}-1)}L_{-1}\overline{L}_{-1}A_{\phi\psi}^{(p,p')},$$

$$\partial_\alpha\partial_\beta\phi^{(p)}\partial_\alpha\partial_\beta\phi^{(p')}\sim\partial_\alpha\phi^{(p)}\partial_\alpha\partial^2\phi^{(p')}\sim(a\overline{a})^{(\Delta A_{\phi\phi}^{(p,p')}-\Delta\phi^{(p)}-\Delta\phi^{(p')}-1)}L_{-1}\overline{L}_{-1}A_{\phi\phi}^{(p,p')},$$

$$\epsilon_{\alpha\beta}\partial_\beta\partial_\sigma\phi^{(p)}\partial_\alpha\partial_\sigma\psi^{(p')}\sim\epsilon_{\alpha\beta}\partial_\alpha\psi^{(p')}\partial_\beta\partial^2\phi^{(p)}\sim(a\overline{a})^{(\Delta A_{\phi\psi}^{(p,p')}-\Delta\phi^{(p)}-\Delta\psi^{(p')})}[L_{-2}\overline{L}_{-1}^2-\overline{L}_{-2}L_{-1}^2]A_{\phi\psi}^{(p,p')},$$

$$\partial_\alpha\partial_\beta\psi^{(p)}\partial_\alpha\partial_\beta\psi^{(p')}\sim\partial^2\psi^{(p)}\partial^2\psi^{(p')}\sim(a\overline{a})^{(\Delta A_{\psi\psi}^{(p,p')}-\Delta\psi^{(p)}-\Delta\psi^{(p')}-1)}A_{\psi\psi}^{(p,p')}, \tag{3.9}$$

with $p,p'\geqslant 0$, except in the last relation, for the product of the type $\psi\psi$, where $p+p'\geqslant 1$. From (3.9) we see clearly that Hopf equations are satisfied if

$$\Delta A_{\psi\psi}^{(0,0)}-2\Delta\psi^{(0)}>0, \tag{3.10a}$$

$$\Delta A_{\psi\psi}^{(p,p')}-\Delta\psi^{(p)}-\Delta\psi^{(p')}-1>0, \tag{3.10b}$$

$$\Delta A_{\psi\phi}^{(p,p')}-\Delta\psi^{(p)}-\Delta\phi^{(p')}-1>0, \tag{3.10c}$$

$$\Delta A_{\phi\phi}^{(p,p')}-\Delta\phi^{(p)}-\Delta\phi^{(p')}-1>0, \tag{3.10d}$$

with $p+p'\geqslant 1$ in (3.10b) and $p,p'\geqslant 0$ in (3.10c) and (3.10d). These equations are the first step in the generalization of unperturbed conformal turbulence, in order to deal with a larger set of primary operators. We will find more inequalities, restricting, then, up to some extent the number of different operators allowed in the theory. Let us now write the conditions for constant enstrophy or energy fluxes through the inertial range. Here we will assume that vacuum expectation values of primary operators are zero. The alternative possibility is discussed in Sec. V. The case of a constant enstrophy flux requires, as commented before, that we compute $\langle\dot\omega(x)\omega(0)\rangle$. From relations (3.4) we get

$$\langle\dot\omega(x)\omega(0)\rangle=\sum_{n,m}^{\infty}\sum_{p=0}^{n}g^{n+m}\langle\{\epsilon_{\alpha\beta}\partial_\alpha\psi^{(p)}\partial_\beta\partial^2\psi^{(n-p)}+g[\partial_\beta\phi^{(p)}\partial_\beta\partial^2\psi^{(n-p)}+\partial^2\phi^{(p)}\partial^2\psi^{(n-p)}]\}|_x\partial^2\psi^{(m)}(0)\rangle. \tag{3.11}$$

