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Quenched disorder, memory, and self-organization

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We use a stochastic description of models with a dynamic in quenched disorder to analyze the mechanism of their self-organization to a critical state in terms of memory effects. We introduce a framework to characterize both memory effects and avalanche events which suggests that self-organization can result in general from memory. This issue is settled by the introduction and the analysis of a model that contains explicitly memory and generalizes the corresponding dynamics in quenched disorder. The model displays a rich behavior and self-organized critical properties for a whole range of the exponent that tunes the strength of memory.

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Many efforts have been recently devoted to uncover the mechanism underlying the tendency of large statistical systems to self-organize into a critical state [1–6]. This issue has a great relevance since self-organized criticality manifests itself in a large variety of phenomena ranging from earthquakes and sandpiles [1,7] to creep phenomena [8], growing interfaces in a disordered medium [9], and biological evolution [10]. The mechanism of self-organization in these systems belong to two main classes whose prototype models are the sandpile model [1] and the biological evolution Bak-Sneppen model (BSM) [10]. The self-organized critical (SOC) state in the latter class is built by a dynamics based on the selection of the minimum of a random disorder field. One distinguishing feature of dynamics in a disordered medium lies in the emergence of memory effects [11]. In this paper we will focus on this aspect of the dynamics using a recently proposed [12] stochastic description of the quenched process which results from addressing the question of how one can describe and reproduce the process without knowing the "microscopic" values of the quenched disorder variables. From this perspective we will identify the dynamical variables involved in a simplified "effective" dynamics. These variables are the age of the quenched variables, that is the time elapsed since their last update, and turn out to be a very useful tool to describe both memory effects and SOC properties in general. This suggests that a SOC state may result as a consequence of memory with no relation to quenched dynamics. This is shown explicitly by the introduction of a model, defined in terms of age variables, which generalizes the qualitative behavior of the corresponding quenched dynamics. The occurrence of a very rich scenario of SOC behavior, wider than that observed in quenched dynamics, suggests that criticality is not a peculiarity of dynamics in quenched disorder but rather it arises in general as a result of memory effects. The model has a general phenomenological nature which allows also for a derivation independent of quenched dynamics. As such it reveals an alternative mechanism for self-organization.

The BSM, introduced to model biological evolution [10], has a very simple definition: At $t=0$ assign a uniformly distributed random variable (RV) $\eta_{i,t=0}$ on each site i of a d-dimensional lattice. At each time t select the smallest RV $\eta_{\min}(t)=\min_i \eta_{i,t}$ and replace it and the RV's on the neighboring sites with newly extracted uniform RV's, leaving all other RV's unchanged. In biological terms, time is measured in mutation events and the species which undergoes mutation is the one with the smallest fitness $\eta_{i,t}$. The system self-organizes to a critical steady state in which almost all RV's are above a certain threshold value η_c. The critical nature of the SOC state can be properly described [5,10] in terms of avalanches, i.e., causally and spatially connected series of events.

The persistence of activity displayed by avalanche events can be connected to memory effects by noticing that evolution will take place more often on recently updated regions than in older ones. This is because a site whose RV has been checked a large number of times in the search for the minimum RV will probably have a large RV. It might still be the smallest in the future but the probability for this to happen gets smaller and smaller as time goes on. A quantitative translation of this observation requires a description of the process which is not based on the values of the RV's $\eta_{i,t}$ but rather on the knowledge of their distributions $p_{i,t}(x)dx = \mathrm{Prob}(x \leq \eta_{i,t} < x+dx)$ [12]. Indeed once the values of $\eta_{i,t}$ are given, the selection of the smallest RV is a deterministic operation (with probability 0 or 1 for each i). On the contrary, if the distributions of $\eta_{i,t}$ are given, each site i will be selected with a probability $\mu_{i,t} = \mathrm{Prob}(\eta_{i,t} < \eta_{j,t}\ \forall j \neq i) = \int_0^1 dx\, p_{i,t}(x) \Pi_{i\neq j} \int_x^1 dy\, p_{j,t}(y)$. Taking the distributions $p_{i,t}(x)$ as dynamic variables makes the effects of memory in the dynamics explicit. It is indeed the distribution $p_{i,t}(x)$, rather than the value of $\eta_{i,t}$, that "remembers," i.e., stores in a conditional way, the events of the past history. This description, proposed originally in Ref. [13] and further refined in Ref. [12], is a *different perspective for quenched growth problems* which eliminates the problem of quenched averages, which are performed by the dynamics itself, at the price of introducing a memory in the dynamics (for more details see Ref. [12]). The use of distributions as dynamical variables makes this description far more complex than the standard one. This raises the question of whether a simplification of this dynamics, which however retains its essential features, is possible. The key observation, in order to find

