Synchronization of chaotic systems and on-off intermittency

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In this paper, a Langevin equation is used for a chaotic system near the synchronization transition. By mapping the motion of the driven system to a random walk, the universal $-3/2$ power law is obtained. It is also shown that the occurrence of on-off intermittency is a common feature of this transition. The numerical study on chaotically driven Duffing oscillators provides clear evidence to support this theoretical investigation. [S1063-651X(96)12708-2]

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I. INTRODUCTION

Recently, Pecora and Carroll have studied the situation in which a state variable of a chaotic system (called a master system) is used as an input to drive a subsystem that is a replica of part of the master system [1]. They found that the driven subsystem sometimes synchronizes to the master system. The occurrence for this synchronization depends on the largest Lyapunov exponent of the driven subsystem. A series of interesting things, such as chaos-hyperchaos transition, chaos-hyperchaos intermittency, etc., was reported by Kapitaniak *et al.* [2–4].

On-off intermittency as a mechanism of bursting has been studied recently [5–11]. It is named for the two-state nature of a chaotic or stochastic motion in which the long period of nearly constant state (laminar phase) is occasionally disrupted by the short time large order burst (burst phase). The most essential character of this kind of intermittency is that the probability distribution of the duration of laminar phases is the $-3/2$ power law function. In this paper, a Langevin equation is used for a quite general dynamical system. The occurrence of the on-off intermittency, or its particular case of chaos-hyperchaos intermittency [2–4], can be investigated through the static solution of the Langevin equation. Mapping the motion of the system to a random walk, we can also show that the $-3/2$ power law is universal for the on-off intermittency near the synchronization transition.

II. LANGEVIN EQUATION AND SYNCHRONIZATION TRANSITION

Consider a chaotic system, $d\vec{z}/dt = F(\vec{z},\mu)$, where $\vec{z}$ is an m-dimensional vector and μ is a set of parameters. Divide the m state variables into two groups via

$$\vec{z} = \begin{bmatrix} \vec{x} \\ \vec{y} \end{bmatrix}, \tag{1}$$

where $\vec{x}$ is m_1-dimensional and $\vec{y}$ is $(m-m_1)$-dimensional. For simplicity we assume that there is only one parameter in this system, and μ stands for the unique parameter. The master system (1) can be written as

$$\frac{d\vec{x}}{dt} = G(\vec{x},\vec{y},\mu), \qquad \frac{d\vec{y}}{dt} = H(\vec{x},\vec{y},\mu) \tag{2}$$

with

$$F(\vec{z},\mu) = \begin{bmatrix} G(\vec{x},\vec{y},\mu) \\ H(\vec{x},\vec{y},\mu) \end{bmatrix}. \tag{3}$$

We refer to $\vec{x}$ as the input or drive, while the driven system is

$$d\hat{\vec{y}}dt = H(\vec{x},\hat{\vec{y}},\mu) \tag{4}$$

Here we call $\hat{\vec{y}}$ the subsystem response.

Assume that the subsystem is synchronized to the master system when $\mu > \mu_c$ and that the synchronization breaks as $\mu \leqslant \mu_c$. Consider an infinite small deviation of $\hat{\vec{y}}$ from $\vec{y}$, i.e.,

$$\hat{\vec{y}}(t) = \vec{y}(t) + \delta\hat{\vec{y}}(t). \tag{5}$$

We get from Eq. (4) that

$$\frac{d\,\delta\hat{\vec{y}}(t)}{dt} = \delta\hat{\vec{y}} \cdot \frac{\partial}{\partial \vec{y}} H(\vec{x},\vec{y},\mu) + \delta\hat{\vec{y}}\,\delta\hat{\vec{y}} : \frac{\partial}{\partial \vec{y}} \frac{\partial}{\partial \vec{y}} H(\vec{x},\vec{y},\mu) + \cdots, \tag{6}$$

where $(\vec{x}(t),\vec{y}(t))$ is a chaotic orbit of the system (2).