In the above expression we may define a ''large-scale'' part as the one that depends solely on the field $\psi^{(0)}$ and a ''small-scale'' part as involving the fields $\phi^{(p)}$ and the other components of ψ. Regarding the large-scale part, we already know from the study of the unperturbed case that the constant enstrophy flux condition is

$$\Delta A_{\psi\psi}^{(0,0)}+\Delta\psi^{(0)}+3=0. \tag{3.12}$$

It is natural to assume, like in the case of unperturbed conformal turbulence, that the correlation functions in the small-scale part may also be evaluated by means of a dimensional argument, where, instead of using the typical large scale parameter L, the correct choice turns out to be the small length scale μ. Assuming, furthermore, that $\mu\to 0$ leads to a well defined limit, we just require the powers of μ in the most relevant terms belonging to the small scale part of (3.11) (contributions that have the lowest power of $a\overline{a}$) to be nonnegative numbers. This discussion may be restated by saying that we will need to select one or both of the conditions

$$\Delta A_{\phi\psi}^{(p,p')}+\Delta\psi^{(p'')}+2\leqslant 0,$$

$$\Delta A_{\psi\psi}^{(p,p')}+\Delta\psi^{(p'')}+3\leqslant 0, \tag{3.13}$$

according to the analysis of the dominant terms in the small scale part of $\langle\dot\omega(x)\omega(0)\rangle$. In the derivation of (3.13) we have used the OPE's computed from the Hopf equations, given by (3.9). Additional care must be taken if it happens that $A_{\psi\psi}^{(p,p')}=\psi^{(p'')}$ or $A_{\phi\psi}^{(p,p')}=\psi^{(p'')}$ for some values of p,p', and p''. In this circumstance it is necessary to have $\Delta\psi^{(p'')}=-3/2$ or $\Delta\psi^{(p'')}=-1$, respectively, to ensure spatially independent correlation functions and hence a constant enstrophy flux.

Let us turn now to the case of a constant energy flux. We have

$$\langle\dot v_\alpha(x)v_\alpha(0)\rangle=-\sum_{n,m}^{\infty}\sum_{p=0}^{n}g^{n+m}\langle\{g^2\partial_\beta\phi^{(p)}\partial_\beta\partial_\alpha\phi^{(n-p)}+g[\epsilon_{\gamma\alpha}\partial_\beta\phi^{(p)}\partial_\beta\partial_\gamma\psi^{(n-p)}+\epsilon_{\sigma\beta}\partial_\sigma\psi^{(p)}\partial_\beta\partial_\alpha\phi^{(n-p)}]+\partial_\alpha\partial^{-2}[\partial^2\psi^{(0)}\partial^2\psi^{(0)}-\partial_\sigma\partial_\beta\psi^{(0)}\partial_\sigma\partial_\beta\psi^{(0)}]\delta_{n,0}+\epsilon_{\sigma\beta}\epsilon_{\gamma\alpha}\partial_\sigma\psi^{(p)}\partial_\beta\partial_\gamma\psi^{(n-p)}\}|_x[\epsilon_{\eta\alpha}\partial_\eta\psi^{(m)}(0)+g\partial_\alpha\phi^{(m)}(0)]\rangle. \tag{3.14}$$

Here we cannot refer immediately to the Hopf equations and formulate a set of conditions, as we did in the constant enstrophy flux case. There are, in (3.14), OPE's that do not appear in (3.4), viz.,

$$\epsilon_{\sigma\beta}\epsilon_{\gamma z}\partial_\sigma\psi^{(p)}\partial_\beta\partial_\gamma\psi^{(p')}\sim(a\bar{a})^{(\Delta A^{(p,p')}_{\psi\psi}-\Delta\psi^{(p)}-\Delta\psi^{(p')}-1)}L_{-1}A^{(p,p')}_{\psi\psi},$$

$$\epsilon_{\sigma\beta}\epsilon_{\gamma z}\partial_\sigma\psi^{(0)}\partial_\beta\partial_\gamma\psi^{(0)}+\partial_z\partial^{-2}[\partial^2\psi^{(0)}\partial^2\psi^{(0)}-\partial_\sigma\partial_\beta\psi^{(0)}\partial_\sigma\partial_\beta\psi^{(0)}]\sim(a\bar{a})^{(\Delta A^{(0,0)}_{\psi\psi}-2\Delta\psi^{(0)})}[\bar{L}_{-1}L_{-2}-4\partial^{-2}L^3_{-1}\bar{L}_{-2}]A^{(0,0)}_{\psi\psi},$$

