Penetration of electromagnetic velocity fields through a conducting wall of finite thickness

Timothy H. Boyer
Department of Physics, City College of the City University of New York, New York, New York 10031
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The penetration of electric and magnetic velocity fields through a conducting wall (with $\epsilon=1$, $\mu=1$) when a nonrelativistic charged particle is traveling outside parallel to the wall is calculated for a good conductor in a perturbation analysis through first order in the velocity ratio $\beta=v/c$. It is found that the magnetic field behind the conducting wall is of a universal character depending only on the displacement of the field point from the charged particle outside the wall; the field is modified by the presence of the conducting wall but is independent of the conductivity, the thickness, or the relative placement of the wall. Within the conductor, the magnetic field depends upon the thickness of the wall but not upon the conductivity of the wall. In front of the conducting wall, the magnetic field is independent of the conductivity or the thickness of the wall. The electric field inside and outside the conducting wall depends upon the conductivity and thickness of the wall. In the region behind the wall, the electric field falls off as r^{-3}. The currents in the conducting wall are independent of conductivity but depend upon the thickness of the wall. The electric field in front of the wall exerts a dragging force on the passing charged particle, providing the energy balance for resistive heating in the wall. This dragging force increases as the thickness of the wall decreases. The present general analysis extends earlier work in the literature that treats the special cases of an infinitely thick conducting wall and of a thin perfectly conducting wall. The ideas involved are unfamiliar to many physicists, who are not aware that electromagnetic velocity fields have an algebraic behavior inside conductors, which is completely different from the familiar exponential damping of electromagnetic wave fields. The ideas are of interest in connection with the electromagnetic shielding of systems from electromagnetic fields of passing charges and in connection with the Aharonov-Bohm effect where charged particles pass close to conducting solenoids. [S1063-651X(96)08906-4]

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I. INTRODUCTION

A. The problem

The exponentially-damped penetration of classical electromagnetic wave fields into conductors is a familiar subject mentioned in all the electromagnetism textbooks. However, only in recent years has it been realized that the penetration of electromagnetic velocity fields is of an entirely different character from that of electromagnetic wave fields [1–4]. Whereas wave fields are exponentially damped in conductors, the velocity fields of charged particles have an algebraic decrease with distance, the electric fields being sharply screened by good conductors, whereas the magnetic fields penetrate even in the limit of a perfect conductor. This new understanding is now referred to in the textbook literature [5] but still finds opponents [6]. Also, there is still some disagreement in the literature [4]. The present calculation for the case of a conducting wall of finite thickness represents a generalization of some previously treated special cases and corresponds to the situation relevant to experiment.

B. Context of the calculation

The existing research literature involves calculations for the penetration of electromagnetic velocity fields in a number of specialized cases. The initial investigation [1] of 1974 uses a nonrelativistic perturbation analysis to treat the penetration of fields into an infinitely thick conducting wall when a charged particle moves outside parallel to the plane surface of the conductor. Furry [2] uses a nonrelativistic perturbation for the case of a thin perfectly conducting wall. Aguirregabiria, Hernandez, and Rivas [3] discuss the velocity fields inside a solid conducting sphere due to the radial motion of a charged particle, again using a nonrelativistic perturbation calculation.

The only nonperturbation analysis is that of Jones [4], who considers a line charge (in order to reduce the spatial dimensions) moving perpendicular to its length and parallel to the face of an infinitely thick conducting wall. Instead of a perturbation in the velocity ratio v/c, he uses Fourier analysis in time; he also allows the conductor to have values of dielectric constant ϵ and relative permeability μ that differ from unity. Jones finds a skin effect when the velocity of the line charge is so high as to be in the Cherenkov radiation region where the charge is moving faster than the speed of light in the medium, but he confirms an algebraic falloff of the magnetic field with distance for lower velocities. However, he arrives at conclusions that are at variance with those found by the perturbation analyses of Refs. [1–3]. Jones finds the penetration results of Ref. [1] only as the zero-velocity limit of his analysis and suggests that the magnetic field in the conductor actually depends upon the inverse square of the conductivity, giving no magnetic field inside the conductor in the limit of perfect (infinite) conductivity. This is in contrast to the results reported in Refs. [1–3], and those to be reported in the present manuscript, which claim that the magnetic field is independent of the wall conductivity. It may be noteworthy that these results involving independence of wall conductivity allow superposition to obtain the steady current limit, which is an exact result of Maxwell's equations; this limit indeed fits the known experimental situation where a magnetic velocity field from a steady current penetrates a good conductor with $\mu=1$ as though the conductor were not present. Jones's Eq. (10) does not give a steady current result for the magnetic field that is independent of conductivity. Also, Jones reports recovery of the appropriate result of Ref. [1] only in the ''zero-velocity'' limit holding other quantities fixed. The actual range of validity of this limit is not suggested.

The original calculation [1] of 1974 was undertaken in connection with possible explanations of the Aharonov-Bohm effect. Experimentalists had accepted the idea that a conduction layer surrounding the solenoid in the effect would eliminate the electric and magnetic fields of the passing charge and so make untenable an explanation for the effect based upon a classical electromagnetic interaction between the passing charge and the solenoid [6]. Indeed, so poorly known is the velocity field penetration that this erroneous argument is repeated exactly in the 1985 review article on the Aharonov-Bohm effect written by Olariu and Popescu [7].

The velocity field penetration problem is also of interest in connection with experiments testing the weak equivalence principle for antimatter [8]. Experiments with antiprotons, negative hydrogen ions, positrons, and electrons under the influence of the earth's gravitational field involve charged particles moving parallel to conducting surfaces. The dragging forces on the external charges associated with Joule heating by currents caused in the walls of the drift tubes must be accounted for experimentally.

C. Present calculation

In the present calculation, we wish to generalize the discussion of Refs. [1,2]. We will give a nonrelativistic perturbation analysis for a point charge moving parallel to a conducting wall of finite thickness. In the limit of an infinitely thick wall, the results of Ref. [1] are recovered, and in the limit of a thin perfectly conducting wall, the results of Ref. [2]. The situation of finite conductivity and finite wall thickness is the natural one when trying to understand the applicability of these ideas to experimental situations. It seems to be of considerable interest that the magnetic velocity fields penetrating a wall of good conductivity have a universal character that is independent of the conductivity, thickness, or detailed placement of the wall.

II. CALCULATION OF THE PENETRATION OF THE VELOCITY FIELDS OF A POINT CHARGE

A. Power series solution of Maxwell's equations

The analysis here for a conducting wall of finite thickness follows a pattern analogous to that used in Ref. [1]. We expect to find a solution of Maxwell's equations as a power series in the velocity parameter $\beta=v/c$ of the charged particle moving parallel to a uniform conducting wall.

The wall here has thickness l, extending between the planes $z=-l$ and $z=0$, while the charge e is located at the position $\xi_d=\mathbf{i}vt+\mathbf{k}d$ a distance d from the wall. In the static limit $\beta=0$, there is a negative surface charge $\sigma^{(0)}_{z=0}$ on the front surface $z=0$ next to the charge e; this surface charge $\sigma^{(0)}_{z=0}$ gives rise to an electric field, which for $z<0$ exactly cancels the electrostatic field of the point charge e. At low velocities we expect small modifications of the electrostatic situation; we expect new surface charges $\sigma^{(1)}_{z=0}$, $\sigma^{(1)}_{z=-l}$ of first order in the parameter β on the front and back surfaces of the conducting wall. These first-order corrections to the surface charge give rise to first-order corrections $\mathbf{E}^{(1)}$ to the electrostatic fields. The new fields in turn produce currents $\mathbf{J}^{(1)}=\mathbf{E}^{(1)}/\eta$ in the conducting wall of resistively η, which then cause first-order magnetic fields $\mathbf{B}^{(1)}$. For simplicity we will take the conducting wall to have unit dielectric constant $\epsilon=1$ and relative permeability $\mu=1$, the same as a vacuum.