It is clear that as μ immediately below μ_c all Lyapunov exponents of the subsystem are negative except the largest one. In this case for an ensemble of randomly chosen initial values of $\hat{\vec{y}}(0)$, the growth of time will drive all the initial phase points to the unique unstable manifold. So for small $\delta\hat{\vec{y}}(t)$ all the phase points of the response subsystems will lie

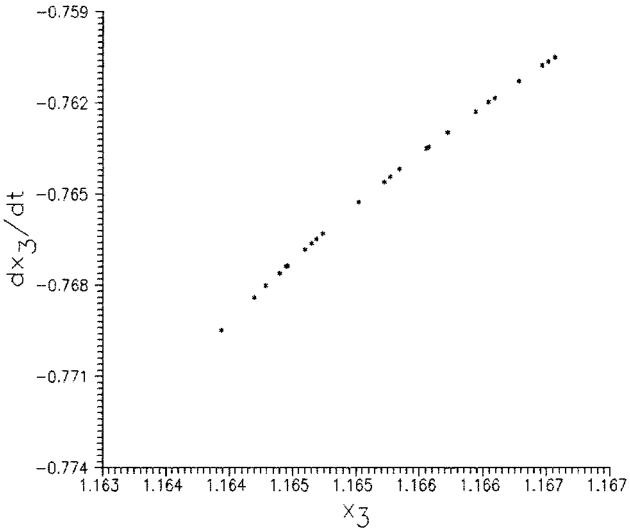

FIG. 1. A snapshot for phase points of response subsystem with only one positive subsystem Lyapunov exponent. The parameters are $\alpha=0.1$, $A=10.0$, $\omega=1.0$, and $\beta=0.120$.

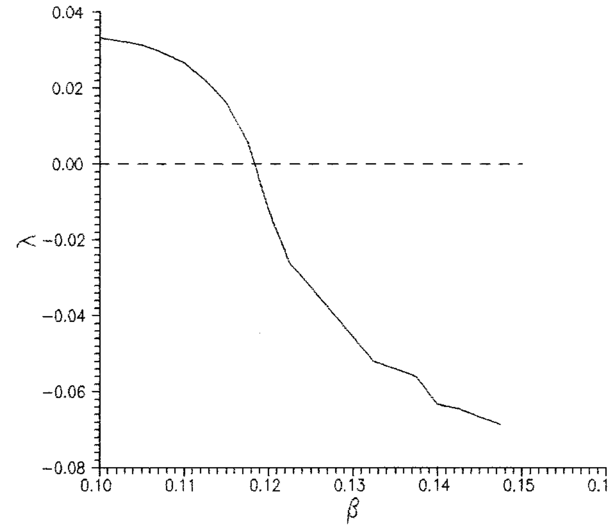

FIG. 2. The largest Lyapunov exponent of response subsystem with $\alpha=0.1$, $A=10.0$, $\omega=1.0$, and $\beta=0.10$–0.15.

on a simple curve after long transient time. Denoting as w the deviation from the state of the master system along this curve, we have