$$\epsilon_{\gamma z}\partial_\beta\phi^{(p)}\partial_\beta\partial_\gamma\psi^{(p')}\sim\epsilon_{\sigma\beta}\partial_\sigma\psi^{(p')}\partial_\beta\partial_z\phi^{(p)}\sim(a\bar{a})^{(\Delta A^{(p,p')}_{\phi\psi}-\Delta\phi^{(p)}-\Delta\psi^{(p')}-1)}L_{-1}A^{(p,p')}_{\phi\psi},$$

$$\partial_\beta\phi^{(p)}\partial_\beta\partial_z\phi^{(p')}\sim(a\bar{a})^{(\Delta A^{(p,p')}_{\phi\phi}-\Delta\phi^{(p)}-\Delta\phi^{(p')}-1)}L_{-1}A^{(p,p')}_{\phi\phi}. \qquad (3.15)$$

In (3.15) we have $\epsilon_{\gamma z}\equiv(\epsilon_{\gamma 1}-i\epsilon_{\gamma 2})/2$. Using the above point-split products, we can get the constant energy flux conditions. First, the equation following from the large-scale part of $\langle\dot{v}(x)v(0)\rangle$,

$$\Delta A^{(0,0)}_{\psi\psi}+\Delta\psi^{(0)}+2=0, \qquad (3.16)$$

and then the inequalities that come from the small-scale terms,

$$\Delta A^{(p,p')}_{\psi\psi}+\Delta\psi^{(p'')}+1+\delta_{p+p',0}\leqslant 0,$$

$$\Delta A^{(p,p')}_{\psi\psi}+\Delta\phi^{(p'')}+1+\delta_{p+p',0}\leqslant 0,$$

$$\Delta A^{(p,p')}_{\phi\phi}+\Delta\psi^{(p'')}+1\leqslant 0,$$

$$\Delta A^{(p,p')}_{\phi\phi}+\Delta\phi^{(p'')}+1\leqslant 0,$$

$$\Delta A^{(p,p')}_{\phi\psi}+\Delta\phi^{(p'')}+1\leqslant 0,$$

$$\Delta A^{(p,p')}_{\phi\psi}+\Delta\psi^{(p'')}+1\leqslant 0, \qquad (3.17)$$

where, analogously to the enstrophy cascade case, only a subset of (3.17) has to be considered. Supplementary relations, like the ones obtained after Eq. (3.13), are in order to avoid possible x-dependent correlation functions. We are led, here, to

$$\Delta\psi^{(p)}=-1 \quad \text{if } A^{(0,0)}_{\psi\psi}=\psi^{(p)},$$

$$\Delta\phi^{(p)}=-1 \quad \text{if } A^{(0,0)}_{\psi\psi}=\phi^{(p)},$$

$$\Delta\psi^{(p'')}=-1/2 \quad \text{if } A^{(p,p')}_{\psi\psi}=\psi^{(p'')} \quad \text{for } p+p'>0,$$
$$\text{or } A^{(p,p')}_{\phi\psi}=\psi^{(p'')} \text{ or } A^{(p,p')}_{\phi\phi}=\psi^{(p'')},$$

$$\Delta\phi^{(p'')}=-1/2 \quad \text{if } A^{(p,p')}_{\psi\psi}=\phi^{(p'')} \quad \text{for } p+p'>0,$$
$$\text{or } A^{(p,p')}_{\phi\psi}=\phi^{(p'')} \text{ or } A^{(p,p')}_{\phi\phi}=\phi^{(p'')}. \qquad (3.18)$$

Once we have some solution at hand, derived from the conditions obtained here, we may associate inertial range exponents to each one of the fields $\psi^{(p)}$ and $\phi^{(p)}$, expressed by $4\Delta\psi^{(p)}+1$ and $4\Delta\phi^{(p)}+1$, respectively. From these values we have to select the one that will appear effectively in experimental situations. This problem is investigated in the following section.