It is natural to use the Coulomb gauge for the quasistatic analysis envisioned here. Then the electrostatic potential Φ is given as an instantaneous integral over the charges

$$\Phi(\mathbf{r},t)=\int\frac{\rho(\mathbf{r}',t)}{|\mathbf{r}-\mathbf{r}'|}d^3r', \tag{1}$$

and the vector potential $\mathbf{A}$ is an integral over the transverse current at the retarded time

$$\mathbf{A}(\mathbf{r},t)=\frac{1}{c}\int\frac{J^{\perp}(\mathbf{r}',t_{\mathrm{ret}})}{|\mathbf{r}-\mathbf{r}'|}d^3r'. \tag{2}$$

Since the current $\mathbf{J}^{\perp}$ is already first order in the velocity parameter β, we can ignore the retardation when evaluating the vector potential $\mathbf{A}$ through first order in β.

A further simplification is possible because we are considering a configuration that moves with constant velocity $\mathbf{v}=\mathbf{i}v$ in the x direction parallel to the wall surface. Thus all functions take the form $f(\mathbf{r},t)=f(x-vt,y,z)$ and therefore all partial time derivatives can be converted to spatial derivatives multiplied by a factor of β,

$$\frac{\partial}{\partial t}f(\mathbf{r},t)=-c\beta\frac{\partial}{\partial x}f(\mathbf{r},t). \tag{3}$$

But then in the Coulomb gauge through first order in β, the vector potential $\mathbf{A}$, which is already first order in β, does not enter the determination of the electric field,

$$\mathbf{E}(\mathbf{r},t)=-\nabla\Phi-\frac{1}{c}\frac{\partial\mathbf{A}}{\partial t}\cong-\nabla\int\frac{\rho(\mathbf{r}',t)}{|\mathbf{r}-\mathbf{r}'|}d^3r'+O(\beta^2). \tag{4}$$

Also, for any gauge through first order in β,

$$\mathbf{B}(\mathbf{r},t)=\nabla\times\mathbf{A}\cong\nabla\times\frac{1}{c}\int\frac{\mathbf{J}(\mathbf{r}',t)}{|\mathbf{r}-\mathbf{r}'|}d^3r'+O(\beta^2). \tag{5}$$

Thus through order β, the electric field can be evaluated from electrostatic theory using the charge density through order β, and the magnetic field can be evaluated using the Biot-Savart integral over the currents arising from these electric fields in the conducting wall, as well as from the charge e.

B. Volume currents and surface charges

In the volume of the conductor, the continuity equation for electric charge becomes

$$0=\frac{\partial\rho}{\partial t}+\nabla\cdot\mathbf{J}=\frac{\partial\rho}{\partial t}+\nabla\cdot(\mathbf{E}/\eta)=\frac{\partial\rho}{\partial t}+\frac{4\pi\rho}{\eta},\qquad(6)$$

so that charges in the volume of the conductor decrease exponentially in time. Hence no charge is expected in the volume of the conductor. On the other hand, the continuity equation relates the surface charges to the normal component of the current **J** in the conductor by

$$0=\frac{\partial}{\partial t}\sigma_{z=0}-J_z(z=0_-)=-c\beta\frac{\partial}{\partial x}\sigma_{z=0}-\frac{1}{\eta}E_z(z=0_-)\qquad(7)$$

and

$$0=\frac{\partial}{\partial t}\sigma_{z=-l}-J_z(z=-l_+)$$
$$=-c\beta\frac{\partial}{\partial x}\sigma_{z=-l}-\frac{1}{\eta}E_z(z=-l_+),\qquad(8)$$

where $z=0_-$ and $z=-l_+$ are just inside the surfaces of the conductor at $z=0$ and $z=-l$, respectively. Now the zero-order terms in β correspond to the familiar electrostatic situation where there is a surface charge σ_e on the front surface of the conductor and none on the back,

$$\sigma^{(0)}_{z=0}=\sigma_e(x,y;\xi_x,d)=\frac{-ed}{2\pi[(x-\xi_x)^2+y^2+d^2]^{3/2}}\qquad(9)$$

and

$$\sigma^{(0)}_{z=-l}=0.\qquad(10)$$

Then from Eqs. (7) and (9), it follows that the electric field just inside the front surface of the conductor at $z=0_-$ is first order in β,

$$E^{(1)}_z(z=0_-)=-c\eta\beta\frac{\partial}{\partial x}\sigma^{(0)}_{z=0}.\qquad(11)$$

On the other hand, Eqs. (8) and (10) indicate that the electric field just inside the back surface of the conductor must vanish through first order in β,

$$E^{(1)}_z(z=-l_+)=0.\qquad(12)$$

C. Electric fields from image charges

The determination of the electric field inside the conductor through first order in β corresponds exactly to solving the electrostatic problem $\nabla^2\Phi=0$ in the region $-l<z<0$ with Neumann boundary conditions given by Eqs. (11) and (12). It is natural to try to construct a Green function for this problem by the repeated reflection of point charge images in the two planes. This gives the formal solution [9] for the electrostatic potential,

$$\Phi(\mathbf{r})=\frac{1}{4\pi}\int\left[\frac{\partial\Phi}{\partial z'}G_N(\mathbf{r},\mathbf{r}')\right]_{z'=0}d\sigma',\qquad(13)$$

$$G_N(\mathbf{r},\mathbf{r}')=\sum_{n=-\infty}^{\infty}\left(\frac{1}{|\mathbf{r}-(\mathbf{r}'+\mathbf{k}2nl)|}+\frac{1}{|\mathbf{r}-(R\mathbf{r}'+\mathbf{k}2nl)|}\right),\qquad(14)$$

where $R\mathbf{r}'=\mathbf{i}x'+\mathbf{j}y'-\mathbf{k}z'$ corresponds to the reflection of $\mathbf{r}=\mathbf{i}x'+\mathbf{j}y'+\mathbf{k}z'$ in the plane $z=0$. Actually, the series in (14) diverges. However, the electric field $\mathbf{E}=-\nabla\Phi$ obtained by differentiating term by term involves a convergent series and is finite.