$$\frac{dw}{dt}=w\Gamma_1(t)+w^2\Gamma_2(t)+w^3\Gamma_3(t)+\cdots, \tag{7}$$

where $\Gamma_1(t)=\Gamma_1(\vec{x},\vec{y},\mu)$, $\Gamma_2(t)=\Gamma_2(\vec{x},\vec{y},\mu)$, $\Gamma_3(t)=\Gamma_3(\vec{x},\vec{y},\mu)$, and so on. In case of small deviation w, the first term $w\Gamma_1(t)$ on the right side of Eq. (6) dominates the motion. As $\mu\approx\mu_c$ the coefficient $\Gamma_1(t)$ changes its sign quite frequently. In the case of $\mu\leqslant\mu_c$, w may not be always confined to very small values. To keep w converged some terms of higher order must be considered on the right side of Eq. (6). Assuming, without loss of generality, that $\Gamma_3(t)<0$ and it does not change its sign, though it is not a constant, we could cut off the series on the right side of Eq. (6) from the term $o(w^4)$. Otherwise, we should search for another higher even order term satisfying $\Gamma_{2n+1}(t)<0$. Moreover, the fluctuation of the nonlinear terms is irrelevant to our discussion. So we let $\Gamma_2(t)=c$ and $\Gamma_3(t)=-d$ where c is a constant and d is a positive constant. An important feature of a chaotic system is that any small difference in initial values will be exponentially enlarged in motion. A correlation time scale τ_0 can be roughly defined as $1/\lambda$, where λ is the largest Lyapunov exponent of the master system. This implies that the coefficient $\Gamma_1(t)$ will lose memory of its initial state for a long enough period of time. With a long sampling interval $T\gg\tau_0$ the chaotic signals $G_n\equiv\Gamma_1(nT)$ will behave as a random noise series. The difference between the chaotic and random series for on-off intermittency is still being researched. However, for on-off intermittency, attention is focused on the asymptotical behavior of the probability distribution of the laminar phase size, mainly for that of large size. Hence, the chaotic series $\Gamma_1(t)$ in Eq. (6) could approximately be replaced by a random white noise below. The feasibility of this approximation can only be tested by the numerical calculation. Letting

$$\Gamma_1(t)=a+b\gamma(t), \tag{8}$$

where a,b are two constants and $\langle\gamma(t)\rangle=0, \langle\gamma(t)\gamma(t')\rangle=\delta(t-t')$, we have

$$\frac{dw}{dt}=aw+cw^2-dw^3+bw\gamma(t). \tag{9}$$

The corresponding Fokker-Planck equation is

$$\frac{\partial\rho(w,t)}{\partial t}=-\frac{\partial}{\partial w}[C(w)\rho(w,t)]+\frac{\partial}{\partial w^2}[D(w)\rho(w,t)] \tag{10}$$

with

$$C(w)=(a+b^2/2)w+cw^2-dw^3, \tag{11}$$

$$D(w)=b^2w^2/2. \tag{12}$$

The static solution is [12]

$$\rho(w)=\begin{cases}\delta(w) & \text{if } a\leqslant 0,\\ N\exp[\Phi(w)] & \text{otherwise,}\end{cases} \tag{13}$$

where N is a constant and

$$\Phi(w)=\left(1-\frac{2a}{b^2}\right)\ln w+\frac{2cw}{b^2}-\frac{dw^2}{b^2}. \tag{14}$$

It shows that for $a<0$, the deviation w will constantly be zero. This implies that the two systems lie on the synchronized state. For $a>0$, the peak of the distribution is still at $w=0$ as long as $a<b^2/2$, but the nonzero value of w already has some probability to occur. This corresponds to the case of on-off intermittency in that the deviation lies at nearly zero value for a long period of time, which is interrupted by the short time burst of nonzero value. With increasing a in value, the largest subsystem Lyapunov exponent will increase in value, and the time during which the system lies

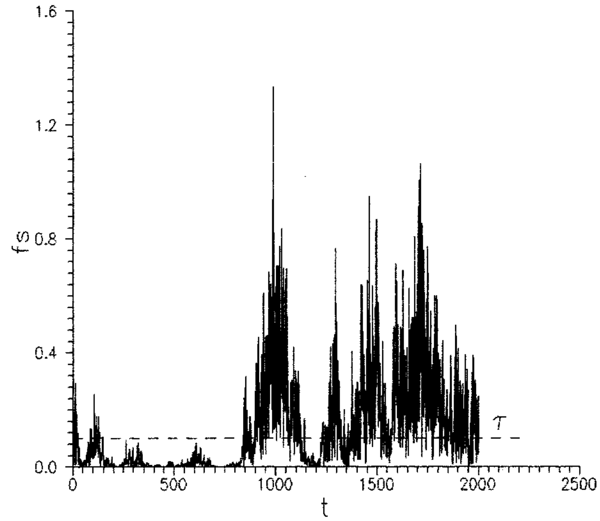

FIG. 3. The deviation of subsystem and master system $fs \equiv \sqrt{(x_2-x_3)^2+(dx_2/dt-dx_3/dt)^2}$ vs time t. The unit of t is the integral step Δt. Here $\beta=0.120$; other parameters are the same as in Fig. 1.

