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Analytic expressions for the stochastic amplitude equation for Taylor-Couette flow

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Expressions for the thermal noise strength in stochastic amplitude equations of quasi-one-dimensional hydrodynamic systems near a pattern-forming instability are given in a generally applicable form. The expressions can be evaluated for any system where the macroscopic equations of motion and the entropy production in terms of generalized forces and fluxes are known and the linear stability analysis can be performed. We apply the results mainly to the Taylor-Couette system (with corotating cylinders) and derive an approximate analytic expression for the noise strength of the amplitude equation near the first threshold. An analytical expression is given also for the mean kinetic energy of the velocity fluctuations. The analytic formulas are easy to evaluate and are of second order in the gap width; their deviation from numerical results is less than 2.5% for a radius ratio of 0.738 and all (co)rotation rates of the outer cylinder. By comparing the mean energy of the fluctuations with the equipartition theorem, we separate equilibrium from nonequilibrium effects.

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I. INTRODUCTION

Near primary instabilities of pattern-forming nonequilibrium systems, the intensity and the correlation lengths and times of thermal fluctuations of the field variables increase, similar to an equilibrium system near a continuous phase transition. Although the effect is very small (in equilibrium, the energy contained in the fluctuations has to be compared with k_BT, where k_B is the Boltzmann constant), it is experimentally accessible and has been measured directly in electrohydrodynamic convection [1,2] and recently in Rayleigh-Bénard convection (RBC) in gasses at elevated pressures [3] and in a narrow channel containing a binary mixture with a negative separation ratio [4]. In addition, fluctuations were measured indirectly, either with the help of a temporal control-parameter ramp [5,6] in RBC in a simple fluid or using noise amplification in the convectively instable regime in RBC in binary mixtures [7] and in Taylor-Couette flow (TCF) with through flow [8–10].

In all relevant systems, the size in at least one direction is so small that in this direction only one mode is slow and has large fluctuations near threshold; the systems are at most quasi-two-dimensional. If the width w in one of the remaining directions, say, y, becomes comparable to (or smaller than) the correlation length of the two-dimensional (2D) patterns, the system becomes quasi-one-dimensional, i.e., only one y mode is slow and has large fluctuations near threshold [11]. This is the case in the long and narrow cells used in the binary-mixture experiments, where $w/d = 2$ in Ref. [7] and $w/d = 0.5$ in Ref. [4]. In TCF, the transition to one dimension corresponds to considering only fluctuations of the azimuthal modes with the lowest threshold, which are the axisymmetric modes for not too large through flow [12,13].

In the theoretical work for RBC in simple fluids [14–16] and binary mixtures [17], and recently for electroconvection [18] and TCF [19,20], one always starts from the basic macroscopic equations that are supplemented with Langevin-noise terms with a noise strength determined from stochastic hydrodynamics [21]. Near threshold one can then derive stochastic amplitude equations with δ-correlated noise strength. Stochastic hydrodynamics in its original formulation by Landau and Lifshitz pertains to equilibrium fluctuations. In fact, all theoretical work is based on the assumption that the system, while being far from its global equilibrium, is nevertheless everywhere near local equilibrium.

Despite the smallness of thermal noise, it is crucial to understand the fundamental issue of how nonequilibrium fluctuations near a dissipative pattern-forming transition, or in general in extended nonlinear systems, are related to equilibrium fluctuations near phase transitions. Can the assumption of local equilibrium be verified? If so, is there an equivalent to the equipartition theorem? Especially puzzling is the fact that theory agrees reasonably with the direct experiments while in the experiments [5,6] using time ramps the fluctuation intensities are about four orders of magnitude larger and in the experiments using noise amplification in the convectively unstable regime the discrepancy is about two orders of magnitude. The latter experiments pertain all to quasi-one-dimensional systems while the theoretical work treated either 2D systems or 1D systems [15,17], where the fields have no variations in the direction y of the width. This corresponds to periodic lateral boundary conditions (BCs) and is equivalent to replacing $\delta(y-y')$ by $1/w$ in the δ-correlated noise strength of the 2D amplitude equation. While this is obviously true for TCF, the nonslip BCs of the velocities in the convection experiments [7,4] are expected to play an important role, especially because there w is of the order of d. Clearly a proper 1D treatment is necessary for a quantitative comparison.

