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Anomalous scaling for passively advected magnetic fields

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The scaling behavior of the covariance of the magnetic field in the three-dimensional kinematic dynamo problem is considered. The velocity is Gaussian and δ-correlated in time; its structure function scales with a positive exponent ξ. No unbounded growth of the magnetic field, i.e., no dynamo effect, occurs for $\xi<1$. A statistically steady state is then attained in the presence of a homogeneous forcing concentrated at the scale L. In the limit of small molecular diffusivities and large L, the correlation function of the magnetic field has an inertial-range scaling exponent $-[(3+\xi)/2]+(3/2)\sqrt{1-(1/3)\xi(\xi+2)}$. This result is extended to the d-dimensional case. In all cases a crucial role is played by zero modes.

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The problem of the scaling laws for a passive scalar advected by a velocity field δ-correlated in time has recently received a great deal of attention [1–7]. A general mechanism for the appearance of anomalous scaling has been identified [3–5]. Nontrivial exponents are associated with zero modes of the closed equations satisfied by the simultaneous correlation functions. The covariance can be determined explicitly and does not show any anomaly [1]. Perturbative calculations indicate that anomalous exponents appear in fourth-order correlations. Our aim is to analyze a similar problem for the kinematic dynamo, i.e., magnetic fields. The major difference is that the magnetic field $\boldsymbol{B}$ can be stretched and $\langle B^2\rangle$ is not conserved in the absence of forcing and dissipation. A nontrivial scaling behavior appears already at the level of second-order correlations. The calculation of the exponent dominating in the inertial range is carried out in a nonperturbative way.

The equations of kinematic dynamo theory are (see, e.g., Refs. [8,9])

$$\partial_t \boldsymbol{B}+\boldsymbol{v}\cdot\nabla\boldsymbol{B}=\boldsymbol{B}\cdot\nabla\boldsymbol{v}+\nu\nabla^2\boldsymbol{B}+\boldsymbol{f}. \tag{1}$$

Here, the velocity $\boldsymbol{v}$, the force $\boldsymbol{f}$ and the magnetic field $\boldsymbol{B}$ are all divergence free. The unforced case will first be considered in the analysis of the turbulent dynamo. The force will then be needed to maintain a statistically stationary state in the cases where no dynamo effect is present.

The velocity is assumed Gaussian, isotropic, δ-correlated in time, and its correlation function is

$$\begin{aligned}\langle v_\alpha(\boldsymbol{x},t)v_\beta(\boldsymbol{x}',t')\rangle &= \delta(t-t')D_{\alpha\beta}(\boldsymbol{x}-\boldsymbol{x}')\\ &\equiv \delta(t-t')[D_{\alpha\beta}(0)-S_{\alpha\beta}(\boldsymbol{r})].\end{aligned} \tag{2}$$

The structure function S scales with exponent ξ (with $0\leqslant\xi\leqslant 2$):

$$S_{\alpha\beta}(\boldsymbol{r})=Dr^{\xi}\left[(\xi+2)\delta_{\alpha\beta}-\xi\frac{r_\alpha r_\beta}{r^2}\right], \tag{3}$$

in the range $\Lambda_{\rm UV}\ll r\ll\Lambda_{\rm IR}$. The cutoffs $\Lambda_{\rm IR}$ and $\Lambda_{\rm UV}$ are the largest and the smallest scales in our problem. The energy $D_{\alpha\alpha}(0)$ diverges $\propto\Lambda_{\rm IR}^{\xi}$. The force $\boldsymbol{f}$ is Gaussian, δ-correlated in time, isotropic, parity invariant and such that

$$\int_{-\infty}^{+\infty}\langle \boldsymbol{f}(\boldsymbol{r},t)\cdot\boldsymbol{f}(0,0)\rangle dt=F\left(\frac{r}{L}\right)\equiv F_L(r). \tag{4}$$