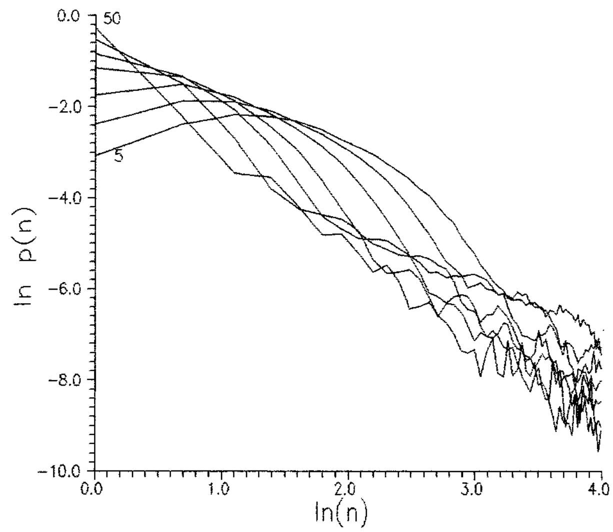

FIG. 4. The transient to a power function of the probability distribution $P(n)$ of laminar phase size n, where n is the number of integral steps throughout the laminar phase. The sampling interval T is of the different values $5\Delta t$, $7\Delta t$, $10\Delta t$, $15\Delta t$, $20\Delta t$, $30\Delta t$, or $50\Delta t$, respectively. Here, Δt is the integral step of time t. The values of the parameters are the same as in Fig. 1.

at laminar phase becomes shorter and shorter, corresponding to the decreasing of the probability $\rho(w)$ at $w=0$.

III. ''GAMBLER RUIN'' PROBLEM AND LAMINAR PHASE DISTRIBUTION

In order to get the distribution of laminar phases, we may map Eq. (9) to a random walk [13]. For on-off intermittency, the main attention was focused on the laminar phase distribution. And during the laminar phase the value of w is so small that the nonlinear terms in Eq. (9) can be ignored:

$$\frac{dw}{dt}=aw+bw\gamma(t). \tag{15}$$

Defining $t_0=0, t_n=n\Delta t, w_n=w(t_n)$, for the given value of w_n, we can get the distribution of w_{n+1} [14]:

$$w_{n+1}=w_n+\sqrt{2D\Delta t}b\gamma(t) \tag{16}$$

and

$$\langle\Delta w_n\rangle=\langle w_{n+1}\rangle-w_n=aw_n\Delta t, \tag{17}$$

$$\sigma^2=\langle\Delta w_n^2\rangle-\langle\Delta w_n\rangle^2=b^2w_n^2\Delta t, \tag{18}$$

where $\langle\ \rangle$ means ensemble average, σ is the stand deviation of series w_n, and $2D=1.0$ for simplicity.

On the other hand, for a random walk along the w axis, each step is of length l. The probability that the walker takes a step at the positive direction is p while at the negative direction it is $q=1-p$. For the given position w_n of the walker, we have the next position w_{n+1}, which satisfies

$$\langle\Delta w_n\rangle=l(p-q) \tag{19}$$

and

$$\sigma^2=4l^2pq. \tag{20}$$

Through the correspondence of the motion of Eq. (15) and the random walk, we have

$$\frac{l(p-q)}{\sqrt{4l^2pq}}=\frac{\langle\Delta w_n\rangle}{\sigma}=\frac{a}{b}\sqrt{\Delta t}\equiv B. \tag{21}$$

Since $p+q=1$, we can get $p=1/2(1-B)$ and $q=1/2(1+B)$ for small B.