The main purpose of this paper is to clarify in a specific system the relation between nonequilibrium fluctuations near a pattern-forming transition and fluctuations near equilibrium transitions by providing for the Taylor-Couette system analytic expressions for the noise strength in the amplitude equation and for the kinetic energy of the fluctuating Taylor vortices. As a by-product we obtain analytic expressions also for the threshold and the deterministic parameters of the amplitude equation and we start with a general expression for the noise strength that can be applied easily to the above convection experiments for physical (no-flux) BCs.

In Sec. II we give general expressions for the noise strength of amplitude equations for effectively one-dimensional systems in terms of the linear operators of the basic equations, the constitutive relations between the generalized forces and fluxes (making up the entropy production), and the eigenfunctions of the critical mode. For illustrative purposes we apply the expression for the noise strength to a model system (stochastic Swift-Hohenberg equation) and then to TCF with axisymmetric vortices. The latter result coincides with that obtained recently by Swift, Babcock, and Hohenberg (SBH) [19] and, with physically plausible assumptions, with that of Deissler [20].

Section III contains the main results. We give for axisymmetric Taylor vortices analytic expressions for the rotation rate at threshold of the inner cylinder (the control parameter) and for the deterministic and stochastic coefficients of the amplitude equation. These formulas, given in terms of the radius ratio R_1/R_2 of the cylinders and the (co)rotation Ω_2 of the outer cylinder, are easy to evaluate with a pocket calculator. They are of second order in the gap width and deviate by less than 2.5% from the results obtained numerically by SBH in the range $\Omega_2 \geq 0$ and $R_1/R_2 \geq 0.738$. The analytic approximations used by SBH exhibit errors up to about 20%. We use our expressions to separate out the different physical effects leading to noise in the TCF system and to derive an analytic formula for the kinetic energy contained in stationary fluctuations. In Sec. IV we compare the expression for the energy with that derived from the equipartition theorem and with the corresponding expressions for RBC and electrohydrodynamic convection (EHC). It is concluded that the energy of stationary fluctuations behaves in analogy to equilibrium systems if one macroscopic field relaxes much slower than all other fields. In contrast, if there is a coupling to a similarly slow relaxing field, the fluctuations and dissipations are no longer balanced.

II. STOCHASTIC AMPLITUDE EQUATIONS FOR QUASI-ONE-DIMENSIONAL SYSTEMS

A. General expressions

Stochastic hydrodynamics [21] relates thermal noise to fluctuations of the fields of the Onsager currents (stresses) $\boldsymbol{J}(\boldsymbol{r},t) = \langle \boldsymbol{J}(\boldsymbol{r},t)\rangle + \tilde{\boldsymbol{J}}(\boldsymbol{r},t)$, where $\langle\,\rangle$ is an ensemble average characterizing the deterministic, macroscopic field and the Gaussian distribution function of the fluctuations is characterized by

$$\langle\, \tilde{\boldsymbol{J}}(\boldsymbol{r},t)\tilde{\boldsymbol{J}}^T(\boldsymbol{r}',t')\,\rangle = k_B(\underline{\boldsymbol{M}} + \underline{\boldsymbol{M}}^T)\delta(\boldsymbol{r}-\boldsymbol{r}')\delta(t-t'). \tag{1}$$

The Onsager matrix $\underline{\boldsymbol{M}}$, given by the (linear) constitutive material equations, relates the currents to the forces, where the forces (strains) are defined by the condition that the sum of the currents times the forces are the local density of the entropy production [22]. Since $\underline{\boldsymbol{M}} + \underline{\boldsymbol{M}}^T$ pertains to the dissipative part of the material equations, Eq. (1) is called the fluctuation-dissipation theorem.

An example is dissipation due to viscosity. The corresponding Onsager currents and forces can be identified, respectively, as the dissipative part of the stress tensor $\underline{\boldsymbol{\sigma}}$ and the velocity gradients divided by the temperature $(\partial_k v_l + \partial_l v_k)/(2T)$. For isotropic, incompressible fluids, the Onsager matrix takes the form $M_{ij,kl} = T\rho_m\nu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$ and Eq. (1) reproduces the well-known Landau-Lifshitz noise terms [21]

$$\langle\, \tilde{\sigma}_{ij}(\boldsymbol{r},t)\tilde{\sigma}_{kl}(\boldsymbol{r}',t')\,\rangle = 2k_B T\rho_m\nu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})\;\delta(\boldsymbol{r}-\boldsymbol{r}')\delta(t-t'), \tag{2}$$

where ν is the kinematic viscosity and ρ_m the mass density.

