Domain growth in a one-dimensional driven diffusive system

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The low-temperature coarsening dynamics of a one-dimensional Ising model, with conserved magnetization and subject to a small external driving force, is studied analytically in the limit where the volume fraction μ of the minority phase is small, and numerically for general μ. The mean domain size $L(t)$ grows as $t^{1/2}$ in all cases, and the domain-size distribution for domains of one sign is very well described by the form $P_l(l) \propto (l/L^3)\exp[-\lambda(\mu)(l^2/L^2)]$, which is exact for small μ (and possibly for all μ). The persistence exponent for the minority phase has the value 3/2 for $\mu \to 0$. [S1063-651X(96)03008-5]

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I. INTRODUCTION

The field of phase-ordering dynamics is by now quite well developed [1]. It deals with the approach to equilibrium of a system quenched from a homogeneous high-temperature phase into a two-phase region. Familiar examples are binary alloys and binary liquids, which are described by a scalar order parameter. Recent work has addressed also cases where the order parameter symmetry is continuous rather than discrete [1]. Especially interesting is the scaling regime that emerges in the late stages of growth. For a scalar order parameter, for example, the domain morphology is apparently time independent if lengths are scaled to a single time-dependent length scale $L(t)$, which represents the typical ''domain size.'' This implies that two-point correlation functions depend on the spatial separation $\mathbf{r}$ of the points only through the ratio $|\mathbf{r}|/L(t)$.

By contrast, the coarsening dynamics of driven systems has been much less studied. A physically relevant example, indeed the motivation for the present work, is the phase separation of a binary liquid under gravity. Numerical simulations of a (physically less realistic) alloy model with gravity suggest the existence of two growing length scales, parallel and perpendicular to the field [2–4], although it has proved difficult to unambiguously extract the time dependence of these length scales.

An independent field of study concerns the stationary properties of these ''driven diffusive systems'' [5]. Here we focus on the nonstationary, coarsening dynamics, which, as we have noted, has attracted relatively little attention thus far. Ultimately, we would like to understand the coarsening of binary liquids under weak gravity, including hydrodynamic effects [6], but as a first step we settle here for a less ambitious goal. Specifically we study a one-dimensional Ising model with conserved dynamics and a driving field E, which favors transport of ''up'' spins to the right (and ''down'' spins to the left). We work in the regime $T \ll E \ll J$, where T and J are the temperature and exchange coupling, respectively. We derive exact results in the limit where one phase occupies a small volume fraction, and numerical results for general volume fractions. The main results are a $\sqrt{t}$ dependence for the mean domain size, and a domain-size distribution for domains of one sign of the form $P_l(l) \propto (l/L^3)\exp(-\lambda l^2/L^2)$, where L is the mean size of domains of that sign. We also show that the recently introduced [7] ''persistence exponent'' θ, which describes the fraction $f(t) \sim t^{-\theta}$ of spins of one phase that have not flipped up to time t (in a sense to be clarified below) is $\theta = 3/2$ for the minority phase in the limit where that phase has a vanishingly small volume fraction. These are the first analytical results for the coarsening dynamics of this driven diffusive system.

The paper is organized as follows. In Sec. II we define the model. In Sec. III we discuss domain growth and dynamical scaling in the model, while Sec. IV deals with the persistence exponent. Section V concludes with a summary and discussion of the results.

II. THE MODEL

The microscopic model we consider is a chain of Ising spins $S_i = \pm 1$ with nearest-neighbor coupling strength J. The system evolves by nearest-neighbor spin-exchange dynamics, with a driving force E that favors motion of up spins to the right over motion to the left. That is, the microscopic processes are

$$++-- \rightleftharpoons +-+- \quad \Delta = 4J - E \quad \text{(i)},$$

$$--++ \rightleftharpoons -+-+ \quad \Delta = 4J + E \quad \text{(ii)},$$

$$++-+ \rightleftharpoons +-++ \quad \Delta = -E \quad \text{(iii)},$$

$$-+-- \rightleftharpoons --+- \quad \Delta = -E \quad \text{(iv)},$$

where the rate for a process from left to right is proportional to $(1/2)[1-\tanh(\Delta/2T)]$. We distinguish between the ''forward'' and ''backward'' versions of the processes depicted above by using $\rightharpoonup$ and $\leftharpoonup$ to denote the process from left to right and right to left, respectively.

We consider the regime $T \ll E \ll J$. This is very different from other studies, which have concentrated on the limit $J/T \to 0$ [8]. The system possesses metastable states consisting of long domains of parallel spins, separated by domain walls. After a long time of order $\exp[(4J-E)/T]$, a process of type (i$\rightharpoonup$) takes place, i.e., a spin splits off from a domain. Processes of type (ii$\rightharpoonup$) are inhibited by a factor $\exp(-2E/T)$ relative to (i$\rightharpoonup$), but even if they occur the sys-

tem quickly relaxes back to the metastable state by the reverse process (ii←), since all the other possible processes are endothermic. Once a process (i→) occurs, the system can relax further by exothermic processes of the kind (iv→), so that the up spin moves to the right, eventually meeting and adhering to a domain wall; the system may also relax exothermically by processes (iii→), so that a down spin moves left, eventually meeting another domain wall. The motion of these free spins is unidirectional, because the reverse processes (iii←) and (iv←) are inhibited by a factor of order $\exp(-E/T)$.