Although we will not use Eqs. (13) and (14) directly, the technique of images that they suggest indeed allows a solution to our problem. We start by ignoring the surface at $z=-l$, assuming it were infinitely far away. The zero-order (electrostatic) solution corresponds to the surface charge $\sigma^{(0)}_{z=0}$ of Eq. (9), which gives rise to electrostatic fields $\mathbf{E}_{-e}(\mathbf{r},\xi_{\pm d})$, which appear to come from a charge $-e$ located at the point $\boldsymbol{\xi}_{+d}=\mathbf{i}\xi_x+\mathbf{k}d$ or $\boldsymbol{\xi}_{-d}=\mathbf{i}\xi_x-\mathbf{k}d$, which is a distance d on the other side of the plane $z=0$ from the field point $\mathbf{r}$. The first-order electric field in (11) arises from the first-order corrections to the surface charges. The surface charge causes electric fields on either side of the wall, which are related by reflection through the wall. Thus the electric field $\mathbf{E}^{(1)(0)}(\mathbf{r},t)$ would arise solely from the surface charge $\sigma^{(1)(0)}_{z=0}$. Here the upper zero is a notation that we will need later and refers to the number of reflections through the plane $z=-l$. Then from Gauss's law and Eq. (11),

$$\sigma^{(1)(0)}_{z=0}=\frac{1}{4\pi}(E^{(1)(0)}_z(z=0_+)-E^{(1)(0)}_z(z=0_-))$$
$$=-\frac{1}{2\pi}E^{(1)(0)}_z(z=0_-)=\frac{c\eta\beta}{2\pi}\frac{\partial}{\partial x}\sigma^{(0)}_{z=0},\qquad(15)$$

since under reflection through the plane $z=0$, the z component of the electric field arising from a surface charge changes sign,

$$E^{(1)(0)}_z(z=0_+)=-E^{(1)(0)}_z(z=0_-).\qquad(16)$$

The electric field associated with the surface charge $\sigma^{(1)(0)}_{z=0}$ follows from Eqs. (4) and (9):

$$\mathbf{E}^{(1)(0)}(\mathbf{r},t)=\int\left[\frac{\sigma^{(1)(0)}_{z=0}(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}\right]_{z'=0}dx'dy'=\frac{-c\eta\beta}{2\pi}\frac{\partial}{\partial\xi_x}\int\left[\frac{\sigma_e(x',y';\xi_x,d)(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}\right]_{z'=0}dx'dy'$$
$$=\begin{cases}\dfrac{c\eta\beta}{2\pi}\dfrac{\partial}{\partial x}\mathbf{E}_{-e}(\mathbf{r},\boldsymbol{\xi}_d), & z<0\\[2ex] \dfrac{c\eta\beta}{2\pi}\dfrac{\partial}{\partial x}\mathbf{E}_{-e}(\mathbf{r},\boldsymbol{\xi}_{-d}), & z>0,\end{cases}\qquad(17)$$

where we have replaced the derivative $(\partial/\partial x)\sigma^{(0)}_{z=0}$ by a derivative $-(\partial/\partial \xi_x)\sigma^{(0)}_{z=0}$ with respect to the ξ_x coordinate of the position $\boldsymbol{\xi}_d=\mathbf{i}\xi_x+\mathbf{k}d$ of the point charge, and then have taken the derivative outside the integral before switching back to an integral with respect to x. The electric field $\mathbf{E}_{-e}(\mathbf{r},\boldsymbol{\xi})$ is the static field of a point charge $-e$ located at $\boldsymbol{\xi}$,

$$\mathbf{E}_{-e}(\mathbf{r},\boldsymbol{\xi})=-e\frac{(\mathbf{r}-\boldsymbol{\xi})}{|\mathbf{r}-\boldsymbol{\xi}|^3}, \tag{18}$$

where here $\boldsymbol{\xi}_d=\mathbf{i}vt+\mathbf{k}d$ and $\boldsymbol{\xi}_{-d}=\mathbf{i}vt-\mathbf{k}d$. The derivative $\partial/\partial x$ in Eq. (17) changes the character of the field over to a dipole field. Thus the electric field in (17) for $z<0$ looks like that of an electric dipole

$$\mathbf{p}=\mathbf{i}\frac{ec\eta\beta}{2\pi} \tag{19}$$

located at the position of the point charge, which is passing the conducting wall. On the opposite side of the wall for $0<z$, the field in (17) looks as though it were caused by the same electric dipole but now located at the point $\boldsymbol{\xi}_{-d}$ found by reflection through the plane $z=0$.

Although the field $\mathbf{E}^{(1)(0)}(\mathbf{r},t)$ serves as a solution for the first-order electric field for the case of an infinitely thick wall $(l\to\infty)$, it violates the condition $E^{(1)}_z(z=-l_+)=0$ of Eq. (12) for the case of a wall of finite thickness. The condition (12) can be achieved by introducing an image electric dipole of the same magnitude (19) at the position $\boldsymbol{\xi}_{-d-2l}=\mathbf{i}vt+\mathbf{k}(-d-2l)$ corresponding to the reflection of the image dipole at $\boldsymbol{\xi}_d$ through the plane $z=-l$. From the analogy with (15) this corresponds to a surface charge

$$\sigma^{(1)(1)}_{z=-l}=\frac{c\eta\beta}{2\pi}\frac{\partial}{\partial x}\sigma_e(x,y;\xi_x,d+l), \tag{20}$$

where $\sigma_e(x,y;\xi_x,d+l)$ is the static surface charge on the plane due to a point charge e a distance $d+l$ away. This surface charge gives rise to an electric field

$$\mathbf{E}^{(1)(1)}(\mathbf{r},t)=\begin{cases}\dfrac{c\eta\beta}{2\pi}\dfrac{\partial}{\partial x}\mathbf{E}_{-e}(\mathbf{r},\boldsymbol{\xi}_d), & z<-l,\\[2ex] \dfrac{c\eta\beta}{2\pi}\dfrac{\partial}{\partial x}\mathbf{E}_{-e}(\mathbf{r},\boldsymbol{\xi}_{-d-2l}), & -l<z.\end{cases} \tag{21}$$

Now the combined surface charges $\sigma^{(1)(0)}_{z=0}$ and $\sigma^{(1)(1)}_{z=-l}$ give rise to electric fields that satisfy the condition (12). However, now the boundary condition (11) is no longer valid because of the surface charge (20) at $z=-l$ giving rise to an electric field at the surface $z=0$. The boundary condition (11) can be reestablished by taking the image of the surface charge (20) in the plane $z=0$, corresponding to a correction surface charge $\sigma^{(1)(2)}_{z=0}$ on the plane $z=0$,

$$\sigma^{(1)(2)}_{z=0}=\frac{c\eta\beta}{2\pi}\frac{\partial}{\partial x}\sigma_e(x,y;\xi_x,d+2l). \tag{22}$$

This new surface charge gives rise to new electric fields. It is clear that this repeated reflection through the two conducting planes will involve an infinite number of iterations. The resulting surface charge on the plane $z=0$ through first order in β is

$$\sigma_{e,\beta,z=0}=\sigma_e(x,y;\xi_x,d)+\frac{c\eta\beta}{2\pi}\frac{\partial}{\partial x}\left(\sum_{n=0}^{\infty}\sigma_e(x,y;\xi_x,d+2nl)\right) \tag{23}$$

and on the plane $z=-l$ is

$$\sigma_{e,\beta\ z=-l}=\frac{c\eta\beta}{2\pi}\frac{\partial}{\partial x}\left(\sum_{n=0}^{\infty}\sigma_e(x,y;\xi_x,d+[2n+1]l)\right). \tag{24}$$