IV. ANALYSIS OF THE CONSTANT FLUX CONDITIONS

We have obtained so far all the conditions necessary to find minimal models related to an enstrophy or energy cascade in a quasi-two-dimensional fluid. In order to explore them, the first observation we can make is that these models must belong to the infinite set of solutions found in the former study of unperturbed conformal turbulence. This follows directly from the conditions that depend only on $\psi^{(0)}$. A strategy of computation could thus be just a numerical analysis of all possible combinations of fields for these previously known minimal models. As straightforward as it may sound, this approach is hardly useful when the number of primary operators becomes large, a fact that happens already for the first few minimal models.

A more interesting computational scheme is provided if we look for solutions of the form

$$\psi=\psi_0+f_a(g)\psi_1,$$

$$\phi=f_b(g)\phi_0, \qquad (4.1)$$

where $f_a(0)=f_b(0)=0$, that is, we are considering solutions with $\psi^{(p)}=\psi_1$ for $p\geqslant 1$ and $\phi^{(p)}=\phi_0$ for any p. This approach is valuable since a little reflection shows that if it is impossible to satisfy the constant flux conditions through any pair of fields ψ_1 and ϕ_0, then there are no further solutions for the model under consideration. Our task, therefore, is to consider the set of minimal models representing conformal turbulence without perturbations, from which the fields ψ_0 may be immediately obtained, and add, according to the new constraints associated with three-dimensional effects, the fields ψ_1 and ϕ_0.

In the study of the inertial range exponents, we may think of at least three limits for $f_{a,b}(g)$: (a) $g\to 0$, that is, $f_{a,b}(g)\to 0$; (b) $f_{a,b}(g)\simeq 1$; and (c) $g\gg 1$, which may be defined as a ''strong-coupling'' regime. In the first case, the perturbations play a negligible role and everything is described by unperturbed conformal turbulence. A competition between exponents shows up in the second case, where the less steep spectral slope will be the most relevant in the limit of higher wave numbers. We see, in this way, that cases (a) and (b) cannot give any of the steeper spectral slopes observed in real experiments. The third case is, in fact, where we have some hope to find a relation with experimental results. It would be unphysical to have $f_{a,b}(g)\to 0$, for large values of g, since in this limit we would recover the unperturbed system. Also, it is unlikely to have $f_{a,b}(g)\to$ const: taking, for instance, Gaussian random forces $f^{(1)}_\alpha$ and $f^{(2)}_\alpha$,

TABLE I. Solutions for the constant enstrophy flux condition. The first six models of unperturbed conformal turbulence were analyzed, all of them yielding possible definitions of ψ_1 and ϕ_0. Here we show the results for the minimal models (2,21), (3,25), and (3,26).