The result of the free up spin moving to the right is for the down domain through which the spin has traveled to have moved bodily one step to the left; the result of a free down spin moving to the left is for the up domain to the left of the wall to take one step to the right. We may therefore map the microscopic dynamics of the lattice of spins onto one for an array of domains. The system then evolves by a domain of up spins moving spontaneously to the right, or a domain of down spins moving to the left. The rates for such processes are independent of the domain size. This mapping is analogous to a mapping by Majumdar and Huse [9] for the low-temperature Kawasaki chain, corresponding to this model in the opposite limit $E=0$.

When domains are of size two spins or less, they can vanish. The microscopic mechanism for this in the Ising spin picture is series of events involving two random walkers that may coalesce, which translates into rather complex transitions in the domain representation. However, when the domains are large, the details of this domain annihilation process are not expected to be important, so we choose to study a model where the simple domain-shifting dynamics applies for domains down to size one, and removing a domain if its size reduces to zero. Simulations of this simplified system permitted much better statistics than would be possible with the true, microscopic system, whilst still giving indistinguishable scaling behavior.

The algorithm used for simulation was the following: (i) set up an array of alternating up and down domains; (ii) pick a domain at random; (iii) if the domain is up, move to the right (i.e., reduce the size of its right neighbor by one, and increase its left neighbor by one), otherwise move to the left; (iv) if one of the neighboring domains is of zero size then remove it, merging its neighbors; (v) update the clock by 1/(number of domains); (vi) repeat steps (ii)–(vi). By virtue of step (v), the time is defined to be measured in units of $\exp(4J-E)/T$, and this convention is adopted throughout the present paper.

III. DOMAIN GROWTH AND DYNAMICAL SCALING

A. Simulation results

Since the number of domains in the model can decrease but not increase, the average domain size must increase monotonically. In a finite system (with periodic boundary conditions) the system will coarsen until there is only one up domain and one down domain; this state will not be stationary, because the dynamics still permits both domain walls to perform correlated random walks. There will be a wide regime of time during which the average domain size is much smaller than the system size, and the system might be expected to display dynamic scaling.

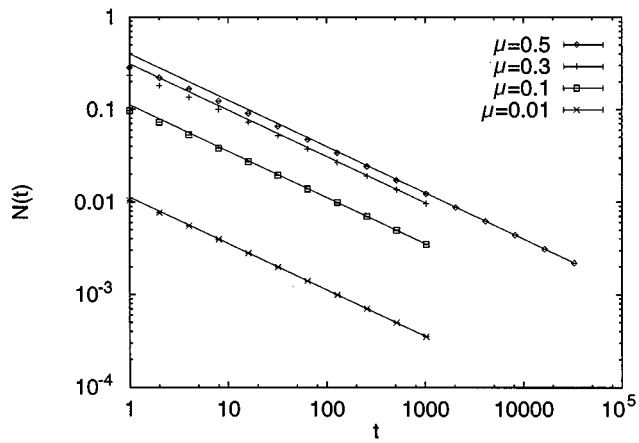

FIG. 1. The average domain density $N(t)$ for four values of the volume fraction μ. The straight lines are plots of the asymptotic prediction from Eq. (10).

Simulations using the domain model described in the previous section were performed, using lattice sizes in the region 10^5–10^6 spins, for times up to 3×10^4 and averaging over several hundred samples. Random initial conditions, where a spin has a probability μ of being up and $1-\mu$ of being down, were used; similar results were found if an ordered initial state was prepared using alternating single up spins and domains of $(1-\mu)/\mu$ down spins. The dynamics conserves the magnetization, so the volume fraction μ remains unchanged. Several different random number generators were used, and the results checked for consistency; the four-register shift generator of Ziff [10] was used for the runs of highest statistics, once it had been established that it gave results consistent with other generators.

Figure 1 shows the average domain density plotted, for different volume fractions μ, as a function of time t on a log-log scale. The straight lines all have gradient -0.50. Figure 2 shows a time-dependent effective exponent, defined as the gradient of a line between successive points in Fig. 1, plotted as a function of $1/\ln(t)$; the results show that the data appear to approach the value -0.50 as $t\to\infty$, though the

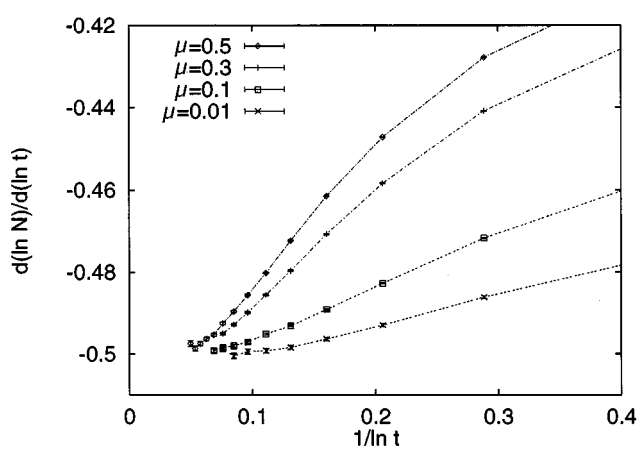

FIG. 2. The effective exponent for the decay of the domain wall density.

