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Array-enhanced friction in the periodic stick-slip motion of nonlinear oscillators

Y. Braiman, F. Family, and H. G. E. Hentschel
Department of Physics, Emory University, Atlanta, Georgia 30322
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We study the nonlinear contribution to stick-slip motion in a weakly coupled discrete one-dimensional array of oscillators subject to a periodic potential. We find a dramatic increase in the friction coefficient of the array compared to that of a single uncoupled oscillator, even though the same constant force f is applied to each oscillator in the array. The sliding friction coefficient is found to diverge as $\eta \propto (\kappa - \kappa_c)^{-1/2}$, where κ_c is the critical value of the coupling constant κ, and shows two dynamical transitions as we increase the external force f applied to each of the oscillators.

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Stick-slip motion is a widely observed phenomenon whose scale ranges from atomic to macroscopic lengths and is of growing interest in physics and related fields. This type of motion is one of the important building blocks in understanding friction and lubrication of materials [1], and it has been the subject of intensive studies in recent years [2]. Besides friction, stick-slip motion is studied in connection with earthquake models and avalanches [3]. Recent experiments [4], as well as theoretical models [5] and studies of spring-block systems [6], indicate that stick-slip motion arises mainly in situations in which the average velocity of the system is low. The transition from creep motion to stick-slip [4], and sliding motion [7] is of interest especially in small systems such as nanostructures, in motion involving boundary lubrication, and other related phenomena. The dynamical mechanisms leading to stick-slip motion are not yet clear and studies of these mechanisms are important in understanding the basic principles of friction at a wide range of length scales from the atomic [8] to the macroscopic [3].

In this work we present the results of our study of stick-slip motion in an intentionally simple system — a one-dimensional (1D) array of nearest-neighbor coupled nonlinear oscillators, all subject to the same external force. Our motivation is to study friction and the underlying mechanism leading to stick-slip motion in the strongly nonlinear regime (where the contribution of nonlinear potential is dominant). In particular, we find that coupling the elements into an array leads to a dramatic increase in the friction coefficient. We have developed [11] a simple formalism to reduce the complexity of the equations and to calculate the velocity and the friction coefficient of the elements in the array. Surprisingly, even in the lowest order, for which dynamics of the array can be described by an effective single-oscillator equation, our analytical approximation shows very good agreement with the numerical solution.

Our starting point is the following (widely used) equation for the motion of N coupled oscillators:

$$m\ddot{x}_j + \gamma\dot{x}_j = -\frac{\partial U}{\partial x_j} - \frac{\partial V}{\partial x_j} + f_j + \xi_j. \tag{1}$$

Here x_j is the coordinate of the j's oscillator, m is the mass, γ is the linear friction coefficient, f_j is the external force, and ξ_j is a random force. The oscillators are subjected to a periodic potential $U(x_j)$ and interact with each other via a pairwise potential $V(x_i - x_j)$. Equation (1) has a very broad range of interpretations, is widely used, and describes various physical systems, depending on the choice of the parameters and the potentials. Equation (1) is used to describe the dynamics of an adsorbate system on a surface [2] (the fric-

tion coefficient is related to the noise and the temperature via the fluctuation-dissipation theorem). In a macroscopic variation, Eq. (1) describes the motion of a set of N coupled oscillators subject to a periodic potential (the Frenkel-Kontorova model; this model has been widely studied in relation to commensurate-incommensurate phase transitions [9]). An example of Eq. (1) applied to Hamiltonian dynamics is given in Ref. [10] and is related to energy transfer in long 1D molecular chains adsorbed on a periodic substrate.

In what follows, we consider a simple variation of Eq. (1), assuming a linear interparticle interaction, a simple periodic substrate potential, zero misfit length (the periodicity of the periodic potential is equal to the equilibrium interparticle spacing), and zero noise. Equation (1) can then be written in the following dimensionless form:

$$\ddot{x}_j+\gamma\dot{x}_j+\sin x_j=f+\kappa(x_{j+1}-2x_j+x_{j-1}) \tag{2}$$

where f is the applied force (the same for all oscillators), $\sin x_j$ represents the periodic potential, and κ is the nearest-neighbor coupling in the array. We consider both periodic and free-end boundary conditions.

We are looking for a stick-slip motion, consequently we focus on the family of wave-propagating solutions defined by

$$x_j(t)=x(t-j\tau)+x(t+j\tau) \tag{3}$$

where τ is a characteristic time scale. This kind of dynamics occurs when we initially excite just one oscillator [$x_N(0)=x_0$ and $\dot{x}_N(0)=\dot{x}_0$], while all the others are initially at rest, i.e., $x_j=\dot{x}_j=0$, $j=1,2,\ldots,N-1$. The excited oscillator initiates a wave propagating across the array in a "falling dominos" type of motion towards the other end of the array which is then reflected back, moving in the opposite direction towards the initially excited oscillator.