The function F is chosen such that its Fourier transform is concentrated mostly around wave numbers $O(1)$. The general form of the covariance of $\boldsymbol{B}$ is (see, e.g., Ref. [10])

$$\langle B_\alpha(\boldsymbol{x},t)B_\beta(\boldsymbol{x}',t)\rangle=G_1(r,t)\delta_{\alpha\beta}+G_2(r,t)\frac{r_\alpha r_\beta}{r^2}, \tag{5}$$

where $\boldsymbol{r}=\boldsymbol{x}-\boldsymbol{x}'$ and the functions G_1 and G_2 are related by

$$\frac{\partial G_1}{\partial r}=-\frac{1}{r^2}\frac{\partial}{\partial r}(G_2 r^2). \tag{6}$$

The second-order correlations are thus defined in terms of a single function; it is convenient to express everything in terms of the trace of the correlation function tensor

$$H(r,t)\equiv 3\,G_1(r,t)+G_2(r,t). \tag{7}$$

We then obtain from (6) and (7)

$$G_1=+\frac{H}{2}-\frac{1}{2r^3}\int_0^r H(\rho)\rho^2 d\rho, \tag{8}$$

$$G_2=-\frac{H}{2}+\frac{3}{2r^3}\int_0^r H(\rho)\rho^2 d\rho. \tag{9}$$

It is easily shown that, on account of the δ correlation in time of $\boldsymbol{v}$ and $\boldsymbol{f}$, the function H satisfies a closed, exact equation. This can be derived using Gaussian integration by parts (see, e.g., Ref. [11], p. 43), which states that, for a Gaussian field $g(x)$, the following equality holds:

$$\langle g(x)\mathscr{F}(g(\cdot))\rangle=\int dx'\langle g(x)g(x')\rangle\left\langle\frac{\delta\mathscr{F}(g(\cdot))}{\delta g(x')}\right\rangle. \tag{10}$$

Here, $\mathscr{F}$ is a generic smooth functional of the field g and $\delta/\delta g(x)$ denotes the functional derivative. By exploiting (10) and the δ correlation in time, we obtain, after lengthy but simple algebra, the following equation:

$$\partial_t H=\frac{2}{r^2}\frac{\partial}{\partial r}\left[(\nu r^2+Dr^{2+\xi})\frac{\partial H}{\partial r}\right]+F_L(r)+2D\xi r^{\xi-2}\times\left[(1+2\xi)H+r\frac{\partial H}{\partial r}+\frac{\xi(\xi-2)}{r^3}\int_0^r H(\rho)\rho^2 d\rho\right]. \quad (11)$$

The homogeneous part of (11) was derived by diagrammatic techniques in Ref. [12] for a generic velocity correlation function.

It is now important to understand whether (11) implies relaxation to a statistically stationary state or if unbounded growth of the magnetic field (dynamo effect) takes place. Let us then consider Eq. (1) [or (11)] with no force. For ξ sufficiently small, we do not expect any dynamo effect. Indeed, for $\xi=0$, Eq. (11) reduces to the same equation as for a passive scalar [1]. The terms responsible for the stretching of the magnetic field are therefore proportional to ξ. In order to analyze the dynamo issue in more detail, it is convenient to perform the following transformation:

$$\psi(r,t)=\frac{(\nu+Dr^{\xi})^{1/2}}{r}\int_0^r H(\rho,t)\rho^2 d\rho, \quad (12)$$

which reduces (11) to a Schrödinger-like form. The energies E of the homogeneous operator are defined by the equation

$$\frac{1}{m(r)}\frac{d^2\psi}{dr^2}+[E-U(r)]\psi=0, \quad (13)$$

where $\psi(r,t)=\psi(r)\exp(-Et)$. The signs have been chosen to be consistent with the quantum mechanical notation. Negative energies E clearly correspond to a dynamo effect. The position-dependent mass $m(r)$ and the potential $U(r)$ are given by

$$m(r)=\frac{1}{2(\nu+Dr^{\xi})}, \quad (14)$$

$$U(r)=\frac{4\nu^2+\nu Dr^{\xi}(8-3\xi-\xi^2)+D^2r^{2\xi}\left(4-3\xi-\frac{3}{2}\xi^2\right)}{r^2(\nu+Dr^{\xi})}. \quad (15)$$