ψ_1	ϕ_0	$4\Delta\psi_1+1$	$4\Delta\phi_0+1$	ψ_1	ϕ_0	$4\Delta\psi_1+1$	$4\Delta\phi_0+1$
minimal model: (2,21)				minimal model: (3,26)			
(1,9)	(1,9)	-7.38	-7.38	(1,6)	(1,6)	-4.96	-4.96
(1,9)	(1,10)	-7.38	-7.57	(1,6)	(1,7)	-4.96	-5.46
minimal model: (3,25)				(1,6)	(1,8)	-4.96	-5.73
(1,6)	(1,6)	-4.8	-4.8	(1,6)	(1,9)	-4.96	-5.77
(1,6)	(1,7)	-4.8	-5.24	(1,6)	(1,10)	-4.96	-5.58
(1,6)	(1,8)	-4.8	-5.44	(1,6)	(1,11)	-4.96	-5.15
(1,6)	(1,9)	-4.8	-5.4	(1,7)	(1,6)	-5.46	-4.96
(1,6)	(1,10)	-4.8	-5.12	(1,7)	(1,7)	-5.46	-5.46
(1,6)	(1,11)	-4.8	-4.6	(1,7)	(1,8)	-5.46	-5.73
(1,7)	(1,6)	-5.24	-4.8	(1,7)	(1,9)	-5.46	-5.77
(1,7)	(1,7)	-5.24	-5.24	(1,7)	(1,10)	-5.46	-5.58
(1,7)	(1,8)	-5.24	-5.44	(1,7)	(1,11)	-5.46	-5.15
(1,7)	(1,9)	-5.24	-5.4	(1,8)	(1,7)	-5.73	-5.46
(1,7)	(1,10)	-5.24	-5.12	(1,8)	(1,9)	-5.73	-5.77
(1,7)	(1,11)	-5.24	-4.6	(1,8)	(1,11)	-5.73	-5.15
(1,8)	(1,7)	-5.44	-5.24	(1,9)	(1,6)	-5.77	-4.96
(1,8)	(1,9)	-5.44	-5.4	(1,9)	(1,8)	-5.77	-5.73
(1,8)	(1,11)	-5.44	-4.6	(1,9)	(1,10)	-5.77	-5.58
(1,9)	(1,6)	-5.4	-4.8	(1,10)	(1,6)	-5.58	-4.96
(1,9)	(1,8)	-5.4	-5.44	(1,10)	(1,7)	-5.58	-5.46
(1,9)	(1,10)	-5.4	-5.12	(1,10)	(1,8)	-5.58	-5.73
(1,10)	(1,6)	-5.12	-4.8	(1,10)	(1,9)	-5.58	-5.77
(1,10)	(1,7)	-5.12	-5.24	(1,10)	(1,10)	-5.58	-5.58
(1,10)	(1,8)	-5.12	-5.44	(1,10)	(1,11)	-5.58	-5.15
(1,10)	(1,9)	-5.12	-5.4	(1,11)	(1,6)	-5.15	-4.96
(1,10)	(1,10)	-5.12	-5.12	(1,11)	(1,7)	-5.15	-5.46
(1,10)	(1,11)	-5.12	-4.6	(1,11)	(1,8)	-5.15	-5.73
(1,11)	(1,6)	-4.6	-4.8	(1,11)	(1,9)	-5.15	-5.77
(1,11)	(1,7)	-4.6	-5.24	(1,11)	(1,10)	-5.15	-5.58
(1,11)	(1,8)	-4.6	-5.44	(1,11)	(1,11)	-5.15	-5.15
(1,11)	(1,9)	-4.6	-5.4				
(1,11)	(1,10)	-4.6	-5.12				
(1,11)	(1,11)	-4.6	-4.6				

with $\langle f_\alpha^{(1)}(\vec{x},t) f_\beta^{(2)}(\vec{x}',t')\rangle = 0$ and $\langle f_\alpha^{(2)}(\vec{x},t) f_\beta^{(2)}(\vec{x}',t')\rangle = D_{\alpha\beta}(|\vec{x}-\vec{x}'|)\delta(t-t')$, it may be proved, from the retarded nature of the diffusion propagator, that $\langle f_\alpha^{(2)}(\vec{x},t) v_\beta(\vec{x}',t)\rangle = g D_{\alpha\beta}(|\vec{x}-\vec{x}'|)$, yielding

$$\frac{\partial}{\partial g}\left[\frac{\langle f_\alpha^{(2)}(\vec{x},t) v_\beta(\vec{x}',t)\rangle}{D_{\alpha\beta}(|\vec{x}-\vec{x}'|)}\right] = 1. \tag{4.2}$$

Let us thus assume that $f_{a,b}(g)$ diverges as $g\rightarrow\infty$. This means that the inertial range exponent derived from ψ_0 may be discarded and we have to analyze only the competition between the exponents obtained from ψ_1 and ϕ_0.

We performed an investigation of the first six minimal models for both the enstrophy and energy cascade cases. In the enstrophy case we found solutions for all the models studied. They are represented in Table I and in Fig. 1. In Table I, we show the fields ψ_1 and ϕ_0 for the minimal models (2,21), (3,25), and (3,26), together with their associated inertial range exponents. As the number of solutions became larger, we had to represent the other three models, (6,55), (7,62), and (8,67) in Fig. 1, where we plotted the most relevant exponents, found from the competition between ψ_1 and ϕ_0, in the strong-coupling regime. We observe, from the results, that there is good agreement with experimental verifications, with the only considerable deviation occurring for the very small set of two solutions for the model (2,21). The solutions, excluding the model (2,21), were organized in Table II, where values of mean exponents and standard deviations are described. It is clearly seen that the perturbed exponents are in general less than the exponents of the unperturbed fluid.