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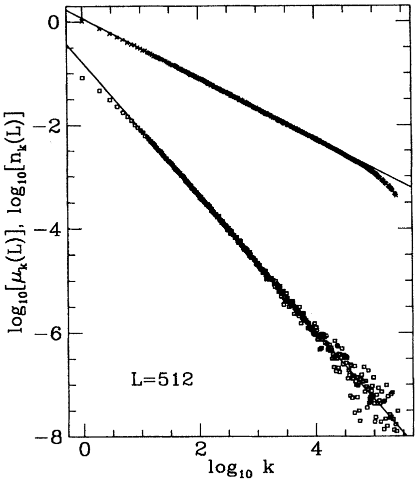

FIG. 1. The distributions $n_k(L)$ ($\times$) and $\mu_k(L)$ ($\square$) for the Bak-Sneppen model. $\mu_k(L)$ was obtained measuring the frequency $p_k(L)=n_k(L)\mu_k(L)$ of events in which a site with age k was selected during the BSM's evolution in the steady state. The fitting lines have slopes $\beta=0.58$ and $\alpha=1.30$, respectively.

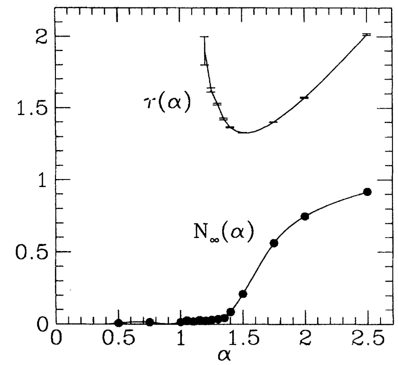

FIG. 2. Fraction of infinite avalanches $N_\infty(\alpha)$ and estimate of the $\tau(\alpha)$ exponent for system sizes $L\leqslant 256$.

simpler dynamical variables, is that RV's with the same age have experienced the same events and hence have the same probability of being selected. The age variable (AV) $k_{i,t}$ is the time elapsed since the last update on site i and

$$k_{i,t+1}=\begin{cases} 0 & \text{if } \eta_{i,t+1}\neq \eta_{i,t} \\ k_{i,t}+1 & \text{else.} \end{cases} \qquad (1)$$

The probability $\mu_{i,t}$ that $\eta_{i,t}$ is the minimum RV will in general depend on its age, on the age $k_{j,t}$ of all other RV's at the same time, and on the age of the last $k_{i,t}$ variables that have been selected in the process. Neglecting the latter dependence, with a crude approximation [12,14], one finds that the probability that a site is selected has a power law dependence on its age: $\mu_{i,t}\sim k_{i,t}^{-\alpha}$ with $\alpha=2$. In this approximation, $\mu_{i,t}$ depends on the ages $k_{j,t}$ of other sites via the normalization condition $\Sigma_i\mu_{i,t}=1$ [12]. Direct numerical simulation confirms this qualitative behavior with an exponent $\alpha=1.30\pm0.02$ (see Fig. 1).

AV's, which result naturally from a simple discussion of the dynamics in a disordered medium, are a useful tool to describe in general both the effects of memory and the scenario of self-organization. AV's can be introduced in any statistical system with sequential update dynamics coupling them to the dynamical variables $\eta_{i,t}$ via Eq. (1). We will denote by $n_{k,t}$ the distribution of AV's, i.e., the number of sites with age k. For a system of linear size L in d dimensions $\Sigma_k n_{k,t}=L^d$. We can define the age of a system as the average age of microscopic variables measured in units of macroscopic time $1/L^d$: $K(L,t)=L^{-2d}\Sigma_i k_{i,t}=L^{-2d}\Sigma_k k n_{k,t}$. The scaling of the age of the system, in the stationary state, with its size $K(L)\sim L^{d\zeta}$ provides a measure of memory. If $\zeta=0$ we will say no memory exists while $\zeta>0$ will signal the presence of large very old regions in the system; the local dynamics will depend on a large period of the past history so that the state of the infinite system will depend on the whole history of the process [15].