We find that for given values of the external force f and the linear friction coefficient γ, there is a critical value of the coupling constant κ_c, below which the linear wave [defined by Eq. (3)] will not propagate across the array. Here we will focus on the behavior of the chain in the close vicinity of this critical coupling as $\kappa\rightarrow\kappa_c$. At $\kappa=\kappa_c$ the motion is localized [11]: at each moment effectively only one oscillator moves driven by the force applied from its neighbors. This force can be to a high degree of accuracy approximated as a constant by assuming $x_{j+1}=y+2\pi$ and $x_{j-1}=y$, where $y=\sin^{-1}f$, the stable fixed point of the single uncoupled oscillator. Thus an approximate one-oscillator equation can be derived,

$$\ddot{x}+\gamma\dot{x}+\sin x=f+\kappa(y+2\pi+y-2x). \tag{4}$$

Equation (4) represents the lowest order in a sequence of n-cluster approximations [11] in which n coupled oscillators are treated exactly in the presence of a force generated by the remaining quiescent oscillators assumed to take the values $x_{\text{right}}=y+2\pi$ and $x_{\text{left}}=y$ representing a linear wave propagating through the array from right to left.

The minimal value of the average velocity v_0 is given by $v_0=2\pi/T_N$, where T_N is the period of oscillation of each single oscillator in the array. For free-end boundary conditions, $T_N=2N\tau$; while for periodic boundary conditions we have $T_N=N\tau$. Therefore writing in general $T_N=mN\tau$ (m is an integer), the minimal velocity is given by $v_0=2\pi/mN\tau$ [12].

We approximate the characteristic time τ for the excitation to be passed between oscillators (a property of the array) by $\tau=T$, where T is the time each oscillator moves separately. Therefore τ and consequently the velocity can be calculated using Eq. (4) together with the matching initial conditions

$$x(0)=\sin^{-1}f=y, \tag{5a}$$

$$\dot{x}(0)=0. \tag{5b}$$

To find T we integrate Eq. (4). To a good approximation, as $\kappa\rightarrow\kappa_c$ this time period is given by the expression [11]:

$$T\approx\gamma\left(\frac{\pi}{\pi-\cos^{-1}f}\right)^{1/2}(\kappa-\kappa_c)^{-1/2}. \tag{6}$$

This leads to the following expression for the minimal velocity of the chain:

$$v_0=\frac{2\pi}{mN\gamma}\left(\frac{\pi-\cos^{-1}f}{\pi}\right)^{1/2}(\kappa-\kappa_c)^{1/2} \tag{7}$$

and, in the leading order the average velocity scales as $(\kappa-\kappa_c)^{1/2}$. The nonlinear friction coefficient defined as

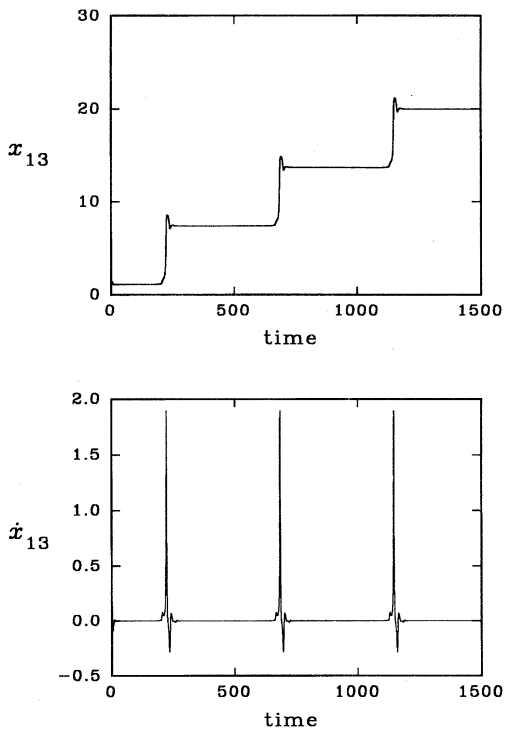

FIG. 1. The time trace of the phase x_{13} (the top plot) and the velocity (the bottom plot) of the 13th oscillator of an $N=25$ oscillator array. The other parameters are $f=0.9$, $\gamma=0.7$, $\kappa=0.02$ and free-end boundary conditions have been imposed. The initial conditions used were $x_j(0)=\dot{x}_j(0)=0$ for all $j=1,\ldots,N-1$, and $x_N(0)=2\pi$; $\dot{x}_N(0)=0$.