The electric fields arising from these surface charges (23) and (24) and from the original point charge through first order in β are

$$\mathbf{E}_{e,\beta}(\mathbf{r},t)=\frac{c\eta\beta}{2\pi}\frac{\partial}{\partial x}\left\{2\sum_{n=0}^{\infty}\mathbf{E}_{-e}(\mathbf{r},\boldsymbol{\xi}_{d+2nl})\right\}\quad \text{for } z<-l, \tag{25}$$

$$\mathbf{E}_{e,\beta}(\mathbf{r},t)=\frac{c\eta\beta}{2\pi}\frac{\partial}{\partial x}\left\{\mathbf{E}_{-e}(\mathbf{r},\boldsymbol{\xi}_d)+\sum_{n=1}^{\infty}[\mathbf{E}_{-e}(\mathbf{r},\boldsymbol{\xi}_{d+2nl})+\mathbf{E}_{-e}(\mathbf{r},\boldsymbol{\xi}_{-d-2nl})]\right\}\quad \text{for } -l<z<0, \tag{26}$$

$$\mathbf{E}_{e,\beta}(\mathbf{r},t)=\mathbf{E}_e(\mathbf{r},\boldsymbol{\xi}_d)+\mathbf{E}_{-e}(\mathbf{r},\boldsymbol{\xi}_{-d})+\frac{c\eta\beta}{2\pi}\frac{\partial}{\partial x}\left\{\mathbf{E}_{-e}(\mathbf{r},\boldsymbol{\xi}_{-d})+2\sum_{n=1}^{\infty}\mathbf{E}_{-e}(\mathbf{r},\boldsymbol{\xi}_{-d-2nl})\right\}\quad \text{for } 0<z, \tag{27}$$

where $\mathbf{E}_e(\mathbf{r},\boldsymbol{\xi}_d)$ is the electrostatic field of a point charge e located at $\boldsymbol{\xi}_d$.

There are a series of checks we can make on our work. First, we note that for the infinitely thick conducting wall $l\to\infty$ the electric field expressions $\mathbf{E}_{-e}(\mathbf{r},\boldsymbol{\xi}_{d+2nl})$ and $\mathbf{E}_{-e}(\mathbf{r},\boldsymbol{\xi}_{-d-2nl})$ vanish so that we recover the results of Ref. [1], Eqs. (30)–(32). Next if we calculated the discontinuity in the normal components of the electric fields at the interface $z=0$ and $-l$, we find the surface charge densities given in (23) and (24). Finally, the fields inside the conductor at $z=0_-$ and $-l_+$ meet the boundary conditions (11) and (12).

D. Electric currents and the magnetic fields

The electric field inside the conducting wall gives rise to currents $\mathbf{J}_c$ according to

$$\mathbf{J}=\mathbf{E}/\eta, \tag{28}$$

so that

$$\mathbf{J}_c(\mathbf{r},t)=\frac{c\beta}{2\pi}\frac{\partial}{\partial x}\left\{\mathbf{E}_{-e}(\mathbf{r},\boldsymbol{\xi}_d)+\sum_{n=1}^{\infty}[\mathbf{E}_{-e}(\mathbf{r},\xi_{d+2nl})+\mathbf{E}_{-e}(\mathbf{r},\boldsymbol{\xi}_{-d-2nl})]\right\} \tag{29}$$

for $-l<z<0$. These currents are independent of the conductivity $1/\eta$ of the wall.

The magnetic fields for our situation arise from these currents in the wall together with the current $\mathbf{J}_e$ of the moving point charge

$$\mathbf{J}_e(\mathbf{r},t)=\mathbf{i}ec\beta\delta^3(\mathbf{r}-\boldsymbol{\xi}_d). \tag{30}$$

Thus the associated magnetic fields through first order in β are those following from the Biot-Savart expression

$$\begin{aligned}\mathbf{B}_{e,\beta}(\mathbf{r},t)&\cong\frac{1}{c}\int\frac{[\mathbf{J}_e(\mathbf{r}',t)+\mathbf{J}_c(\mathbf{r}',t)]\times(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}d^3r'\\&=\frac{e\beta\mathbf{i}\times(\mathbf{r}-\boldsymbol{\xi}_d)}{|\mathbf{r}-\boldsymbol{\xi}_d|^3}\\&\quad+\frac{1}{c}\int dx'\int dy'\int_{z'=-l}^{z'=0}dz'\frac{\mathbf{J}_c(\mathbf{r}',t)\times(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}.\end{aligned} \tag{31}$$

E. Magnetic fields and Maxwell's curl equations

Although it may be feasible to evaluate the integral in (31) directly, it is also possible to find the magnetic field by recognizing familiar solutions to Maxwell's equations. Since the integrations do not seem elementary, we will follow the second route in a manner analogous to that followed in Ref. [1].

In the region $z<-l$ behind the conducting wall, the curl equation for the magnetic field gives

$$\nabla\times\mathbf{B}=\frac{4\pi}{c}\mathbf{J}+\frac{1}{c}\frac{\partial}{\partial t}\mathbf{E}\cong\mathbf{0}. \tag{32}$$

This occurs since there are no currents $\mathbf{J}$ in this region of empty space and since the term $(1/c)\partial\mathbf{E}/\partial t=-\beta\partial\mathbf{E}/\partial x$ must be second order in β from Eq. (25). It follows from Eq. (32) that the magnetic field in this region is given to first order in β by the gradient of a scalar function

$$\mathbf{B}_{e,\beta}=-\nabla\phi\quad\text{for}\ \ z<-l. \tag{33}$$

In the region $-l<z<0$ inside the conducting wall, the curl equation for the magnetic field receives a contribution from the currents in (29),

$$\begin{aligned}\nabla\times\mathbf{B}&=\frac{4\pi}{c}\mathbf{J}_c+O(\beta^2)\\&\cong\frac{4\pi}{c}\frac{c\beta}{2\pi}\frac{\partial}{\partial x}\left\{\mathbf{E}_{-e}(\mathbf{r},\boldsymbol{\xi}_d)+\sum_{n=1}^{\infty}[\mathbf{E}_{-e}(\mathbf{r},\boldsymbol{\xi}_{d+2nl})\right.\\&\quad\left.+\mathbf{E}_{-e}(\mathbf{r},\boldsymbol{\xi}_{-d-2nl})]\right\}.\end{aligned} \tag{34}$$

Now we can satisfy this equation by writing

$$\begin{aligned}\mathbf{B}_{e,\beta}=\boldsymbol{\beta}\times\left\{2\left(\mathbf{E}_e(\mathbf{r},\boldsymbol{\xi}_d)+\sum_{n=1}^{\infty}[\mathbf{E}_e(\mathbf{r},\boldsymbol{\xi}_{d+2nl})\right.\right.\\\left.\left.+\mathbf{E}_e(\mathbf{r},\boldsymbol{\xi}_{-d-2nl})]\right)\right\}-\nabla\phi,\end{aligned} \tag{35}$$

since

$$\nabla\times(2\boldsymbol{\beta}\times\mathbf{E}_e)=2\boldsymbol{\beta}\nabla\cdot\mathbf{E}_e-2\boldsymbol{\beta}\cdot\nabla\mathbf{E}_e=2\beta\frac{\partial}{\partial x}\mathbf{E}_{-e}; \tag{36}$$

we have $\nabla\cdot\mathbf{E}_e=0$ in the region when the source point $\boldsymbol{\xi}$ is outside the region.