In the energy case, an interesting fact happened: most of the models studied did not yield any solution for the fields ψ_1 and ϕ_0. Only the model (10,59), represented in Table III, gave solutions, all of them with inertial range exponents

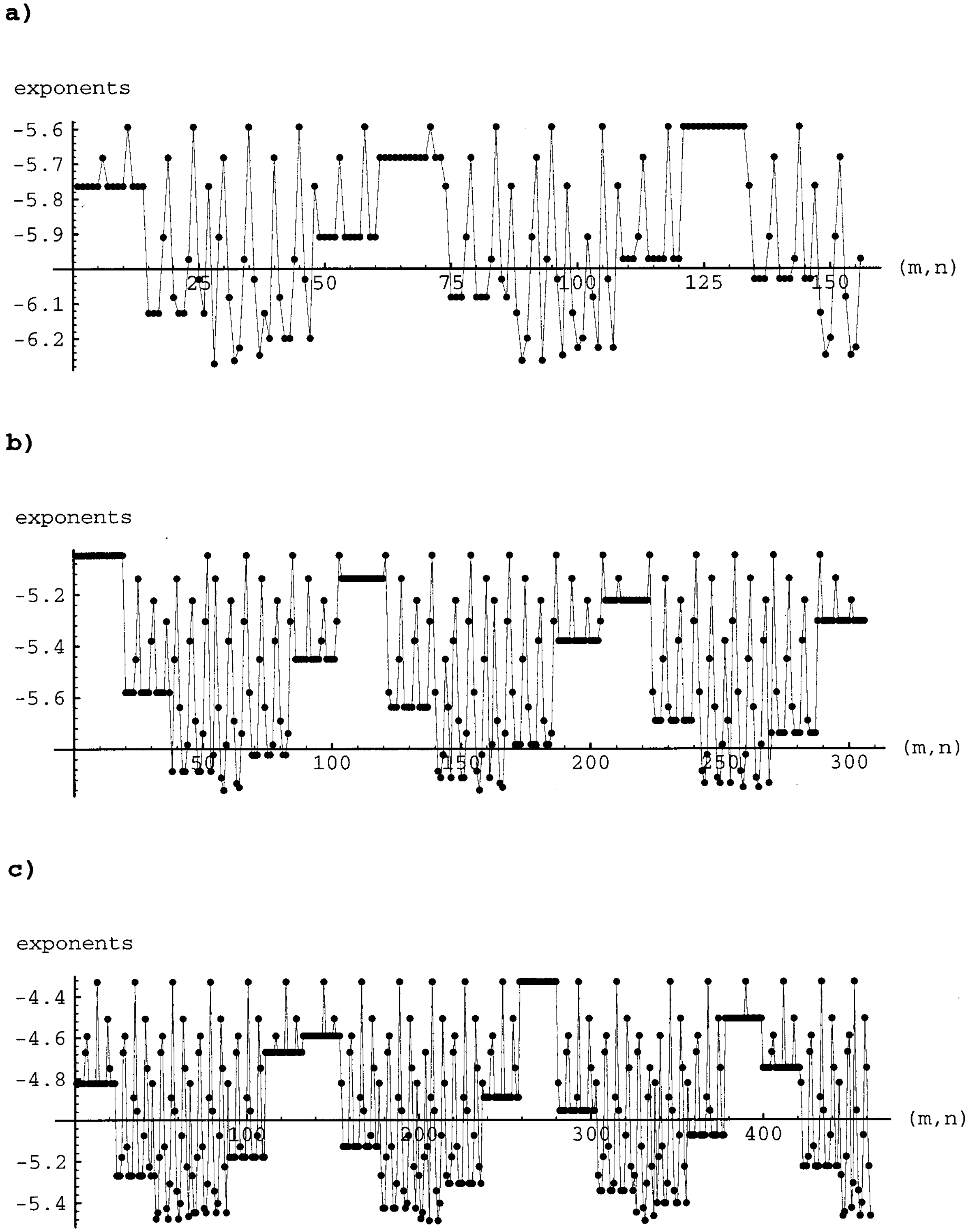

FIG. 1. Graphic representation of the inertial range exponents in the enstrophy cascade case. (a), (b), and (c) refer to the minimal models (6,55), (7,62), and (8,67), respectively. The horizontal axis, labeled by (m,n), represents the most relevant field between ψ_1 and ϕ_0, in the strong-coupling regime. The ordering of fields is the same as in the tables.

close to -3.0, which do not support the conjecture of a Kolmogorov exponent $-5/3$ for the range of lower wave numbers. However, more theoretical and experimental work is necessary in order to arrive at a conclusive answer on this point.