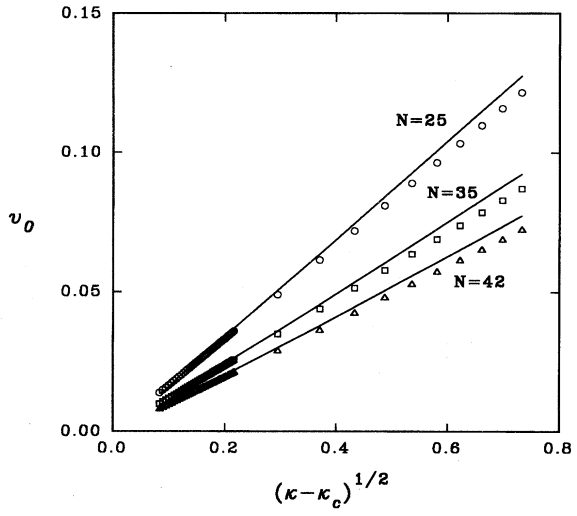

FIG. 2. The numerically calculated time averaged velocity across the oscillators v_0 as the function of $(\kappa-\kappa_c)^{1/2}$ for three array lengths $N=25,35,42$ (from the top to the bottom). The circles ($N=25$), squares ($N=35$), and triangles ($N=42$) denote analytically predicted values of the average velocity. The other parameters are as in Fig. 1.

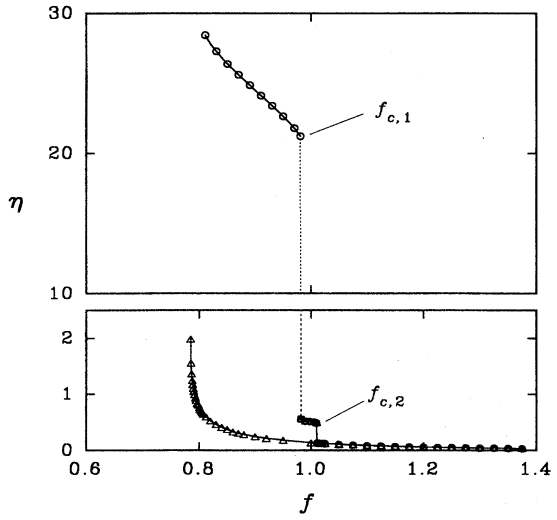

FIG. 3. The friction coefficient $\eta=(f/v_0-\gamma)/\gamma$ as a function of the applied force f. The triangles (the bottom curve) correspond to a single uncoupled oscillator, while the circles (the top curve) correspond to the chain ($N=25$) with free-ends boundary conditions. The lines plotted are to guide the eye. The other parameters are as in Fig. 1 and $\kappa=0.1$.

$\eta=(f/v_0-\gamma)/\gamma$ in consequence diverges as $\eta\propto(\kappa-\kappa_c)^{-1/2}$ and is given by the expression

$$\eta=\frac{mNf}{2\pi}\left(\frac{\pi}{\pi-\cos^{-1}f}\right)^{1/2}(\kappa-\kappa_c)^{-1/2}-1 \tag{8}$$

where the critical value of the coupling $\kappa_c(f,\gamma)$ is a function of the forcing f and linear friction γ and an upper bound is given by the high linear friction limit

$$\kappa_c(f,\gamma)\leq\frac{(1-f)}{2(\pi-\cos^{-1}f)}+\frac{(1-f)^2}{4(\pi-\cos^{-1}f)^3}+\cdots \tag{9}$$

where the right hand side of Eq. (9) is valid as $f\to 1$.

Equation (8) indicates a dramatic increase in friction coefficient of an array compared to the friction coefficient of a single uncoupled oscillator, even though the same constant force is applied to each oscillator in array. In fact, as $\kappa\to\kappa_c$, the friction coefficient is proportional to the size of the chain $\eta\propto N$.

We performed numerical simulations on arrays containing $N=18,25,35,42,50$ oscillators. The external force was set to $f=0.9$, while the dissipation $\gamma=0.7$. The coupling constant κ was varied from 0 to 1. Simulations were carried out for both periodic and free-end boundary conditions. The initial conditions were chosen in the following way: $x_N=2\pi$ and $\dot{x}_N=0$. The other oscillators were set initially $x_j=\dot{x}_j=0$. To demonstrate this stick-slip motion, we present in Fig. 1 the phase x_{13} of the 13th oscillator (the top plot) and the phase derivative $\dot{x}_{13}$ of the same oscillator (the bottom plot). The time series of the phase x_{13} and the velocity $\dot{x}_{13}$ indicate the existence of two different kinds of solutions, one which is virtually quiescent, and the other one showing a fast 2π jump.