Finally, in the region $0<z$ in front of the conducting wall, the curl equation for the magnetic field takes the form

$$\begin{aligned}\nabla\times\mathbf{B}&=\frac{4\pi}{c}e\mathbf{v}\delta^3(\mathbf{r}-\boldsymbol{\xi}_d)+\frac{1}{c}\frac{\partial}{\partial t}\mathbf{E}\\&=4\pi e\beta\delta^3(\mathbf{r}-\boldsymbol{\xi}_d)-\beta\frac{\partial}{\partial x}([\mathbf{E}_e(\mathbf{r},\boldsymbol{\xi}_d)\\&\quad+\mathbf{E}_{-e}(\mathbf{r},\boldsymbol{\xi}_{-d})]+O(\beta^2).\end{aligned} \tag{37}$$

This has the solution to first order in β,

$$\mathbf{B}_{e,\beta}(\mathbf{r},t)=\boldsymbol{\beta}\times\{\mathbf{E}_e(\mathbf{r},\boldsymbol{\xi}_d)+\mathbf{E}_{-e}(\mathbf{r},\boldsymbol{\xi}_{-d})\}-\nabla\phi. \tag{38}$$

Maxwell's curl equation for the magnetic field is satisfied in all of space by Eqs. (33), (35), and (38), except possibly at the boundaries $z=0$, $z=-l$. The equation holds for all space if we can show that the tangential components of $\mathbf{B}$ are continuous at the two boundaries. The function $-\nabla\phi$ has continuous tangential components, provided all the sources for ϕ are confined to the planes $z=0$, $z=-l$. All the other terms for the magnetic field in (33), (35), and (38) involve expressions of the form $\boldsymbol{\beta}\times\mathscr{E}(\mathbf{r},t)$. Since $\boldsymbol{\beta}=\mathbf{i}\beta$, we need to check only that the z components of the expressions $\mathscr{E}(\mathbf{r},t)$ are continuous across the boundaries. However, this indeed holds from the relationship between the expressions for $\mathbf{B}$ and the boundary condition (11) and (12). Thus the z component in the curly brackets in (35)

$$\left\{2\left(\mathbf{E}_e(\mathbf{r},\boldsymbol{\xi}_d)+\sum_{n=1}^{\infty}[\mathbf{E}_e(\mathbf{r},\boldsymbol{\xi}_{d+2nl})+\mathbf{E}_e(\mathbf{r},\boldsymbol{\xi}_{-d-2nl})]\right)\right\} \tag{39}$$

vanishes at $z=-l$, whereas at $z=0$ this bracketed expression has the same z component as $2\mathbf{E}_e(\mathbf{r},\boldsymbol{\xi}_d)$, which agrees with the z component of the curly bracket in Eq. (38),

$$\{\mathbf{E}_e(\mathbf{r},\boldsymbol{\xi}_d)+\mathbf{E}_{-e}(\mathbf{r},\boldsymbol{\xi}_{-d})\}. \tag{40}$$

F. Magnetic fields and Maxwell's divergence equations

Since the Maxwell curl equation for $\mathbf{B}$ is satisfied in all space through first order in β, we need only determine the scalar function ϕ so as to satisfy Maxwell's divergence equation in order to arrive at a complete solution. The requirement $\nabla\cdot\mathbf{B}=\mathbf{0}$ in all space takes the form from (33), (35), and (38),

$$\nabla\cdot(\boldsymbol{\beta}\times\mathcal{E}-\nabla\phi)=0) \tag{41}$$

or

$$\nabla^2\phi=\nabla\cdot(\boldsymbol{\beta}\times\mathcal{E}), \tag{42}$$

where $\mathcal{E}$ is a sum of point charge electrostatic fields. However, away from the boundaries, the source term in Eq. (42) becomes

$$\nabla\cdot(\boldsymbol{\beta}\times\mathcal{E})=\mathcal{E}\cdot(\nabla\times\boldsymbol{\beta})-\boldsymbol{\beta}\cdot(\nabla\times\mathcal{E})=\mathbf{0}, \tag{43}$$

since $\boldsymbol{\beta}$ is a constant and since $\nabla\times\mathcal{E}=\mathbf{0}$ for an electrostatic field. Therefore from Eq. (42), we see that ϕ must satisfy Laplace's equation $\nabla^2\phi=0$ except possibly at the boundaries $z=0$ and $z=l$.

At these boundaries, Maxwell's equation $\nabla\cdot\mathbf{B}=\mathbf{0}$ becomes the condition that the normal component of $\mathbf{B}$ is continuous. Thus from Eqs. (33), (35), and (38), we require at $z=0$,

$$\mathbf{k}\cdot[\boldsymbol{\beta}\times\{\mathbf{E}_e(\mathbf{r},\boldsymbol{\xi}_d)+\mathbf{E}_{-e}(\mathbf{r},\boldsymbol{\xi}_{-d})\}-\nabla\phi]_{z=0_+}=\mathbf{k}\cdot\left[\boldsymbol{\beta}\times\left\{2\left(\mathbf{E}_e(\mathbf{r},\boldsymbol{\xi}_d)+\sum_{n=1}^{\infty}[\mathbf{E}_e(\mathbf{r},\boldsymbol{\xi}_{d+2nl})+\mathbf{E}_e(\mathbf{r},\boldsymbol{\xi}_{-d-2nl})]\right)\right\}-\nabla\phi\right]_{z=0_-}, \tag{44}$$

and at $z=-l$,

$$\mathbf{k}\cdot\left[\beta\times\left\{2\left(\mathbf{E}_e(\mathbf{r},\boldsymbol{\xi}_d)+\sum_{n=1}^{\infty}[\mathbf{E}_e(\mathbf{r},\boldsymbol{\xi}_{d+2nl})+\mathbf{E}_e(\mathbf{r},\boldsymbol{\xi}_{-d-2nl})]\right)\right\}-\nabla\phi\right]_{z=-l_+}=\mathbf{k}\cdot[-\nabla\phi]_{z=-l_-}. \tag{45}$$

These expressions can be simplified by noting $\boldsymbol{\beta}=\mathbf{i}\beta$ and using

$$\mathbf{k}\cdot(\boldsymbol{\beta}\times\mathcal{E})=(\mathbf{k}\times\boldsymbol{\beta})\cdot\mathcal{E}=\beta\mathcal{E}_y. \tag{46}$$

Then we find that Eqs. (44) and (45) become

$$-\left.\frac{\partial\phi}{\partial z}\right|_{z=0_+}+\left.\frac{\partial\phi}{\partial z}\right|_{z=0_-}=\frac{2e\beta y}{[(x-\xi_x)^2+y^2+d^2]^{3/2}}+\sum_{n=1}^{\infty}\frac{4e\beta y}{[(x-\xi_x)^2+y^2+(d+2nl)^2]^{3/2}} \tag{47}$$

and

$$-\left.\frac{\partial\phi}{\partial z}\right|_{z=-l_+}+\left.\frac{\partial\phi}{\partial z}\right|_{z=-l_-}=\sum_{n=1}^{\infty}\frac{-4e\beta y}{\{(x-\xi_x)^2+y^2+(d+[2n-1]l)^2\}^{3/2}}. \tag{48}$$

The determination of ϕ is analogous to solving an electrostatic problem, with the surface charges $\sigma_{(47)}$ and $\sigma_{(48)}$ implied by Eqs. (47) and (48). The function $-\nabla\phi$ is an integral over the surface charges

$$-\nabla\phi(\mathbf{r},t)=\int dx'\int dy'\frac{\sigma_{(47)}[\mathbf{i}(x-x')+\mathbf{j}(y-y')+\mathbf{k}z]}{[(x-x')^2+(y-y')^2+z^2]^{3/2}}+\int dx'\int dy'\frac{\sigma_{(48)}[\mathbf{i}(x-x')+\mathbf{j}(y-y')+\mathbf{k}(z+l)]}{[(x-x')^2+(y-y')^2+(z-z')^2]^{3/2}}. \tag{49}$$

Although it may be possible to carry out the integrals in (49) directly, it is also feasible [10] to solve for $-\nabla\phi$ by recognizing the surface charges in (47) and (48) as being related to the images of certain line charges stretching to spatial infinity from the image points $\mathbf{i}\xi_x+\mathbf{k}[\pm(d+2nl)]$ on either side of the planes $z=0$ and $z=l$.