We consider 1D systems that are infinite and translationally invariant in z and bounded, with a cross section C, in the complementary two transverse directions $r_\perp = (r_{\perp 1}, r_{\perp 2})$. The $r_\perp$ coordinates are allowed to be curvilinear, for instance $r_\perp = (r,\phi)$ in the Taylor system in cylindrical coordinates (r,ϕ,z). Further, we consider fluctuations below the threshold of a hydrodynamic instability where a linear description is valid and restrict ourselves to autonomous systems where the linear part of the basic macroscopic equations for the fluctuations $\boldsymbol{u}(\boldsymbol{r},t)$ can be written in the symbolic form

$$[\underline{\boldsymbol{S}}^R(\boldsymbol{\nabla},\boldsymbol{r}_\perp)\partial_t + \underline{\boldsymbol{L}}^R(\boldsymbol{\nabla},\boldsymbol{r}_\perp)]\boldsymbol{u}(\boldsymbol{r},t) = \underline{\boldsymbol{D}}(\boldsymbol{\nabla})\tilde{\boldsymbol{J}}(\boldsymbol{r},t). \tag{3}$$

The components u_α of $\boldsymbol{u}$ represent the deviations of the macroscopic fields (velocity components, pressure, temperature, etc.) from the unstructured state. The linear matrix-differential operators $\underline{\boldsymbol{S}}^R$ and $\underline{\boldsymbol{L}}^R$ depend on a control parameter R and, in general (for example, in TCF or non-Boussinesq RBC), on the transverse position $r_\perp$. The stochastic terms contain the fluctuations of the Onsager currents in the form $\underline{\boldsymbol{D}}\tilde{\boldsymbol{J}}$, where $\underline{\boldsymbol{D}}$ is typically proportional to a matrix-differential operator for conservation laws and a matrix for balance equations of nonconserved quantities such as the director field [18,23]. For the Navier-Stokes equations this term is $\boldsymbol{\nabla}\underline{\tilde{\boldsymbol{\sigma}}}$; see Eq. (10).

If the bifurcation to a pattern with critical wave vector k_c (in the infinite direction) and critical frequency ω_c is continuous and nondegenerate, the physical fields can be described near threshold by a slowly varying amplitude $A(z,t)$,

$$\boldsymbol{u}(\boldsymbol{r},t) = A(z,t)\boldsymbol{f}(r_\perp)e^{i(k_c z-\omega_c t)} + \text{c.c.} + (\text{higher-order terms}), \quad (4)$$

where $\boldsymbol{f}(r_\perp)e^{i(k_c x-\omega_c t)}$ is the eigenfunction of the critical mode. Below threshold $\epsilon := R/R_c - 1 < 0$, but neither too far ($-1 \ll \epsilon$) nor too close ($\epsilon < -\epsilon_{\rm NL}$, where $\epsilon_{\rm NL}$ is some positive quantity characteristic of nonlinearities "at" threshold [24] that can be set equal to zero for all practical purposes), the amplitude obeys the linear stochastic amplitude equation

$$\tau_0(\partial_t + v_g\partial_z)A(z,t) = (\epsilon + \beta^2\partial_z^2)A(z,t) + \tau_0\sqrt{Q}\,\Gamma(z,t), \quad (5)$$

where the correlation length $\xi_0 = \text{Re}\beta$. The deterministic coefficients $\tau_0^{-1} = \partial_\epsilon \text{Re}\lambda$, $v_g = \partial_k\omega$, and $\xi_0^2 = -\frac{1}{2}\tau_0\partial_k^2\lambda$ (τ_0 and ξ_0 can be complex) come from the expansion of the linear modal growth rate $\lambda(\epsilon,k) = \text{Re}\lambda(\epsilon,k) - i\omega(\epsilon,k)$ around the threshold values $\epsilon = 0$ and $k = k_c$ with $\lambda(0,k_c) = -i\omega_c$. The complex, Gaussian unit-noise source $\Gamma(z,t)$ satisfies $\langle\,\Gamma\,\rangle = \langle\,\Gamma\Gamma\,\rangle = \langle\,\Gamma^*\Gamma^*\,\rangle = 0$ and $\langle\,\Gamma^*(z,t)\Gamma(z',t')\,\rangle = \delta(z-z')\delta(t-t')$.