Let us illustrate this definition with some specific examples: consider the Ising model with Metropolis dynamics [16]: $\eta_{i,t}=\pm1$ are spin variables and at each time one site is selected at random and it is flipped with a probability related to Boltzmann weights. On average $\eta_{i,t}$ will flip once every L^d events so that $\zeta=0$. The same conclusion can be drawn for all Markovian systems. Consider next the prototype model of SOC, the sandpile model [1]: $\eta_{i,t}=0,1,\ldots$ is the height of the sandpile on site i. The dynamics comprises either toppling events, when some $\eta_{i,t}\geqslant 2d$ relaxes ($\eta_{i,t+1}=0$) distributing one sand unit to each of its neighbor sites $i+\delta$ ($\eta_{i+\delta,t+1}=\eta_{i+\delta,t}+1$), or random sand addition events, when all $\eta_{i,t}<2d$. Defining AV $k_{i,t}$ through Eq. (1), we found $\zeta=0$ also in this case. On the contrary for the $d=1$ BSM we found $\zeta=1.46\pm0.03$. The distribution $n_{k,t}$, averaged over realizations is shown in Fig. 2 and it follows a scaling behavior

$$n_k(L)=\langle n_{k,t}\rangle=k^{-\beta}f(k/L^{1+\zeta}) \quad \text{for } k>0 \qquad (2)$$

with $\beta=0.58\pm0.01$. The exponent β coincides with the exponent $\tilde{d}$ introduced in Ref. [17] which describes the scaling of the number of updates on a given site in a period of time T with T. The definition of $K(L,t)$ and the normalization of $n_{k,t}$ imply that β is also related to $\zeta=\beta/(1-\beta)$, in excellent agreement with numerical results.

If ζ is the indicator of the relevance of memory, the self-organized nature is usually related to the occurrence of avalanche events. An avalanche event is made up of a spatially and causally connected sequence of events. Causal relation can be identified by looking at the sequence $\eta_{\min}(t)=\min_i\eta_{i,t}$ of selected values [5,6,10]. If, e.g., $\eta_{\min}(t+1)<\eta_{\min}(t)$, the site selected at time $t+1$ must have been updated at time t and hence the two events are causally related. This causal (and spatial) connection may persist for a long period of time and actually avalanches of all sizes can occur in the BSM [5,6,10]: the probability that an avalanche lasts a time s has a power law dependence $N(s)\sim s^{-\tau}$ on s. A rather intuitive and, to some extent, less ambiguous [18] definition of avalanches is possible in terms of the variables

k_i. Consider the avalanche started at time t_0. This will be active at time t_0+s if all sites $i(t)$ selected at times $t_0<t\leq t_0+s$ had an age $k_{i(t),t}\leq t-t_0$. Indeed in this case all the selected sites have been updated by the process initiated at time t_0. This avalanche will terminate at time t_0+s when a site with age $k>s$ will be selected and will contribute to the statistics $N(s)$ of avalanches of size s. Note that this method can be applied to any system, including the sandpile model.

Having shown that AV's allows a complete description of memory and self-organization we can explore the connection between these two phenomena by introducing a different model. The model, inspired by our discussion on quenched dynamics, is defined as follows: an AV $k_{i,t}$ is assigned to each site $i=1,\ldots,L$ of a linear lattice with periodic boundary conditions. Initially $k_{i,0}=0\ \forall i$. At each time step t one site is selected with a probability $\mu_{k_{i,t},t}=\mu_{0,t}(1+k_{i,t})^{-\alpha}$ which depends only on the age $k_{i,t}$ of site i. When a site i is selected, its age and that of its neighbors is set back to zero ($k_{i+\delta,t+1}=0$ for $\delta=0,\pm 1$) while all other ages are increased by 1 ($k_{j,t+1}=k_{j,t}+1$ for $|j-i|>1$). The power law dependence of the selection probability on k generalizes the situation observed in the BSM (see Fig. 1). The constant $\mu_{0,t}$ is fixed by the normalization $\Sigma_k n_{k,t}\mu_{k,t}=1$. This condition introduces a nonlocal dependence of the selection probability which is of the same type as that implied by the search of a global minimum in the BSM, as discussed above. The requirement behind this condition (as in the BSM) is that only one site is selected at each time step. This would correspond to an extremely slow parallel dynamics, in which the probability of selecting simultaneously two or more sites is zero, and where the time is measured in selection events. This limit of "slow driving" underlies most (if not all) models of SOC. In general, the model describes a slowly driven process with single update events selected from a power law waiting time distribution. Contrary to the BSM, which has a microscopic definition, the age model defined here has a phenomenological nature. It requires the measure of the "phenomenological" parameter α but no assumption on the microscopic mechanism of the dynamics or on the interaction. The definiton of the age model is also closer in spirit to the theory of evolution, where usually the fitness of a species is determined by its age [19]. For a finite L the system gets to a steady state that is characterized by a distribution of counters $n_k(L)=\langle n_{k,t}\rangle$ for which we shall assume the scaling form Eq. (2). Three counters are updated at each time step, so $n_{0,k}=n_0(L)=3$.