The electric field $\mathbf{E}_\lambda(\mathbf{r};\boldsymbol{\xi}_d,+\infty)$ of a line charge λ per unit length beginning at $\boldsymbol{\xi}_d=\mathbf{i}\xi_x+\mathbf{k}d$ and running to spatial infinity parallel to the $+z$ axis is given by

$$\frac{1}{\lambda}\mathbf{E}_\lambda(\mathbf{r};\boldsymbol{\xi}_d,+\infty)=\frac{[\mathbf{i}(x-\xi_x)+\mathbf{j}y](z-d)-\mathbf{k}[(x-\xi_x)^2+y^2]}{[(x-\xi_x)^2+y^2][(x-\xi_x)^2+y^2+(z-d)^2]^{1/2}}+\frac{\mathbf{i}(x-\xi_x)+\mathbf{j}y}{(x-\xi_x)^2+y^2}. \tag{50}$$

The electric field $\mathbf{E}_\lambda(\mathbf{r};\boldsymbol{\xi}_{-d},-\infty)$ of a line charge λ per unit length beginning at $\boldsymbol{\xi}_{-d}=\mathbf{i}\xi_x+\mathbf{k}(-d)$ and running to spatial infinity parallel to the $-z$ axis is found by reversing the signs of z and $\mathbf{k}$ in (50),

$$\frac{1}{\lambda}\mathbf{E}_\lambda(\mathbf{r};\xi_{-d},-\infty)=\frac{-[\mathbf{i}(x-\xi_x)+\mathbf{j}y](z+d)+\mathbf{k}[(x-\xi_x)^2+y^2]}{[(x-\xi_x)^2+y^2][(x-\xi_x)^2+y^2+(z+d)^2]^{1/2}}+\frac{\mathbf{i}(x-\xi_x)+\mathbf{j}y}{(x-\xi_x)^2+y^2}. \tag{51}$$

The electrostatic field

$$\mathbf{E}=\begin{cases} -\dfrac{\partial}{\partial y}\mathbf{E}_\lambda(\mathbf{r};\boldsymbol{\xi}_d,+\infty), & z<0 \\ -\dfrac{\partial}{\partial y}\mathbf{E}_\lambda(\mathbf{r};\boldsymbol{\xi}_{-d},-\infty), & 0<z \end{cases} \tag{52}$$

is generated by a surface charge

$$\sigma=\frac{1}{4\pi}[E_z(z=0_+)-E_z(z=0_-)]$$
$$=-\frac{1}{2\pi}\frac{\lambda y}{[(x-\xi_x)^2+y^2+d^2]^{3/2}}, \tag{53}$$

which is precisely of the form of the surface charge terms appearing in Eqs. (47) and (48). Accordingly, it is possible to read off the magnetic field by relating the field functions of the form (50) and (51) through the surface charge (53) to the discontinuities (47) and (48).

G. Results for the magnetic fields

In the region $z<-l$ behind the conducting wall, the magnetic field is

$$\mathbf{B}_{e,\beta}(\mathbf{r},t)=-e\beta\frac{\partial}{\partial y}\left(\frac{1}{\lambda}\mathbf{E}_\lambda(\mathbf{r};\boldsymbol{\xi}_d,+\infty)+2\sum_{n=1}^{\infty}\frac{1}{\lambda}\mathbf{E}_\lambda(\mathbf{r};\boldsymbol{\xi}_{d+2nl},+\infty)\right)$$
$$+e\beta\frac{\partial}{\partial y}\left(2\sum_{n=1}^{\infty}\frac{1}{\lambda}\mathbf{E}_\lambda(\mathbf{r};\boldsymbol{\xi}_{d+[2n-2]l},+\infty)\right)$$
$$=e\beta\frac{\partial}{\partial y}\left(\frac{1}{\lambda}\mathbf{E}_\lambda(\mathbf{r};\boldsymbol{\xi}_d,+\infty)\right),\quad z<-l, \tag{54}$$

where the first term involving $-e\beta$ comes from (47) and the second involving $+e\beta$ comes from (48), and the x component of $\boldsymbol{\xi}$ is understood as $\xi_x=vt$. In the region $-l<z<0$ inside the conducting wall, the magnetic field is

$$\mathbf{B}_{e,\beta}(\mathbf{r},t)=\boldsymbol{\beta}\times\left\{2\left(\mathbf{E}_e(\mathbf{r},\boldsymbol{\xi}_d)+\sum_{n=1}^{\infty}[\mathbf{E}_e(\mathbf{r},\boldsymbol{\xi}_{d+2nl})+\mathbf{E}_e(\mathbf{r},\boldsymbol{\xi}_{-d-2nl})]\right)\right\}$$
$$-e\beta\frac{\partial}{\partial y}\left(\frac{1}{\lambda}\mathbf{E}_\lambda(\mathbf{r};\boldsymbol{\xi}_d,+\infty)+2\sum_{n=1}^{\infty}\frac{1}{\lambda}\mathbf{E}_\lambda(\mathbf{r};\boldsymbol{\xi}_{d+2nl},+\infty)\right)+e\beta\frac{\partial}{\partial y}\left(2\sum_{n=1}^{\infty}\frac{1}{\lambda}\mathbf{E}_\lambda(\mathbf{r};\boldsymbol{\xi}_{-d-2nl},-\infty)\right)$$
$$=\beta\times\left\{2\left(\mathbf{E}_e(\mathbf{r},\boldsymbol{\xi}_d)+\sum_{n=1}^{\infty}[\mathbf{E}_e(\mathbf{r},\boldsymbol{\xi}_{d+2nl})+\mathbf{E}_e(\mathbf{r},\boldsymbol{\xi}_{-d-2nl})]\right)\right\}-e\beta\frac{\partial}{\partial y}\left(\frac{1}{\lambda}E_\lambda(\mathbf{r};\boldsymbol{\xi}_d,+\infty)\right)$$
$$-2e\beta\sum_{n=1}^{\infty}\frac{\partial}{\partial y}\left(\frac{1}{\lambda}\mathbf{E}_\lambda(\mathbf{r};\boldsymbol{\xi}_{d+2nl},+\infty)-\frac{1}{\lambda}\mathbf{E}_\lambda(\mathbf{r};\boldsymbol{\xi}_{-d-2nl},-\infty)\right),\quad -l<z<0 \tag{55}$$

where the expression has been rewritten to assure the convergence of the infinite series. In the region $0<z$ in front of the conducting wall, the magnetic field is