V. BOUNDARY PERTURBATIONS

It is worth understanding what happens when boundary effects are supposed to have some influence on the problem of conformal turbulence. Below, we obtain the set of condi-

TABLE II. Statistical data related to the solutions found for the constant enstrophy flux condition, in the strong-coupling regime, where a comparison is made with the unperturbed values of the inertial range exponents.

Minimal model	Exponent (g=0)	Mean exponent ($g\neq 0$)	Standard deviation
(3,25)	-4.6	-4.90	0.28
(3,26)	-4.23	-5.25	0.27
(6,55)	-3.73	-5.89	0.21
(7,62)	-4.03	-5.46	0.28
(8,67)	-4.51	-4.90	0.34

tions needed to account for it, leaving a numerical analysis for future investigations.

The basic modification here is that we have to study further OPE's in the conditions of constant enstrophy or energy fluxes, found in Sec. III, since now VEV's of single operators do not necessarily vanish. In this way, let us define the primary operator $A^{(p,p')p''}_{(O_1O_2)O_3}$ as the one with the lowest dimension appearing in the OPE $(O_1^{(p)}O_2^{(p')})O_3^{(p'')}$, where the product of $O_1^{(p)}$ and $O_2^{(p)}$ was computed first. The conditions we are looking for must be obtained from the analysis of the x dependence of the dominant terms in (3.11) and (3.14). In the situation of a constant enstrophy flux, the large-scale part of (3.11) gives

$$\Delta A^{(0,0)}_{\psi\psi}+\Delta\psi^{(0)}-\Delta A^{(0,0)0}_{(\psi\psi)\psi}+3=0, \tag{5.1}$$

which is nothing other than the condition established in Sec. II, in a diferent notation. On the other hand, the small-scale part of (3.11) gives one or both of the conditions

TABLE III. Solutions for the constant energy flux condition. The analysis of the first six models of unperturbed conformal turbulence showed that most of them were ''blocked'' by the presence of perturbations. The only solution obtained corresponds to the model (10,59).

Minimal model: (10,59)			
ψ_1	ϕ_0	$4\Delta\psi_1+1$	$4\Delta\phi_0+1$
(1,6)	(1,6)	-3.07	-3.07
(1,6)	(2,12)	-3.07	-3.06
(1,6)	(3,18)	-3.07	-3.05
(1,6)	(4,24)	-3.07	-3.04
(2,12)	(1,6)	-3.06	-3.07
(2,12)	(2,12)	-3.06	-3.06
(2,12)	(3,18)	-3.06	-3.05
(2,12)	(4,24)	-3.06	-3.04
(3,18)	(1,6)	-3.05	-3.07
(3,18)	(2,12)	-3.05	-3.06
(3,18)	(3,18)	-3.05	-3.05
(4,24)	(1,6)	-3.04	-3.04
(4,24)	(2,12)	-3.04	-3.06
(4,24)	(4,24)	-3.04	-3.04

$$\Delta A^{(p,p')}_{\phi\psi}+\Delta\psi^{(p'')}-\Delta A^{(p,p')p''}_{(\phi\psi)\psi}+2=0,$$

$$\Delta A^{(p,p')}_{\psi\psi}+\Delta\psi^{(p'')}-\Delta A^{(p,p')p''}_{(\psi\psi)\psi}+3=0. \tag{5.2}$$