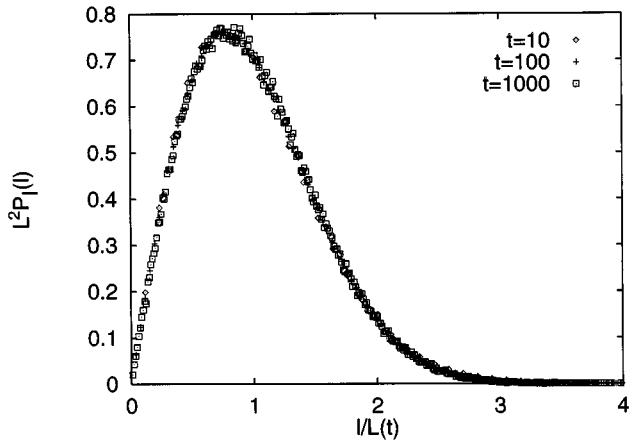

FIG. 3. Scaling plot of the domain-size distribution for volume fraction $\mu=0.5$.

convergence is slower for larger values of μ. The characteristic domain size therefore increases as $t^{0.50}$.

Figure 3 is a scaling plot of the domain-size distribution $P_l(l)$ (defined as the number of domains of size l per lattice site) for a simulation with $\mu=0.5$. The average domain size $L(t)$ is defined by $L^{-1}=\Sigma_l P_l(l)$. The data show good collapse to scaling, even for short times. Figure 4 shows the same data plotted in the form $\ln[L^3P_l(l)/l]$ versus $(l/L)^2$. The linear behavior evident in the plot suggests scaling of the form

$$P_l \propto \frac{l}{L^3(t)} \exp\{-\lambda[l/L(t)]^2\}. \tag{1}$$

For the case $\mu \neq 0.5$, the average sizes of up and down domains differs by a factor $\mu/(1-\mu)$, but nevertheless the up and down domain-size distributions P_+ and P_- are both found to satisfy independently scaling of the form (1).

Figure 5 shows the scaling of the equal-time two-point correlation function $C(x,t)=\langle S_i(t)S_{i+x}(t)\rangle$. Although the data appear to collapse to a scaling form, the approach appears slower than was the case for P_l, suggesting that

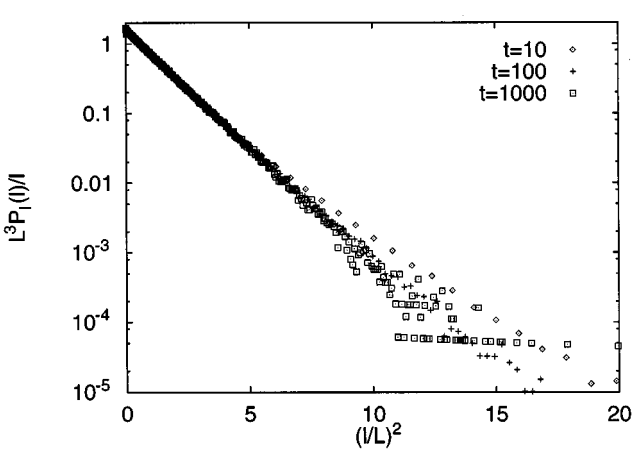

FIG. 4. Fit of the data in Fig. 3 to the form (1).

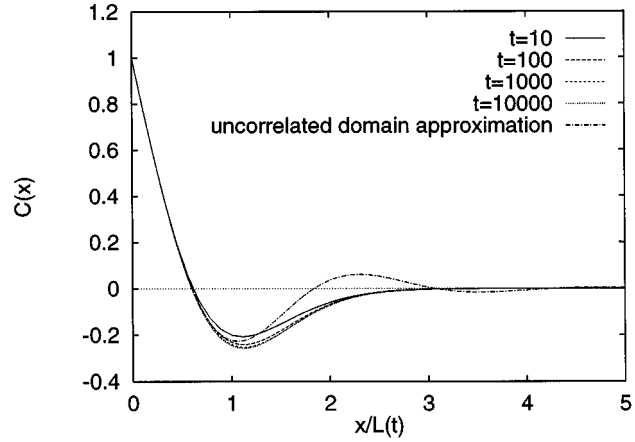

FIG. 5. Scaling plot of the equal-time spin-spin correlation function for $\mu=0.5$. The prediction assuming uncorrelated domains is shown for comparison.

$P_l(l)$ is a more natural quantity to describe the system. The apparent simple form for P_l suggests that the scaling state might be very simple, for instance, there might be no correlations between domains in the scaling limit. The structure factor $S(q)$ [the Fourier transform of $C(x)$] for a system consisting of uncorrelated domains may be shown to be [11]

$$S(q)=\frac{4}{Lq^2}\frac{1-|\widetilde{P}_l(q)|^2}{|1+\widetilde{P}_l(q)|^2}, \tag{2}$$

where $\widetilde{P}_l(q)$ is the Fourier transform of P_l. The inverse Fourier transform of (2), where $\widetilde{P}_l$ was calculated by assuming the simple form (1), is plotted in Fig. 5 for comparison. The discrepancy with the simulation data shows that strong correlations need to be taken into account in this system, even though simulations measured only $\sim$3–5 % correlations between the sizes of neighboring domains.