$$\mathbf{B}_{e,\beta}(\mathbf{r},t)=\boldsymbol{\beta}\times\{\mathbf{E}_e(\mathbf{r},\boldsymbol{\xi}_d)+\mathbf{E}_{-e}(\mathbf{r},\boldsymbol{\xi}_{-d})\}-e\beta\frac{\partial}{\partial y}\left\{\frac{1}{\lambda}\mathbf{E}_\lambda(\mathbf{r};\boldsymbol{\xi}_{-d},-\infty)+2\sum_{n=1}^{\infty}\frac{1}{\lambda}\mathbf{E}_\lambda(\mathbf{r};\boldsymbol{\xi}_{-d-2nl},-\infty)\right\}$$
$$+e\beta\frac{\partial}{\partial y}\left\{2\sum_{n=1}^{\infty}\frac{1}{\lambda}\mathbf{E}_\lambda(\mathbf{r};\boldsymbol{\xi}_{-d-2nl},-\infty)\right\}$$
$$=\boldsymbol{\beta}\times\{\mathbf{E}_e(\mathbf{r},\boldsymbol{\xi}_d)+\mathbf{E}_{-e}(\mathbf{r},\boldsymbol{\xi}_{-d})\}-e\beta\frac{\partial}{\partial y}\left\{\frac{1}{\lambda}\mathbf{E}_\lambda(\mathbf{r};\boldsymbol{\xi}_{-d},-\infty)\right\},\quad 0<z. \tag{56}$$

We can check our results (54), (55), and (56) for the magnetic field by verifying that they satisfy Maxwell's equations in all space. Also, in the limit in which the conducting wall becomes infinitely thick $l\to\infty$, the results go over to those of Ref. [1].

III. LIMIT OF A LINE CHARGE MOVING PARALLEL TO ITS LENGTH

A. Limit of a steady current

By adding the point charge solutions obtained above, one can obtain limiting configurations involving line charges and steady currents where the results correspond to familiar experimental situations. For example, the case of a sequence of charges moving one after the other suggests a steady current whose magnetic field is known to penetrate through a good conductor with $\mu=1$ as though the conductor were not present.

B. Electric fields of a moving line charge

A line charge moving parallel to its length can be obtained by starting with a point charge at $\boldsymbol{\xi}_d'=\mathbf{i}(vt+x')+\mathbf{k}d$ rather than at $\boldsymbol{\xi}_d=\mathbf{i}vt+\mathbf{k}d$, replacing the charge e by $\lambda dx'$

where λ gives the charge per unit length, and then integrating in x'. From Eq. (23), the surface charge on the front surface of the conductor becomes that appropriate for a line charge λ, while the first-order terms, which are odd in x, do not contribute in the integral. Since on the back surface the surface charge is entirely odd in x, the surface charge at $z=-l$ vanishes in this line charge limit.

Similarly, the electric field existing in the line charge limit has no contribution from the terms odd in x. Thus from Eqs. (25), (26), and (27), we see that the only electric fields arising in this case are from the static fields $\mathbf{E}_e(\mathbf{r},\boldsymbol{\xi}_d)+\mathbf{E}_{-e}(\mathbf{r},\boldsymbol{\xi}_{-d})$, so that we recover the static line charge limit with no electric field inside or behind the conductor.

C. Magnetic fields of a moving line charge

The magnetic field involves integrals of the following forms. Terms arising from the curl equation involve

$$\int_{-\infty}^{\infty} dx' \boldsymbol{\beta}\times\mathbf{E}_e(\mathbf{r},\boldsymbol{\xi}'_d)=e\beta\int_{-\infty}^{\infty}dx'\frac{-\mathbf{j}(z-d)+\mathbf{k}y}{[(x-x')^2+y^2+(z-d)^2]^{3/2}}=\left[\frac{e\beta[-\mathbf{j}(z-d)+\mathbf{k}y](x'-x)}{[y^2+(z-d)^2][(x-x')^2+y^2+(z-d)^2]^{1/2}}\right]_{x'=-\infty}^{x'=\infty}$$
$$=\frac{2e\beta[-\mathbf{j}(z-d)+\mathbf{k}y]}{y^2+(z-d)^2}. \tag{57}$$

Terms introduced from the divergence equation involve

$$\int_{-\infty}^{\infty}dx'\frac{\partial}{\partial y}\left(\frac{1}{\lambda}\mathbf{E}_\lambda(\mathbf{r};\boldsymbol{\xi}'_d,+\infty)\right)=\int_{-\infty}^{\infty}dx'\left\{\frac{\mathbf{k}y}{[(x-x')^2+y^2+(z-d)^2]^{3/2}}-\frac{2\mathbf{i}(x-x')y}{[(x-x')^2+y^2]^2}\right.$$
$$-\frac{\mathbf{i}(x-x')y(z-d)[3(x-x')^2+3y^2+2(z-d)^2]}{[(x-x')^2+y^2]^2[(x-x')^2+y^2+(z-d)^2]^{3/2}}$$
$$-\frac{\mathbf{j}(z-d)[-(x-x')^2+y^2]}{[(x-x')^2+y^2]^2[(x-x')^2+y^2+(z+d)^2]^{1/2}}$$
$$\left.-\frac{\mathbf{j}y^2(z-d)}{[(x-x')^2+y^2][(x-x')^2+y^2+(z-d)^2]^{3/2}}+\frac{\mathbf{j}[(x-x')^2-y^2]}{[(x-x')^2+y^2]^2}\right\}. \tag{58}$$

The terms in (58) odd in $x-x'$ all vanish by symmetry leaving

$$\int_{-\infty}^{\infty}\frac{\partial}{\partial y}\left(\frac{1}{\lambda}\mathbf{E}_\lambda(\mathbf{r};\boldsymbol{\xi}'^{,+\infty}_d)\right)dx'=\left\{\frac{\mathbf{k}y(x'-x)}{[y^2+(z+d)^2][(x-x')^2+y^2+(z-d)^2]^{1/2}}-\frac{\mathbf{j}(z-d)(x'-x)}{[(x-x')^2+y^2][(x-x')^2+y^2+(z-d)^2]^{1/2}}\right.$$
$$\left.-\frac{\mathbf{j}(z-d)(x'-x)}{[y^2+(z-d)^2][(x-x')^2+y^2+(z-d)^2]^{1/2}}-\frac{\mathbf{j}(x'-x)}{(x-x')^2+y^2}\right\}_{x'=-\infty}^{x'=\infty}$$
$$=\frac{-2\mathbf{j}(z-d)+2\mathbf{k}y}{y^2+(z-d)^2}. \tag{59}$$

Similarly, the integral

$$\int_{-\infty}^{\infty}\frac{\partial}{\partial y}\left(\frac{1}{\lambda}\mathbf{E}_\lambda\left(\mathbf{r};\boldsymbol{\xi}'_{-d},-\infty dx'=\frac{2\mathbf{j}(z+d)-2\mathbf{k}y}{y^2+(z+d)^2}\right.\right. \tag{60}$$

is related to (59) by reversing the signs of z and $\mathbf{k}$.