An immediate consequence of (15) is that the potential is everywhere repulsive for ξ sufficiently small. The ground state energy E_0 is therefore non-negative, as can be shown by using its variational expression [13]. This expression is the same as for an ordinary Schrödinger equation, except for the positive definite denominator. It is then sufficient to analyze the problem of the existence of bound states for a particle of unit mass in a potential $V(r)=m(r)U(r)$. The asymptotic behaviors of $V(r)$ are

$$m(r)U(r)\sim\frac{2}{r^2}\quad \text{small } r;\quad \frac{(2-\frac{3}{2}\xi-\frac{3}{2}\xi^2)}{r^2}\quad \text{large } r. \quad (16)$$

The potential is always repulsive at small distances, becoming attractive at infinity for $\xi\simeq 0.915$. The range where no dynamo effect occurs actually extends up to $\xi=1$. Let us indeed recall some known results of quantum mechanics concerning an attractive potential varying at infinity as $-c/r^2$ (see, e.g., Ref. [14], §35). For $c>1/4$, the particle "falls" and arbitrarily large negative energies may be involved. On the other hand, for $c<1/4$ the number of levels of negative energy is finite and depends on the form of the potential at finite distances. It is also known that a potential of the form $-c/r^2$ (with $c<1/4$) in the whole space does not possess any negative energy level. The eigenmode corresponding to $E=0$ has indeed no node and thus corresponds to the ground state. This result carries over to our case. The critical value 1/4 corresponds to $\xi=1$ in the asymptotic expression (16). Furthermore, for $\xi<1$, the asymptotically subdominant terms in (15) are both repulsive. The function $m(r)U(r)$ is thus everywhere larger than $-c/r^2$, with $c<1/4$. It follows that the energies E in (13) are non-negative for $\xi<1$. Note finally that a modified semiclassical analysis indicates that negative energies appear indeed for $\xi>1$, i.e., the value $\xi=1$ is the threshold for the dynamo effect [15].

Knowing that no dynamo effect is present for $\xi<1$, it makes now sense to analyze the behavior of the correlation function in the statistically steady state maintained by the force **f**. The single-time correlations do not depend on time and the function ψ, defined in (12), satisfies

$$\mathscr{M}_2\psi\equiv\frac{d^2\psi}{dr^2}-m(r)U(r)\psi=-\frac{m(r)(\nu+Dr^{\xi})^{1/2}}{r}\int_0^r F_L(\rho)\rho^2 d\rho. \quad (17)$$

We are interested in the scaling behavior of $H(r)$ in the inertial range $\eta\ll r\ll L$. The dissipation scale is $\eta=O((\nu/D)^{1/\xi})$ and we shall consider the leading-order behavior for large L's and small ν's. A nontrivial scaling behavior can take place due to zero modes (functions annihilated by $\mathscr{M}_2$). This general mechanism was recently proposed in Refs. [3–5]. A crucial remark is that the zero modes are not globally acceptable solutions because they do not satisfy the appropriate "boundary conditions" (correct large-scale and small-scale behavior). However, in the presence of the forcing, it may be possible to match at L a zero mode with a solution of the inhomogeneous problem and thereby satisfy the large-scale boundary condition. This is precisely what happens here. Let us first consider (17) with *no external forcing*. The operator $\mathscr{M}_2$ has two scaling zero modes in the range $\eta\ll r\ll\Lambda_{\mathrm{IR}}$. The scaling exponents s_1 and s_2 are

$$s_{1,2}=\frac{1}{2}\pm\frac{3}{2}\sqrt{1-(1/3)\xi(\xi+2)}. \quad (18)$$

Furthermore, it follows from (16) that in the far dissipation range there is a regular zero mode which scales as r^2 [a constant for $H(r)$]. The problem now is to understand which exponent in (18) corresponds to the regular zero mode. In principle, a full asymptotic matching should be carried out. For this, a new space variable rescaled by η should be introduced, the resulting equation being solved and matched with the power-law solutions in the inertial range. Similarly, the power-law solutions should be matched at Λ_{IR} with the decaying solution obtained for $r \gg \Lambda_{IR}$. Note that in this region the dynamics is governed by transport equations with effective transport coefficients. In our case with no helicity, and therefore no α effect [8], the equations are essentially the same as for a passive scalar with an eddy diffusivity $\nu_E \gg \nu$. In practice, rather than going through this cumbersome procedure, we prefer using the following simpler and more physical arguments [16] which are inspired by §35 of Ref. [14]. The general solution of the homogeneous equation $\mathscr{M}_2 \psi = 0$ with zero molecular diffusivity is