B. Solution for domain density assuming scaling

The numerical data suggest that the scaling function for the domain-size distribution is the same for minority and majority domains, and is independent of μ. Using this as an assumption, we can calculate the average domain size. The number of domain walls changes by a domain of size one shrinking to nothing, and so we expect that $P_\pm(l_\pm)\sim l_\pm$ for $l_\pm \to 0$. We therefore assume a scaling form

$$P_\pm(l_\pm,t)=\alpha_\pm \frac{l_\pm}{L_\pm^3(t)} F\left(\frac{l_\pm}{L_\pm(t)}\right), \tag{3}$$

where $\alpha_\pm$ is chosen so that $F(0)=1$, and $F'(0)=0$. The rate at which up domains vanish is $P_+(1)$, which approaches $\partial P_+/\partial l_+$ in the continuum limit. Since each domain vanishing event removes two domain walls, the rate of decay of the number $N(t)$ of domain walls per site is

$$-\frac{dN}{dt}=2\left(\frac{\partial P_+}{\partial l_+}\bigg|_{l_+=0}+\frac{\partial P_-}{\partial l_-}\bigg|_{l_-=0}\right)=2\left(\frac{\alpha_+}{L_+^3}+\frac{\alpha_-}{L_-^3}\right). \tag{4}$$

The domain-size distribution is normalized to the density of domains, and there are the same number of up as down domains, so

$$N/2=\int_0^\infty P_+(l_+)dl_+=\frac{\alpha_+ f_1}{L_+}, \tag{5}$$

$$N/2=\int_0^\infty P_+(l_-)dl_-=\frac{\alpha_- f_1}{L_-}, \tag{6}$$

where $f_1\equiv\int_0^\infty xF(x)dx$. The densities of up and down spins are

$$\mu=\int_0^\infty l_+P_+(l_+)dl_+=\alpha_+ f_2, \tag{7}$$

$$1-\mu=\int_0^\infty l_-P_-(l_-)dl_-=\alpha_- f_2, \tag{8}$$

where $f_2\equiv\int_0^\infty x^2F(x)dx$. Substituting for $L_\pm$ and $\alpha_\pm$ from (5)–(8) into (4), and integrating, we find the following asymptotic result as $t\to\infty$:

$$N(t)=\left\{\frac{2f_1^3}{f_2^2t[\mu^{-2}+(1-\mu)^{-2}]}\right\}^{1/2}. \tag{9}$$

For the particular case $F(x)=\exp(-\lambda x^2)$, suggested by the data, we have $f_1=1/(2\lambda)$, $f_2=\pi^{1/2}\lambda^{-3/2}/4$, and

$$N(t)=\frac{2}{\{\pi t[\mu^{-2}+(1-\mu)^{-2}]\}^{1/2}}. \tag{10}$$

The straight lines in Fig. 1 are, in fact, plots of Eq. (10) for appropriate values of μ. The excellent agreement of the data with the prediction confirms both the predicted μ dependence and also the simple form for $F(x)$.

C. Solution for $P_\pm$ in the limit $\mu\to 0$

The simulations are in excellent agreement with the scaling hypothesis, and with the simple form for the scaling function (1). We would like to have an *ab initio* explanation for these results. Unfortunately, we were only able to solve the dynamics in the ''off-critical'' limit $\mu\to 0$ (or, equivalently, $\mu\to 1$).

In the limit $\mu\to 0$, it is necessary only to consider the motion of the minority spins, i.e., the motion of the majority domains. This is because the size of each domain performs a random walk until it either dies (shrinks to zero size) or coalesces with the nearest domain of the same type. To coalesce with another domain, the intervening domain of the opposite type has to shrink to zero size. The time scale for the vanishing of majority domains will therefore be much longer (by a factor $\sim\mu^{-2}$) than for minority domains. The dynamics will therefore progress primarily by minority domains shrinking to zero size, and never coalescing, whereas the majority domains' sizes only change appreciably due to coalescence.

Consider a particular minority domain containing n up spins at time t. In the limit where coalescence is forbidden, the domain changes size by (a) an up spin arriving from the next domain to the left or (b) an up spin splitting off the domain, and moving to the next domain on the right. It is also possible for a down spin to move from the right to the left of the domain, but this does not change the value of n. Notice that the dynamics is independent of where the neighboring domains are, whether they are vanishing or coalescing. Once the size of the domain reduces to zero, the domain ceases to exist. The master equation for the probability $P(n,t)$ of the size being n at time t is

$$\frac{dP(n,t)}{dt}=P(n+1,t)+P(n-1,t)-2P(n,t) \tag{11}$$

for $n\geqslant 1$, with $P(0,t)=0$. In the continuum limit this master equation approaches the diffusion equation, whose solution with $P(n,t=0)=\delta(n-n_0)$ is

$$P(n,t)=(4\pi t)^{-1/2}\left\{\exp\left[-\frac{(n-n_0)^2}{4t}\right]-\exp\left[-\frac{(n+n_0)^2}{4t}\right]\right\}, \tag{12}$$

$$=(\pi t)^{-1/2}\sinh\left(\frac{nn_0}{2t}\right)\exp\left(-\frac{n^2}{4t}-\frac{n_0^2}{4t}\right). \tag{13}$$

In the scaling limit $t\to\infty$, $n\to\infty$, with $n/\sqrt{t}$ fixed, this reduces to

$$P(n,t)\to\frac{nn_0}{2\sqrt{\pi t^3}}\exp\left(-\frac{n^2}{4t}\right). \tag{14}$$