The stick-slip dynamics of the chain can be understood based on the description of the dynamics of the uncoupled oscillator. If $f_{\min}<f<1$ ($f_{\min}$ is the minimal value of the force to obtain nonzero average velocity solution), two distinct solutions are possible, depending on initial conditions. The first is the fixed point, defined by $\sin x=f$. This solution corresponds to the static solution, $v_{\rm av}=0$ when all the particles in the chain are inside their potential wells. The second solution is the limit cycle, corresponding to a running solution $v_{\rm av}>0$ with the particles hopping over the potential maxima. The rightmost oscillator will initiate the motion of its neighbors in a chainlike dynamics of the following kind: each oscillator will make a fast 2π flip corresponding to a jump between neighboring potential wells and will then stay quiescent during the long time interval.

In Fig. 2 we have plotted the average velocity of the oscillators as a function of the coupling constant $(\kappa-\kappa_c)^{1/2}$ for three different lengths of arrays $N=25,35,42$, with the same single-oscillator parameters $f=0.9$ and $\gamma=0.7$. The initial conditions are the same as in Fig. 1. The solid lines show numerically calculated values, while the points are the analytical predictions based on Eq. (7) for free-end boundary conditions. The values of κ are varied between $0.02<\kappa<0.5$, and κ_c was found from the numerical simulations. In fact, we found that in a very narrow range of κ there exists an in-phase solution of the array (not shown in Fig. 2), resulting in high average velocity of a chain (equal to the average velocity of uncoupled oscillator). This resonance is also predicted by the equation for a single oscillator [Eq. (4)]. The in-phase solution then disappears leading to stick-slip motion of the oscillators. Our theoretical expression Eq. (7) is valid as $\kappa\to\kappa_c$, but the numerical agreement extends into the high coupling constant κ regime. This agreement can be understood on the basis of an analysis valid far from critical cou-

pling [11] which predicts a $\kappa^{1/2}$ dependence for the velocity. Finally, we observe that at coupling $\kappa \approx 1$, the "running" solution, initiated by the motion of rightmost oscillator, ceases to exist and the average velocity is zero.

Figure 3 shows the average nonlinear friction coefficient $\eta=(f/v_0-\gamma)/\gamma$ of the array (normalized to γ) as a function of applied force f for $\kappa=0.1$. Each point on the curve was calculated using the same set of initial conditions for each value of the external force f (see Fig. 1). The curve shows the presence of two dynamical transitions at f_{c1} and f_{c2} and its resulting effect on the average value of the sliding friction. If the external force is low, $f_{\min}<f<f_{c1}$, the dynamics of an array shows stick-slip motion. As we increase the force, the array undergoes a dynamical transition at f_{c1} to a new kind of dynamics valid for forcing in the range $f_{c1}<f<f_{c2}$, where the oscillators form two separate clusters consisting of alternate oscillators; the oscillators in each cluster being almost in-phase, but out of phase with the oscillators forming the other cluster. Finally, for even larger external forces $f>f_{c2}$, the dynamics is "in-phase," and all the oscillators move together. We numerically calculated the transition points f_{c1} and f_{c2} for larger arrays, up to $N=150$ oscillators, and found that the transition points do not vary with N. We note that the transitions observed in our model express the passage to different dynamical attractors and are not ordering phase transitions (such as commensurate solid–incommensurate fluid, fluid–commensurate solid, fluid–incommensurate solid, etc.) [2].

We have also compared our results on the dynamics of the coupled chain to the dynamics of an uncoupled nonlinear oscillator. The bottom line shows the friction coefficient for a single uncoupled oscillator as a function of the applied force, while the top curve applies to the entire chain. The notable difference in dynamics of the coupled array and the uncoupled oscillator is that in a very broad range of forcing $f_{\min}<f<f_{c1}$ the friction coefficient for the array is considerably higher than for a single oscillator, consequently, the coupling acts as a source of additional friction.

In summary, we have studied the periodic stick-slip dynamics in a discrete chain of coupled nonlinear oscillators. The mechanism of stick-slip motion in our example is related to the periodic transition from a "stick" dynamics (characterized by the stable fixed point of the single uncoupled oscillator in which each particle forming the chain is stuck in its potential well) to "slip" motion of the oscillator (the limit cycle) in the chain corresponding to motion over the periodic potential. The nonlinear dynamics leads to a fundamentally different friction law, in particular when the driving force is barely larger than the minimal force needed to start motion. The friction coefficient grows with the number of elements N in the array and scales as $(\kappa-\kappa_c)^{-1/2}$, and the average velocity calculated from Eq. (7) exhibits the $(\kappa-\kappa_c)^{1/2}$ dependence. These predictions are in excellent agreement with the numerically calculated values.

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[11] The details will be presented in Y. Braiman, F. Family, and H. G. E. Hentschel (unpublished).

[12] In fact, an array admits the velocities which are the multiples of the minimal velocity $v=nv_0$. These values of the velocities can be obtained by manipulating the initial condition of the elements in an array, keeping the same all other parameters.