In terms of a random walk, the laminar phase of size s can be defined by

$$w_1\leqslant\tau, w_2\leqslant\tau, \ldots, w_s\leqslant\tau, w_{s+1}>\tau$$

for a suitably shifted time origin, where τ is the threshold of the laminar phase. In order to make sure that we are not beginning at the inner part of another laminar phase, a backward condition $w_0>\tau$ is also needed. So the probability distribution of laminar phases is:

$$P(s)=\text{prob}\{w_0>\tau|w_1\leqslant\tau, w_2\leqslant\tau, \ldots, w_s\leqslant\tau, w_{s+1}>\tau\}. \tag{22}$$

The probability

$$\lambda_s=\text{prob}\{w_1\leqslant\tau, w_2\leqslant\tau, \ldots, w_s\leqslant\tau, w_{s+1}>\tau\} \tag{23}$$

is in fact the special case of the ''Gambler ruin'' problem [15,16]. A gambler plays a series of independent games against a bank with infinite resources. Once he is ruined, the game will be terminated. In each game, the gambler either loses or wins one dollar, and the probability that the bank wins is $q=(1-B)/2$, while the probability that the gambler wins is $p=(1+B)/2$. Denoting the net dollars that the bank had won after the nth game as w_n, the ruined series w_n of the gambler is typically

$$w_1, w_2, \ldots, w_s, \ldots.$$

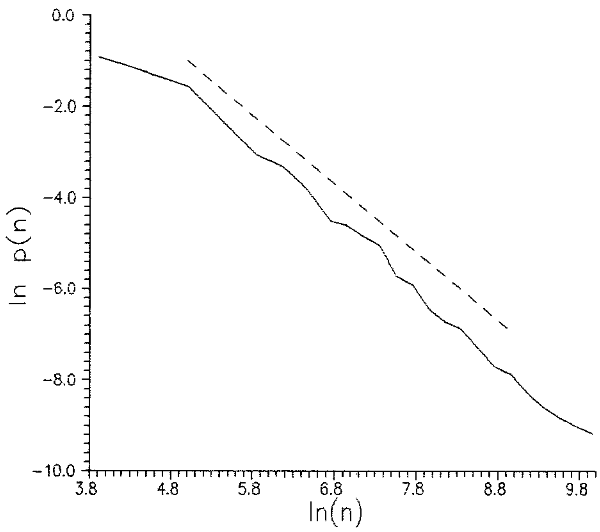

FIG. 5. The distribution of laminar phase size. The symbols $P(n)$ and n are of the same meaning as Fig. 4. The sampling interval T is $10\Delta t$. The dashed line of slope $-3/2$ is plotted to guide the eyes. A coarse grain technique is used to get this result that the occurrence probability of length from $100n\Delta t$ to $100(n+1)\Delta t$ is summed up and regarded as probability of length $100(n+1/2)\Delta t$.

For the reason that once he is ruined the game will be terminated, if the gambler starts out with a finite capital of z dollars, the series will be typically

$$w_1 \leqslant z, w_2 \leqslant z, \ldots, w_s \leqslant z, w_{s+1} > z.$$

This is the same series of a laminar phase of size s if the threshold $\tau = z$. From these above, we can say that the ''Gambler ruined'' problem and laminar phase distribution are just two sides of one coin.