In a random initial state, the number of domains of size n_0 per lattice site is $(1-\mu)^2\mu^{n_0}$. The up domain size distribution at a time t in the scaling limit is therefore

$$P_+(l_+)=\sum_{n_0}(1-\mu)^2\mu^{n_0}P(l_+,t) \tag{15}$$

$$\to\frac{\mu l_+}{2\sqrt{\pi t^3}}\exp\left(-\frac{l_+^2}{4t}\right), \tag{16}$$

which is of the form (1), with $L=2t^{1/2}$. The total domain density (twice the density of up domains) is

$$N(t)=\int_0^\infty P_+(l_+)dl_+=\frac{2\mu}{\sqrt{\pi t}}, \tag{17}$$

which approaches (10) in the limit $\mu\to 0$.

We may calculate the size distribution of the majority domains from the probability that a given region contain no domain walls. Consider a region of M lattice sites containing n minority spins; these spins need not necessarily all belong to the same domain. Then the number of minority spins changes by spins entering the region at the left and coalescing with the leftmost domain wall in the region, and spins splitting off the rightmost domain wall. Once n reduces to zero, however, any spins entering the region from the left will simply pass through the region, and it will remain empty. We have again assumed the limit $\mu\to 0$, by not allowing for any domains to move out of the region or to coalesce with domains outside the region.

Under these conditions, the probability $P_M(n,t)$ obeys a master equation of the form (11), whose solution with $n=\mu M$ at $t=0$ is

$$P_M(n,t)=(4\pi t)^{-1/2}\left\{\exp\left[-\frac{(n-\mu M)^2}{4t}\right]-\exp\left[-\frac{(n+\mu M)^2}{4t}\right]\right\}. \tag{18}$$

The probability P_E of the region M being empty is then

$$P_E(M,t)=1-\int_0^\infty P_M(n,t)dn \tag{19}$$

$$=1-(4\pi t)^{-1/2}\int_{-\mu M}^{\mu M}\exp\left(-\frac{u^2}{4t}\right)du. \tag{20}$$

Consider now a region of m sites, sitting inside a domain of size l sites. The number of positions that it can occupy within the region is $(l-m)\Theta(l-m)$, where Θ is the Heaviside function. The probability that a randomly chosen interval m lies within a domain of size in the range l to $l+dl$ is therefore $(l-m)\Theta(l-m)P_l(l)dl$, so the probability that a region of size m contains no domain walls is

$$P_E(m)=\int_m^\infty (l-m)P_l(l)dl. \tag{21}$$

Differentiating twice, we have $P_E''(m)=P_l(m)$.

The result for the distribution of majority domains is therefore

$$P_-(l_-)=\left.\frac{\partial^2 P_E(M)}{\partial M^2}\right|_{M=l_-} \tag{22}$$

$$=\frac{\mu^3 l_-}{2\sqrt{\pi t^3}}\exp\left(-\frac{\mu^2 l_-^2}{4t}\right), \tag{23}$$

which is of the form (1) with $L_-=L_+/\mu$. This shows that the calculation is only valid to lowest order in μ, since the conservation of magnetization implies that $L_-/L_+=(1-\mu)/\mu$.

IV. PERSISTENCE EXPONENT

It was recently found that, in a one-dimensional (1D) Ising model evolving at zero temperature from a disordered state under Glauber dynamics, the probability that a given spin has never flipped up to time t decays as $t^{-\theta}$ [7]. The value of the persistence exponent θ depends upon the magnetization, and whether the spin is in the majority or minority phase. The dynamics for a spin in a phase with volume fraction μ is the same as for a q-state Potts model with symmetric initial condition, with $q=1/\mu$ [12], leading to the result that θ takes values in the range 0–1 as μ decreases from 1 to 0, with $\theta=3/8$ for $\mu=1/2$ [13].

For a 1D driven diffusive system, there are two kinds of persistence that may be considered. The first concerns the probability that a spin in the microscopic Ising representation has never changed its value. Whenever a domain wall emits a spin, that spin moves rapidly through a domain, causing each of the spins in that domain to flip twice. Since this spin emission is a Markov process, the probability that a given spin has never flipped decays exponentially.

A more interesting kind of persistence to investigate is the probability that a given spin has never belonged to another *stable* domain. That is, we discount the rapid flipping due to spin motion through a domain, and consider only the case where an entire domain has migrated towards the site in question. This kind of coarse graining in time would also be necessary when studying the Glauber-Ising model at low but nonzero temperatures, where short-lived thermally activated flipping of spins within the interiors of domains occurs, in order to recover the true zero-temperature persistence behavior.

A. Analytical results for $\mu\to 0$

Let us consider a test site initially within a minority domain, a distance n_1 sites from the left domain wall and n_2 sites from the right wall. We define n_1 and n_2 such that a spin in a domain of size unity has $(n_1,n_2)=(1,1)$. The dynamics causes the domain walls to wander stochastically, and eventually one of the walls will cross the test site; i.e., the test site will have flipped.