Then in each of the regions, we find the magnetic field integral

$$\int_{-\infty}^{\infty}\lambda dx'\frac{1}{e}\mathbf{B}_{e,\beta}(\mathbf{r},t)=\frac{2\lambda v}{c}\,\frac{-\mathbf{j}(z-d)+\mathbf{k}y}{y^2+(z-d)^2}, \tag{61}$$

which is precisely the magnetic field of a steady current $I=\lambda v$ of a line charge λ moving along its length with velocity $\mathbf{v}=\mathbf{i}v$. In the case of the region $z<-l$ behind the conducting wall where $\mathbf{B}_{e,\beta}$ is given by (54), the integral involves (59). In the case of the region $0<z$ in front of the conducting wall where $\mathbf{B}_{e,\beta}$ is given by (55), the integral involves a cancellation between

$$\int_{-\infty}^{\infty} dx' \boldsymbol{\beta}\times\mathbf{E}_e(\mathbf{r},\boldsymbol{\xi}'_{-d})$$

and

$$\int_{-\infty}^{\infty} dx' \left(-e\beta \frac{\partial}{\partial y} \left\{ \frac{1}{\lambda} \mathbf{E}_\lambda(\mathbf{r};\boldsymbol{\xi}'_{-d},-\infty) \right\} \right),$$

leaving the integral of the form (57). In the region $-l<z<0$ inside the conducting wall where $\mathbf{B}_{e,\beta}$ is given by (56), there is a similar cancellation of integrals:

$$\int_{-\infty}^{\infty} dx' \boldsymbol{\beta}\times 2\mathbf{E}_e(\mathbf{r},\boldsymbol{\xi}'_{d+2nl}) \quad \text{cancels with} \int_{-\infty}^{\infty} dx' e\beta \frac{\partial}{\partial y}\left(2\frac{1}{\lambda}\mathbf{E}_\lambda(\mathbf{r};\boldsymbol{\xi}'_{d+2nl},+\infty)\right),$$

$$\int_{-\infty}^{\infty} dx' \boldsymbol{\beta}\times 2\mathbf{E}_e(\mathbf{r},\boldsymbol{\xi}'_{-d-2nl}) \quad \text{cancels with} \int_{-\infty}^{\infty} dx' (-e\beta) \frac{\partial}{\partial y}\left(2\frac{1}{\lambda}\mathbf{E}_\lambda(\mathbf{r};\boldsymbol{\xi}'_{-d-2nl},-\infty)\right),$$

and

$$\int_{-\infty}^{\infty} dx' \boldsymbol{\beta}\times 2\mathbf{E}_e(\mathbf{r},\boldsymbol{\xi}'_{d}) \quad \text{is half canceled by} \int_{-\infty}^{\infty} dx' (-e\beta) \frac{\partial}{\partial y}\left(\frac{1}{\lambda}\mathbf{E}_\lambda(\mathbf{r};\boldsymbol{\xi}'_{d},+\infty)\right),$$

leaving exactly the required result (61). Thus, in the limit of a steady current, we recover the familiarly observed result that the magnetic field penetrates through a conducting wall as though the wall were not present.

IV. DISCUSSION OF RESULTS

A. Retarding force on the passing charge

The charge passing the conducting wall gives rise to electric currents inside the wall. These dissipate energy in Joule heating. In order to satisfy the ideas of energy conservation contained in Maxwell's equations, there must be an electric retarding force on the passing charge. Indeed, the electric fields associated with the changes in surface charges beyond the electrostatic situation provide the required electric force.

The electric field acting on the passing charge removes energy from the charge, which must be associated with a loss of particle kinetic energy or with energy provided by an external force on the particle. The power of the electric field on the charged particle follows from Eq. (27) as

$$\begin{aligned} P_{em} &= \mathbf{F}_{em}\cdot\mathbf{v} = c\boldsymbol{\beta}\cdot e\mathbf{E}_{e,\beta}(\boldsymbol{\xi}_d,t) \\ &= ce\beta \frac{c\eta\beta}{2\pi}\frac{\partial}{\partial x}\left\{ \mathbf{E}_{-e}(\mathbf{r},\boldsymbol{\xi}_{-d}) + 2\sum_{n=1}^{\infty} \mathbf{E}_{-e}(\mathbf{r},\boldsymbol{\xi}_{-d-2nl})\right\}_{\mathbf{r}=\boldsymbol{\xi}_d} \\ &= -\frac{\eta c^2 e^2 \beta^2}{2\pi}\left[\frac{1}{[2d]^3} + 2\sum_{n=1}^{\infty}\frac{1}{[2d+2nl]^3}\right]. \end{aligned} \tag{62}$$

Indeed, the electric field is removing energy from the passing charge. We note here that for fixed η and β, the smallest retarding force occurs for an infinitely thick conducting wall where $l\to\infty$. As l decreases, the perturbation approximation puts a limitation on how small l can be.

B. The approximations

The solution for Maxwell's equations given here is not exact but rather is a perturbation approximation. The terms retained in Maxwell's equations correspond to a power-series expansion in the velocity ratio $\beta=v/c$ for the passing point charge. Thus we require the low-velocity condition

$$|\beta| \ll 1. \tag{63}$$

Next we regarded the surface charge correction $\sigma^{(1)}$ as small compared to the electrostatic surface charge $\sigma^{(0)}$,

$$|\sigma^{(1)}| \ll |\sigma^{(0)}|. \tag{64}$$

Comparing the expressions in Eqs. (9) and (15), we see that this condition will be satisfied, provided

$$\frac{c\eta\beta}{d} \ll 1. \tag{65}$$

This condition can always be achieved for fixed η and d, provided the particle velocity is made sufficiently small.

On the other hand, we may also regard this as a small-distance limit on the separation d of the point charge from the conducting wall when β is set equal to 1. Since for good conducting metals the resistivity η is of the order of 2×10^{-8} ohm-meters, which corresponds [11] to $\frac{2}{9}\times10^{-17}$ sec in Gaussian units, the value $c\eta$ becomes $\frac{2}{3}\times10^{-7}$ cm, so that d is limited by atomic dimensions.

Finally, the correction to the surface charge involves a sum over the images, so that we must require

$$\left| \frac{c\eta\beta}{2\pi}\frac{\partial}{\partial x}\left(\sum_{n=0}^{\infty} \sigma_e(x,y;\xi_{d+2nl})\right)\right| \ll |\sigma_e(x,y;\xi_d)|. \tag{66}$$

Thus, in addition to the condition (65), we must require that l be not much smaller than the limit on d. However, again atomic dimensions are involved for good metallic conductors.

C. Discussion of results for velocity fields

Our calculations show some results that are natural extensions of the conclusions found in Refs. [1–3], and some that seem surprising. The algebraic rather than exponential falloff of the electric and magnetic velocity fields inside and outside the conducting wall is consistent with earlier work. Thus the velocity fields penetrate even good conductors to an extent that is not anticipated by those looking to a skin-depth approximation.

Inside the conducting wall, the electric and magnetic fields depend upon the thickness of the wall. The magnetic field and the volume currents inside the wall are independent of the conductivity of the wall.

Noteworthy in the results is the fact that the magnetic field on the far side of a conducting wall is independent of the conductivity of the conducting wall, is independent of the thickness of the wall, and is independent of the placement of the wall. Furry [2] noted earlier that the penetrating magnetic field was independent of the relative placement of the wall for the special case of a thin perfectly conducting wall. Only the distance from the passing charge determines the character of the magnetic field once any intervening plane conducting wall is present. One notes that the magnetic velocity field beyond the conductor is not the same as that for a point charge in vacuum. The conducting wall does modify the magnetic field, but the modification is of a universal character independent of the conductivity, thickness, and placement of the wall, provided all the approximations (63)–(66) are valid.

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