$$\psi = A r^{s_1} + B r^{s_2}, \tag{19}$$

where A and B are two arbitrary constants. We consider now a neighborhood of the origin having radius $O(\eta)$, where we replace the effective potential $V(r) = m(r) U(r)$ by the constant $V(\eta)$. The regular solution of the Schrödinger equation is explicitly known and can be patched with (19) at η. It is then easy to find that the ratio

$$\frac{B}{A} \simeq \text{const} \times \eta^{s_1 - s_2}. \tag{20}$$

Then, going to the limit $\eta \to 0$, we find that $B/A \to 0$. Equation (20) clearly indicates that the zero mode which is regular in the dissipation range corresponds to the exponent s_1 in the inertial range. The same argument repeated at the infrared cutoff $\Lambda_{IR} \to \infty$ leads to

$$\frac{B}{A} \simeq \text{const} \times \Lambda_{IR}^{s_1 - s_2}. \tag{21}$$

The conclusion stemming from (20) and (21) is that there is no nontrivial zero mode which satisfies the boundary conditions at both zero and infinity: the zero mode with exponent s_1 (in the inertial range) does not match correctly at large scales, while the other one is not appropriate at small scales. However, *in the presence of the forcing*, the situation at large scales is modified. Let us indeed define the new function $\phi = r^{-s_1} \psi$. Equation (17) in the range $\eta \ll r \ll \Lambda_{IR}$ reduces then to a form which can be immediately integrated. The general solution is

$$\psi = a r^{s_1} + b r^{s_2} - \frac{r^{s_1}}{2\sqrt{D}} \int_0^r \rho^{-2 s_1} d\rho \int_0^\rho (\rho')^{-s_2 - \xi/2} d\rho' \int_0^{\rho'} \tilde{\rho}^2 F_L(\tilde{\rho}) d\tilde{\rho}, \tag{22}$$

where a and b are two arbitrary constants to be fixed by imposing the boundary conditions. The parameter b is fixed by imposing the boundary condition on the ultraviolet side, i.e., that the most divergent term r^{s_2} does not appear in the expansion of ψ for $r \ll L$. This gives $b = 0$. By requiring that the most divergent term r^{s_1} does not appear in the expansion of ψ for $r \gg L$, we obtain

$$a = \frac{1}{2\sqrt{D}} \int_0^\infty \rho^{-2 s_1} d\rho \int_0^\rho (\rho')^{-s_2 - \xi/2} d\rho' \int_0^{\rho'} \tilde{\rho}^2 F_L(\tilde{\rho}) d\tilde{\rho}. \tag{23}$$

The behavior in the inertial range $\eta \ll r \ll L$ of the trace $H(r)$ of the covariance is

$$H(r) \simeq \frac{1}{2D} \left[r^{\gamma} \left(\frac{L^{4 - s_1 - \xi/2}}{s_1 - s_2} \frac{s_1 + 1 - \xi/2}{s_1 - 1 + \xi/2} \times \int_0^\infty \rho^{3 - s_1 - \xi/2} F(\rho) d\rho \right) - r^{2-\xi} \frac{F(0)}{3} \frac{(5 - \xi)}{(3 + s_2 - \xi/2)(3 + s_1 - \xi/2)} \right]. \tag{24}$$

The dominant contribution in (24) scales with the exponent

$$\gamma = -\frac{3+\xi}{2} + \frac{3}{2}\sqrt{1 - (1/3)\xi(\xi+2)} \simeq -\xi - \frac{\xi^2}{3} + O(\xi^3), \tag{25}$$

which is universal, i.e., depends neither on ν nor on the detailed form of F. The constant is, however, nonuniversal. Note that the expansion in (25) coincides with the one which would be obtained by the perturbation technique used in [4].