We shall assume that this domain does not coalesce with another domain before one of the domain walls reaches the test site. This assumption will certainly be valid in the limit $\mu\to 0$. Then the three processes that cause the domain walls to move are (i) an up spin joins onto the left-hand edge, $(n_1,n_2)\to(n_1+1,n_2)$; (ii) an up spin splits off the right-hand edge, $(n_1,n_2)\to(n_1,n_2-1)$; (iii) a down spin splits off the right-hand edge and moves through the domain to the left-hand edge, $(n_1,n_2)\to(n_1-1,n_2+1)$. The master equation for the joint probability $P(n_1,n_2,t)$ is then

$$\frac{dP(n_1,n_2,t)}{dt}=P(n_1-1,n_2)+P(n_1+1,n_2-1) \tag{24}$$

$$+P(n_1,n_2+1)-3P(n_1,n_2), \tag{25}$$

which becomes, in the continuum limit $n_1\to x_1$, $n_2\to x_2$,

$$\frac{\partial P(x_1,x_2,t)}{\partial t}=\left\{\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}-\frac{\partial^2}{\partial x_1\partial x_2}\right\}P. \tag{26}$$

Making the change of variable

$$x=x_1+x_2 \tag{27}$$

$$y=\frac{1}{\sqrt{3}}(x_1-x_2), \tag{28}$$

Eq. (26) becomes

$$\frac{\partial P(x,y,t)}{\partial t}=\left\{\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right\}P. \tag{29}$$

When either of the domain walls reaches the test site, the test site flips. Therefore, if we want P to represent the conditional probability that the site has not flipped, we must solve Eq. (26) with boundary condition $P=0$ along $x_1=0$ and $x_2=0$. This corresponds to solving the diffusion Eq. (29) with boundary condition $P=0$ along the lines $y=\pm x/\sqrt{3}$, for the region $x\geqslant y\sqrt{3}$, $x\geqslant -y\sqrt{3}$—that is, in a wedge of angle $\pi/3$ with absorbing boundaries.

The diffusion equation is readily solved in a wedge of angle ψ with absorbing boundaries, and it is known that the survival probability for a random walker under such conditions decays as $t^{-\pi/2\psi}$ [14]. After transforming Eq. (29) into polar coordinates r and θ, where

$$r=(x^2+y^2)^{1/2}, \tag{30}$$

$$\theta=\phi+\pi/6, \tag{31}$$

$$\tan\phi=y/x, \tag{32}$$

we need to solve the diffusion equation for the region $0\leqslant\theta\leqslant\pi/3$, with $P=0$ on $\theta=0$ and $\theta=\pi/3$. The appropriate solution, starting from $r=r_0$, $\theta=\theta_0$ is

$$P(r,\theta,t)=\frac{3}{\pi}\sum_m \int_0^\infty d\lambda e^{-\lambda t}\sin(3m\theta)\sin(3m\theta_0) \times J_{3m}(r_0\lambda^{1/2})J_{3m}(r\lambda^{1/2}) \tag{33}$$

$$=\frac{3}{\pi t}\sum_m \sin(3m\theta)\sin(3m\theta_0) \times I_{3m}\left(\frac{rr_0}{2t}\right)\exp\left(-\frac{r_0^2+r^2}{4t}\right), \tag{34}$$

where we have used Eq. (6.615) of [15]. To find the persistence probability, we need to evaluate $P_P(t)=\int_0^\infty r\,dr\int_0^{\pi/3}d\theta P$. Performing the integrals [15], and taking the limit $t\to\infty$ (where the dominant contribution comes from the term $m=1$ in the sum) we find

$$P_P\to\frac{1}{2\sqrt{\pi}}\sin(3\theta_0)\left(\frac{r_0^2}{4t}\right)^{3/2}. \tag{35}$$

On substituting for the initial conditions in terms of $x=n_1^0+n_2^0$, $y=(n_1^0-n_2^0)/\sqrt{3}$, Eq. (35) reduces to $P_P(t)=(1/4\sqrt{\pi})n_1^0n_2^0(n_1^0+n_2^0)t^{-3/2}$.

In order to calculate the persistence probability, we need to average over the possible values of n_1^0 and n_2^0. The probability in the initial state of a given ''up'' spin having precisely n_1^0-1 consecutive up neighbors to its left is $(1-\mu)\mu^{n_1^0-1}$, so the probability of an initial configuration (n_1^0,n_2^0) is $(1-\mu)^2\mu^{n_1^0+n_2^0-2}$. Summing over $n_1^0\geqslant 1$ and $n_2^0\geqslant 1$ gives the final result,

$$P_P(t)=\frac{1}{2\sqrt{\pi t^3}}\frac{1+\mu}{(1-\mu)^3}. \tag{36}$$

One might attempt to perform a similar calculation for a site within a majority domain. Here, however, it is important to take account not only of the fact that the walls of the domain can move, but also that the neighboring domains can vanish before the test spin has flipped. Contributions from domains that have coalesced will therefore be important, and the necessity to include information about domain-domain correlations makes the calculation extremely complicated, if not intractable.