The probability that the gambler is ruined after precisely s games is

$$\pi(z,s) = \frac{z}{s}\begin{pmatrix} s \\ \frac{s-z}{2} \end{pmatrix} p^{(s-z)/2} q^{(s+z)/2}. \tag{24}$$

So the probability λ_s is

$$\pi(z=1,s) = \frac{s^{-3/2}}{\sqrt{2\pi}}\sqrt{\frac{1+B}{1-B}}(1-B^2)^{s/2}. \tag{25}$$

The probability for $w_0 > \tau$ is simply $q/2 = (1-B)/4$, and then we have the probability distribution of the duration of laminar phases:

$$P(s) = \frac{1-B}{4}\pi(z=1,s) = \frac{s^{-3/2}}{\sqrt{2\pi}} e^{-B^2 s/2}. \tag{26}$$

It can be seen that for $a=0$, i.e., $B=0$, corresponding to the critical state where the largest subsystem Lyapunov exponent becomes zero, we have

$$P(s) \propto s^{-3/2}, \tag{27}$$

the pure power law of exponent $-3/2$, while for $a>0$ we have

$$P(s) \propto s^{-3/2} \exp(-s/s^*) \tag{28}$$

with

$$s^* = \frac{2}{B^2} = \frac{2b^2}{a^2\Delta t}. \tag{29}$$

This shows that for a positive value of the largest subsystem Lyapunov exponent, the distribution of laminar phases has an exponent decay at large size laminar phases.

IV. NUMERICAL EXPERIMENT ON DUFFING OSCILLATORS

At below, we will give an example studied by numerical calculation. It is based on the fourth-order Runge-Kutta method with integral step $\Delta t = 0.01$.

The system we studied is two chaotically driven Duffing oscillators:

$$\frac{d^2x_1}{dt^2} + \alpha\frac{dx_1}{dt} + x_1^3 = A\cos\omega t, \tag{30}$$

$$\frac{d^2x_2}{dt^2} + \beta\frac{dx_2}{dt} + x_2^3 = x_1, \tag{31}$$

$$\frac{d^2x_3}{dt^2} + \beta\frac{dx_3}{dt} + x_3^3 = x_1, \tag{32}$$

where Eq. (30) and Eq. (31) are the master system while the response subsystem is Eq. (32) and, in principle, there may be many response subsystems. And in this case x_1, dx_1/dt, and t stand for $\vec{x}$, x_2 and dx_2/dt stands for $\vec{y}$ in Eq. (2), while x_3 and dx_3/dt stand for $\hat{\vec{y}}$. Here only some components of $\vec{x}$ appear in the response subsystem while the others are irrelevant. The value of the parameters are $\alpha=0.1, A=10.0$, and $\omega=1.0$ (the famous Japanese attractor [17]). A snapshot of phase points of 25 response subsystems with $\beta=0.120$ is shown in Fig. 1. It can be seen that the deviation is indeed along a simple curve although the orientation of the curve changes with time. With the decreasing of β from 0.05 to 0.15, the two oscillators x_2 and x_3 can go from synchronization to nonsynchronization. This can be seen from the largest subsystem Lyapunov exponent (see Fig. 2). For $\beta=0.120$, just above the zero point of the largest subsystem Lyapunov exponent, the deviation $fs \equiv \sqrt{(x_2-x_3)^2+(dx_2/dt-dx_3/dt)^2}$ is shown in Fig. 3. It is truly of the on-off nature. The probability distribution of laminar phases was also calculated (see Fig. 4). The sampling time interval T is taken to be $5\Delta t$, $7\Delta t$, $10\Delta t$, $15\Delta t$, $20\Delta t$, $30\Delta t$, or $50\Delta t$, respectively. As T increases in value, the transient of the distribution to a good power law relation is obvious. The power law fitting of the distribution is just of exponent $-3/2$ (see Fig. 5).

V. CONCLUSION

In this paper, we derived a Langevin equation from a quite general system, and showed that the occurrence of the on-off intermittency or its particular case of chaos-hyperchaos intermittency is a common feature at the sychronization transition. By mapping the motion of the driven system to a random walk, the probability distribution of laminar phase distribution is obtained. The numerical calculation of chaotic driving Duffing oscillators shows clear evidence to support this theoretical conclusion.

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