The previous results can be generalized to the d-dimensional case ($d \geq 2$). Equation (1) governs again the dynamics of $\boldsymbol{B}$, which is a d-dimensional vector field [17]. The definition (3) of the structure function S becomes

$$S_{\alpha\beta}(\boldsymbol{r}) = D r^{\xi} \left[(d + \xi - 1) \delta_{\alpha\beta} - \xi \frac{r_\alpha r_\beta}{r^2} \right]. \tag{26}$$

The transformation reducing the equation for $H(r)$ to a Schrödinger-like form is

$$\psi(r,t) = \frac{\left(\nu + D \frac{(d-1)}{2} r^{\xi} \right)^{1/(d-1)}}{r^{(d-1)/2}} \int_0^r H(\rho, t) \rho^{d-1} d\rho, \tag{27}$$

where the position-dependent mass $m(r)$ and the potential $U(r)$ are

$$m(r) = \frac{1}{2\nu + D(d-1) r^{\xi}}, \tag{28}$$

$$U(r) = \frac{1}{r^2} \left[(d+1)(d-1)\nu^2 + \nu D r^{\xi} [(d+1)(d-1)^2 - 2\xi(d+\xi)(d-2)] + D^2 r^{2\xi} \left(\frac{(d+1)(d-1)^3}{4} - \xi d (d-2)(d+\xi-1) \right) \right] \frac{1}{2\nu + D(d-1) r^{\xi}}. \tag{29}$$

It is particularly interesting to consider the two-dimensional (2D) case. The effective potential $V = mU$ reduces for $d=2$ to $V = 3/4r^2$. Since the potential is everywhere repulsive there is no dynamo effect. This is in agreement with the general antidynamo theorem valid for 2D magnetic fields [9]. The vector potential $\boldsymbol{A} = A\hat{z}$ satisfies indeed a passive scalar equation. The covariance $C(r)$ of A can then be calculated exactly and $H=-\nabla^2 C(r)$. On the other hand, we can apply the same procedure used previously for the 3D case. The dominant behavior of $H(r)$ in the inertial range is

$$H(r)\simeq -r^{-\xi}\frac{L^2(2-\xi)}{2D}\int_0^\infty \rho F(\rho)\ln(\rho)d\rho, \qquad (30)$$

which coincides indeed with $-\nabla^2 C(r)$. The identity $\int_0^\infty \rho F(\rho)d\rho=0$, valid for 2D solenoidal random fields (see Ref. [10]), has been used in deriving (30). Note that a simple dimensional argument allows one to predict the scaling $r^{2-\xi}$ for $C(r)$ and thus the exponent $-\xi$ appearing in (30). It follows that anomalous inertial-range scaling for $H(r)$ should be defined as a power-law behavior r^γ, with $\gamma \neq -\xi$. In this sense, anomalous scaling disappears in the infinite-dimensional limit. The exponent of $H(r)$ in the d-dimensional case is indeed

$$\gamma=\frac{d}{2}\left(1-4\xi\frac{(d-2)(d+\xi-1)}{d(d-1)^2}\right)^{1/2}-\frac{d}{2}-\frac{\xi}{d-1}$$
$$\simeq -\xi-2\frac{\xi^2}{d}+O\left(\frac{1}{d^2}\right). \qquad (31)$$

Note finally that the same arguments used in 3D give a range of ξ where no dynamo effect takes place which enlarges with d (for $d\geqslant 3$).

We have thus shown that in the kinematic dynamo problem a nontrivial scaling behavior appears already at the level of second-order correlations. The scaling exponent arises from the balance between the stretching and the damping due to the eddy diffusivity and is associated with a zero mode. This is in agreement with the general mechanism recently proposed in Refs. [3–5]. The exponent is universal, i.e., depends neither on the diffusivity nor on the detailed form of the large-scale forcing. However, the constants appearing in the correlation functions are nonuniversal.

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