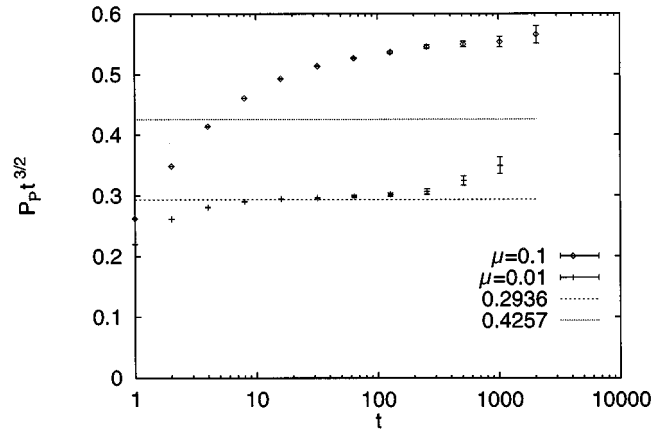

FIG. 6. Bias plot for the persistence probability of minority spins.

B. Simulation results

The persistence probability for both majority and minority spins was measured for a range of values of μ. While the measured values of the exponent α for the minority domains was found to be close to 1.5 for μ small, for $\mu=0.5$ there appeared to be some evidence that the asymptotic behavior was governed by a different exponent. However, the slow onset of the asymptotic regime for intermediate values of μ, together with the large value of the exponent (>1, compared with $\leqslant 1$ for the Glauber case [13]), led to unavoidably poor statistics in the asymptotic regime, and hence it was not possible to establish reliable values for the exponent.

Figure 6 shows a plot of $P_P(t)t^{3/2}$ against time for minority spins, for $\mu=0.01$. For comparison, the constant (≈ 0.2936) predicted by Eq. (36) is also shown. There is reasonable agreement of the simulations with the independent-domain prediction, suggesting that it is a good approximation for small μ, for the time regime measured. We interpret the deviation from the constant at longer times as more likely to be due to statistics than a true systematic effect. Similar data for $\mu=0.1$ are shown, and deviations from the constant predicted by (36) (≈ 0.4257) are already quite marked.

The effective persistence exponent $d(\ln P_P)/d(\ln t)$ for minority spins is plotted as a function of $1/\ln t$ in Fig. 7, for several values of μ. The purpose of this plot is to investigate whether there are any underlying trends in the exponent. When a function of the form $f(t)\propto t^a(\ln t)^b$ is plotted on a graph of this kind, the curve approaches the f axis linearly with gradient b and intercept a. The behavior for data that converge asymptotically to a power law is for the slope of such a plot to level off to zero in the asymptotic regime. For small values of μ, the exponent seems to have settled down

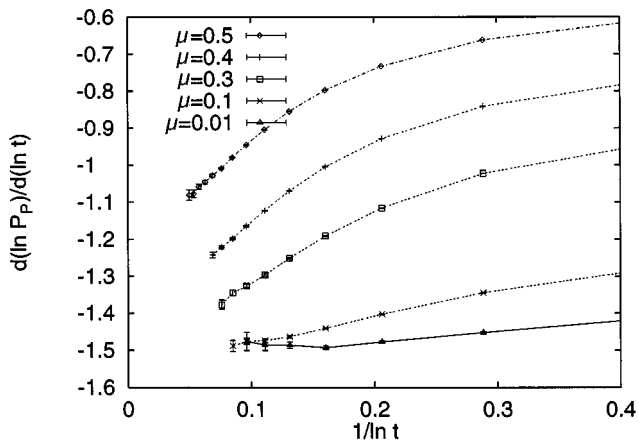

FIG. 7. Effective exponent for the persistence probability of minority spins.

to a value close to -1.5. However, for μ close to 0.5 there still appears to be some trend in the effective exponent, suggesting that the asymptotic regime has not been reached. Only for $\mu=0.5$ is there strong evidence that the data converge to an exponent different from -1.5. Improved statistics and longer times would be needed to give an unequivocal conclusion, but we would not be able to do this using our current techniques since each set of data required approximately 1 week of CPU time.

The results for the persistence of the majority species are even more problematic, because for $\mu\neq 0.5$ too few spins were found to have to flipped for the asymptotic regime to have been reached.

The deviations from the free-domain picture must be due to the fact that some of the minority domains coalesce. It is possible to estimate the number of minority domains that coalesce from the rate at which majority domains vanish. From Eq. (4), using (5)–(8), we find

$$-\frac{dN_+}{dt}=\frac{(1-\mu)^2+\mu^2}{\mu^2}\left.\frac{\partial P_-}{\partial l_-}\right|_{l_-=0}, \tag{37}$$

where $N_+=N/2$ is the number of minority (up) domains per site. Integrating, we find

$$\int_t^\infty dt\left.\frac{\partial P_-}{\partial l_-}\right|_{l_-=0}=\frac{\mu^2}{(1-\mu)^2+\mu^2}N_+(t). \tag{38}$$

The term on the left-hand side is the total number of majority domains that vanish between time t and infinity, which is equal to the number of minority-domain coalescences. Some domains may coalesce more than once, so this is an upper bound on the number of minority domains that coalesce at least once after time t. The fraction of minority domains that coalesce after time t is therefore less than $\mu^2/[\mu^2+(1-\mu)^2]$, which vanishes as μ^2 for small μ.

Measurements from the numerical simulations confirmed the picture that the fraction of domains that coalesce is small for μ small. Nevertheless, it is found that the dominant contribution to persistence in the long-time limit comes from spins in domains that have coalesced, thus explaining the deviations from the free-domain approximation.