A similar analysis for the case of an energy cascade yields, for the large- and small-scale parts of (3.14), respectively,

$$\Delta A^{(0,0)}_{\psi\psi}+\Delta\psi^{(0)}-\Delta A^{(0,0)0}_{(\psi\psi)\psi}+2=0 \tag{5.3}$$

and

$$\Delta A^{(p,p')}_{\psi\psi}+\Delta\psi^{(p'')}-\Delta A^{(p,p')p''}_{(\psi\psi)\psi}+1+\delta_{p+p',0}=0,$$

$$\Delta A^{(p,p')}_{\psi\psi}+\Delta\phi^{(p'')}-\Delta A^{(p,p')p''}_{(\psi\psi)\phi}+1+\delta_{p+p',0}=0,$$

$$\Delta A^{(p,p')}_{\phi\phi}+\Delta\psi^{(p'')}-\Delta A^{(p,p')p''}_{(\phi\phi)\psi}+1=0,$$

$$\Delta A^{(p,p')}_{\phi\phi}+\Delta\phi^{(p'')}-\Delta A^{(p,p')p''}_{(\phi\phi)\phi}+1=0,$$

$$\Delta A^{(p,p')}_{\phi\psi}+\Delta\phi^{(p'')}-\Delta A^{(p,p')p''}_{(\phi\psi)\phi}+1=0,$$

$$\Delta A^{(p,p')}_{\phi\psi}+\Delta\psi^{(p'')}-\Delta A^{(p,p')p''}_{(\phi\psi)\psi}+1=0. \tag{5.4}$$

The computation of inertial range exponents is also modified. We now have to consider all possible combinations such as $\psi^{(p)}\psi^{(p')}$ and $\phi^{(p)}\phi^{(p')}$ in the evaluation of the velocity-velocity correlation function. The observed inertial range exponent must be obtained from $2\Delta\psi^{(p)}+2\Delta\psi^{(p')}-2\Delta A^{(p,p')}_{\psi\psi}+1$ or $2\Delta\phi^{(p)}+2\Delta\phi^{(p')}-2\Delta A^{(p,p')}_{\phi\phi}+1$.

VI. CONCLUSION

The problem of two-dimensional turbulence was investigated, taking into account the presence of three-dimensional perturbations. They were introduced in an effective way, represented by random forcing terms that act at small scales in the two-dimensional Navier-Stokes equations, as suggested by experimental observations. A coupling constant g, related to the strength of these additional forces, allowed us to write a power expansion for the velocity field, containing also a compressible part. An infinite set of equations was found by just collecting terms with the same powers of g. The components $\psi^{(p)}$ and $\phi^{(p)}$, appearing in the power expansion of the velocity field, were assumed to be primary operators of some conformal minimal model. We obtained, then, from point-split products of operators, a group of conditions in order to have a solution of the Hopf equations and to reproduce the situation of a constant enstrophy or energy flux through the inertial range. In the constant flux conditions, large- and small-scale terms were defined and evaluated by means of an extension of the dimensional argument employed formerly in the study of analogous correlation functions. An analysis of the first six minimal models of unperturbed conformal turbulence was performed, showing that the picture of a constant enstrophy cascade is in good agreement with experimental data, yielding inertial range exponents, for the strong coupling regime, $g\gg 1$, very close to the ones observed in the laboratory. Regarding the energy cascade case, we noticed that most of the minimal models considered in our study were unable to give solutions for the perturbed system. Only one solution was obtained, with inertial range exponents around -3.0. It would be interesting to investigate further minimal models for the energy case, in order to see if a closer connection with the results indicated in experiments could be reached.

From Tables I and III, and Fig. 1, we see clearly that there are many solutions, differing by just one of the fields ψ_1 or ϕ_0, that give exactly the same inertial range exponents in the strong-coupling regime. It is tempting, then, to conjecture that one could find ''plateaus'' for the spectral slopes, while varying some set of external parameters. This question is contained, of course, in the deeper problem of how to match large-scale properties of the fluid with the minimal models describing the inertial range.

A point that deserves attention is the crossover between unperturbed conformal turbulence and the results obtained in the strong-coupling regime. A bridge between these two situations may be investigated not only by varying g, as we did in Sec. IV, but also through $\mu \rightarrow 0$, when the effects of small-scale three-dimensional perturbations on the constant enstrophy or energy fluxes become negligible. Finally, it is important to stress that a standard direct numerical simulation of Eqs. (3.4) and (3.5), up to some level n in their hierarchy, would be an interesting way to study the above questions and the physical assumptions addressed in the present work.

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