Consider first the $\alpha=0$ case. A fraction $3/L$ of the sites with counter k are updated while the others increase their k value, hence $n_{k+1}(L)\cong n_k(L)(1-3/L)$ and $n_k(L)\cong 3\exp(3k/L)$ so $\zeta=\beta=0$. Since $\mu_0=1/L$ at all times, the probability of a connected event of s steps also vanishes exponentially with s, i.e., $\tau(\alpha=0)=0$. For $\alpha=0$ there are neither memory nor SOC. The same behavior persists up to $\alpha=1$. This results from focusing on a site with $k_i=k$ and considering the probability $P_k(s)$ that it will not be selected in the next s steps, under the condition that in this period it will not be updated because of its neighbors. It is not difficult to check that $P_k(s)=\Pi_{j=k+1}^{k+s}(1-\mu_0 j^{-\alpha})\rightarrow 0$ as $s\rightarrow\infty$ for $\alpha<1$ which means that any site, if it is not updated by its neighbors, will surely be selected sooner or later. The average age of sites is then at most of order L so that $\zeta(\alpha<1)=\beta(\alpha<1)=0$. The occurrence of SOC can be excluded as well for $\alpha<1$. The existence of avalanche events on all length and time scales requires μ_0 to be finite as $L\rightarrow\infty$. Using Eq. (2) for $n_k(L)$, it is easy to see that $\mu_0=[\Sigma_k n_{k,t}(k+1)^{-\alpha}]^{-1}\rightarrow 0$ for $L\rightarrow\infty$ if $\alpha+\beta<1$, i.e., if $\alpha<1$.

TABLE I. Exponents $\beta(\alpha)$ and $\zeta(\alpha)$. The former was obtained from the slope of $\log n_k(L)$ vs $\log k$ and confirmed by collapsing data for $L=64$, 128, and 256. For the latter we evaluated the slope of $\log K(L)$ vs $\log L$ plots.

α	β	ζ
1.10	0.3 ± 0.1	0.5 ± 0.2
1.20	0.48 ± 0.01	0.90 ± 0.04
1.30	0.58 ± 0.01	1.40 ± 0.06
1.40	0.619 ± 0.005	1.53 ± 0.03
1.50	0.613 ± 0.005	1.47 ± 0.03
1.75	0.571 ± 0.005	1.31 ± 0.03
2.00	0.545 ± 0.005	1.17 ± 0.02
2.50	0.510 ± 0.005	1.06 ± 0.02