V. SUMMARY

The low-temperature coarsening dynamics of a driven diffusive system—the 1D Ising model with a driving force E satisfying $T\ll E\ll J$—has been studied by a combination of analytical and numerical techniques. Compelling evidence for a mean domain size growing as $t^{1/2}$, and a domain-size distribution of the form (1), has been presented. These results are exact in the limit where one phase occupies a vanishingly small volume fraction μ. In the same limit, the persistence exponent for the minority phase is $\theta=3/2$. The limit of zero volume fraction was studied by Lifshitz and Slyozov [16] to predict the growth exponent ($=1/3$) for the case without driving force in general dimension, and it is hoped that the approach of the present paper also might be usefully extended to higher dimensions.

The random-walk character of the domain dynamics suggests $t^{1/2}$ growth generally, and this is borne out by the simulation results (Figs. 1 and 2). The simulations also lend strong support (Figs. 3 and 4) to the scaling distribution (1) for general μ. By contrast, the value of θ away from the small-μ limit is difficult to determine numerically, due to slow convergence to the asymptotic regime (Fig. 7). For $\mu=0.5$, however, the results seem to be inconsistent with $\theta=3/2$, suggesting that the exponent θ may depend continuously on μ, as is the case for the 1D Ising model with Glauber dynamics [12].

In the absence of the bias field E, the kinetics reduce to conventional Kawasaki dynamics, for which the mean domain size in the limit $J\gg T$ far from equilibrium grows as $t^{1/3}$ [17]. The difference between these models lies in the probability $q(l)$ that a spin, having split off from a domain, will reach the next domain wall a distance l away before returning to its original position. For the limit of very strong bias, studied in the present paper, we have $q(l)=1$, whereas for the unbiased case we have $q(l)=1/l$ [18]. Since $O(l^2)$ such processes are necessary for a domain to vanish, the typical lifetime of a domain is l^2 and l^3 for the strongly biased and unbiased cases, respectively, leading to the growth laws $L\sim t^{1/2}$ and $L\sim t^{1/3}$. How does the system behave for intermediate values of the driving force? If we have a small bias giving an average drift velocity v towards the far wall, then as l diverges $q(l)$ will saturate to some small but finite value. It is possible to find an explicit form for $q(l)$ in the continuum limit by generalizing the approach of Cordery, Sarkar, and Tobochnik [18] to the case of biased diffusion [19]. For brevity, we shall instead use the following scaling argument: in the time $\tau\sim l^2$ for the spin to have diffused a length l, the bias gives rise to a drift $v\tau\sim vl^2$. This drift is of order l when $vl=O(1)$, so the scaling form consistent with $q\to l^{-1}$ as $vl\to 0$ is $q(l)=l^{-1}f(vl)$, with $f(0)=$ const. For q independent of l as $l\to\infty$, we must have $f(x)\sim x$ as $x\to\infty$, so $q(l)\sim v$ as $vl\to\infty$. In the limits where vL is either large or small, we can translate this into a growth law by writing

$$\frac{1}{t}\sim\frac{q(L)}{L^2}\sim\begin{cases}L^{-3} & \text{for} \quad vL\ll 1\\ vL^{-2} & \text{for} \quad vL\gg 1.\end{cases} \tag{39}$$

We conclude, therefore, that any finite bias eventually gives rise to a $t^{1/2}$ growth law as $t\to\infty$—provided, of course, that

$J/T \gg E/T$, so that the system does not saturate to equilibrium before this growth law appears. For weak bias, $v = E/T$, the two growth regimes are

$$L \sim \begin{cases} t^{1/3} & \text{for } L \ll T/E \\ (Et/T)^{1/2} & \text{for } L \gg T/E. \end{cases} \tag{40}$$

[It should be remembered that t is measured in units of $\exp(4J-E)/T$]. Similar arguments have been used by Cadilhe and Privman [20] in relation to the dynamic critical behavior in this type of system.

It is tempting to propose an experimental setup that might be described by the present one-dimensional model. If two immiscible fluids of differing density are stirred and placed in a vertical tube, then they will typically be able to slide past each other and separate hydrodynamically. However, if the tube is sufficiently narrow then it is possible for a state where the denser fluid is above the lighter fluid to be metastable; the loss in gravitational potential energy if the interface is tilted slightly can be stabilized by the increase in interfacial energy as the area of the interface increases. A state consisting of alternating quasi-one-dimensional ''domains'' can therefore be metastable. There now arises the question of adding a noise source to create droplets at the surface. Simply shaking the tube will tend to induce tilting of the interface, which will lead to hydrodynamic instability. The driving vibrations therefore have to be of a wavelength much smaller than the width of the tube in order to create droplets without exciting the ''sloshing'' mode. We leave the problems of finding an ultrasonic source of high enough intensity, and of preparing the intial condition, as challenges for the keen experimenter.

In a separate paper [21] we will present results for the $T=0$ dynamics of a deterministic 1D scalar field model, defined by the modified Cahn-Hilliard equation $\partial_t \phi = -\partial_x^2(\partial_x^2 \phi + \phi - \phi^3) + E\phi\partial_x\phi$. In the small-$E$ limit, the coarsening dynamics of this driven diffusive model also exhibit a scaling distribution for domain sizes, and a $t^{1/2}$ growth of the mean domain size. Both models, the stochastic Ising model considered here and the deterministic model, are of great interest in higher dimensions. The approach of looking at the limit of small volume fraction μ, which proved so successful here, may well be fruitful in elucidating the behavior of these models in general dimension D.

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