Let us now consider the opposite case: $\alpha=\infty$. In this case $\mu_k=0\ \forall k>0$ and $\mu_0=1/3$. The model describes a random walk on a $d=1$ lattice. It is not difficult to find $\zeta(\infty)=1$ and $\beta(\infty)=1/2$. The evolution is a single connected event: every avalanche lasts for an infinite time. For a finite $\alpha\gg 1$ it is convenient to generalize the avalanche distribution to account for infinite avalanches: $N(s)=(1-N_\infty)N_f(s)+N_\infty\delta_{s,\infty}$ where N_∞ is the fraction of avalanches that never stop. These are all those avalanches that began before time $t-\max_i k_{i,t}$, which are still active, and which cannot be terminated by any selection event. An avalanche of size s is terminated when a site with $k_i>s$ is selected. This will occur with probability $P_{\text{stop}}(s)=\Sigma_{k>s}n_{k,t}\mu_{k,t}$. We can estimate the probability that an avalanche is still active as $P_{\text{act}}(s,t)\cong\Pi_{j=1}^{s-1}[1-P_{\text{stop}}(s-j,t-j)]$. Since $P_{\text{stop}}(s)\sim s^{1-\alpha-\beta}$, we expect $N_\infty=\lim_{s\rightarrow\infty}P_{\text{act}}(s)>0$ for $\alpha+\beta>2$. If $N_\infty>0$, the probability to observe an avalanche of size s will be $N(s)\cong P_{\text{act}}(s)P_{\text{stop}}(s)\cong N_\infty P_{\text{stop}}(s)$ which implies $\tau=\alpha+\beta-1$. There is a value $\alpha_c=2-\beta(\alpha_c)$ above which a finite fraction of infinite avalanches ($N_\infty>0$) coexist with a distribution $N_f(s)$ of finite avalanches with an exponent $\tau(\alpha)=\alpha+\beta(\alpha)-1\rightarrow 1$ as $\alpha\rightarrow\alpha_c^+$ the system is in this sense *supercritical.*

In the interval $\alpha\in[1,\alpha_c]$, where $N_\infty(\alpha)=0$, the usual scenario of SOC is expected to apply. The $\tau(\alpha)$ exponent is expected to diverge as $\alpha\rightarrow 1$ to signal the onset of the noncritical phase $\alpha\in[0,1]$. The appearance of infinite avalanches at α_c could be accounted for by the divergence of the normalization of $N(s)$ as $\tau(\alpha)\rightarrow 1^+$ for $\alpha\rightarrow\alpha_c^-$. In summary we expect that $\alpha_c=2-\beta(\alpha_c)=\sup\{\alpha;N_\infty(\alpha)=0\}$ and that $\tau(\alpha)$ reaches a minimum $\tau=1$ at α_c.

This scenario is supported by computer simulation: (1) for $\alpha<1$, as expected, μ_0 vanishes as $L\rightarrow\infty$ and $K(L)$ is finite as $L\rightarrow\infty$ [for $\alpha=1$ the best fit suggests $K(L)\sim\ln L$]. (2) Table I lists the values obtained for the exponents β and ζ by numerical simulations of the model for sizes up to

$L=256$ and $\alpha>1$. The statistical uncertainty gets large as $\alpha=1$ is approached from above. The relation $\zeta=\beta/(1-\beta)$ is satisfied fairly well. $\alpha+\beta(\alpha)$ gets bigger than 2 for $\alpha_c \cong 1.4$. In this region $\beta(\alpha)$ reaches a maximum. (3) Approximately at α_c, as seen in Fig. 2, $N_\infty(\alpha)$ becomes positive. For the sizes we could study ($L\leq 256$) the distribution $N_f(s)$ is not still relaxed to its asymptotic behavior and the estimates of τ shown in Fig. 2 are only meant to reproduce the qualitative behavior of $\tau(\alpha)$. For large α, where a stable behavior of $N_f(s)$ was found with L, we found $\tau(\alpha)\cong\alpha+\beta(\alpha)-1$. Notice that τ reaches a minimum approximately in the same region where $\alpha+\beta\approx 2$ and N_∞ starts to increase. (4) We performed simulations on the BSM as well. The agreement of $\beta(1.30)$ and $\zeta(1.30)$ with the exponents measured for the BSM is remarkable and, together with the vanishing of $N_\infty \sim L^{-1.0}$, suggests that the BSM belongs to the region $[1,\alpha_c]$. The poor quality of our statistics of avalanches did not allow us to identify the BSM and our $\alpha \simeq 1.30$ model in a conclusive way. It is interesting to note that the BSM is very close to the boundary α_c beyond which catastrophic events (infinite avalanches) occur.

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[18] The condition $\eta_{\min}(t+1)<\eta_{\min}(t)$ is sufficient but not necessary for causal connection of two successive events. Indeed it might occur, with eventually vanishing probability, that $\eta_{\min}(t+1)$ has been just updated and it is larger than $\eta_{\min}(t)$. The condition which uses AV's is clearly both sufficient and necessary for causal relation.

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