An adaptation to one dimension of the derivation of the noise intensity Q along the lines of Ref. [18] leads to

$$Q = \frac{1}{C}\frac{[\boldsymbol{f}^\dagger, \underline{\boldsymbol{O}}(\boldsymbol{\nabla}_\perp, ik_c, r_\perp)\boldsymbol{f}^\dagger]}{|[\boldsymbol{f}^\dagger, \underline{\boldsymbol{S}}^{R_c}(\boldsymbol{\nabla}_\perp, ik_c, r_\perp)\boldsymbol{f}]|^2}, \quad (6)$$

where the noise-correlation matrix $\underline{\boldsymbol{O}}(\boldsymbol{\nabla}_\perp, ik, r_\perp)$ of the stochastic forces $\underline{\boldsymbol{D}}(\boldsymbol{\nabla})\tilde{\boldsymbol{J}}$ in the basic equations for modes k is given by

$$\underline{\boldsymbol{O}}(\boldsymbol{\nabla}_\perp, ik, r_\perp) = k_B\underline{\boldsymbol{D}}(\boldsymbol{\nabla}_\perp, ik)(\underline{\boldsymbol{M}} + \underline{\boldsymbol{M}}^T)\underline{\boldsymbol{D}}^\dagger(\boldsymbol{\nabla}_\perp, ik). \quad (7)$$

The square brackets in (6) denote the scalar product

$$[\boldsymbol{\phi}, \boldsymbol{\psi}] := \frac{1}{C}\int_C d^2r_\perp\; \phi_\alpha^*(r_\perp)\psi_\alpha(r_\perp) \quad (8)$$

for vector fields $\boldsymbol{\phi}$ and $\boldsymbol{\psi}$ defined in the cross section C of the system (summation over doubly occurring indices is implied here and in the following). The Hermitian conjugate $\underline{\boldsymbol{D}}^\dagger$ of $\underline{\boldsymbol{D}}$ and all following Hermitian conjugates are defined with respect to this scalar product. The functions $\boldsymbol{f}(r_\perp)$ and $\boldsymbol{f}^\dagger(r_\perp)$ are the transverse parts of the eigenfunctions of the deterministic problem (3) at threshold and of its adjoint, respectively.

Equations (6) and (7) for the thermal noise of the amplitude equation (5) for 1D systems with nondegenerate bifurcations are the main results of this section and will be used throughout the rest of this paper. Boundary conditions and an eventual through flow are contained implicitly via the eigenfunctions and the specific form of $\underline{\boldsymbol{L}}^R$. Multiplying (6) with any scalar product $[f_\alpha, f_\beta]$ of the components of the transverse eigenfunctions, or linear combinations thereof, makes the right-hand side independent of the normalizations of both $\boldsymbol{f}$ and $\boldsymbol{f}^\dagger$ and also independent of dilations or contractions of the cross section (as long as the system remains quasi-one-dimensional). This means that cross-section-integrated fluctuation intensities, leading with (4) and $\langle\,|A|^2\rangle \propto Q$ to such expressions, are independent of the transverse size, as in equilibrium systems. An example is the line density of the mean kinetic energy of the velocity fluctuations $\partial_z\langle E\rangle = \rho_m/2\int_C d^2r_\perp\langle\,\boldsymbol{v}^2\rangle$, which is proportional to $[\boldsymbol{f}_v, \boldsymbol{f}_v]Q$ and will be calculated for TCF in Sec. III.

A trivial application of Eq. (6) consists of deriving the noise term of the amplitude equation from the stochastic Swift-Hohenberg equation

$$\tau_0^{\rm SH}\partial_t\psi(z,y,t) = [\epsilon - \tilde{\xi}_0^4(k_c^2 + \boldsymbol{\nabla}_2^2)^2]\psi + \tau_0^{\rm SH}\xi(z,y,t),$$

$$\langle\,\xi(z,y,t)\xi(z',y',t')\rangle = Q^{\rm SH}\delta(z-z')\delta(y-y')\delta(t-t') \quad (9)$$

as basic "macroscopic" equation for the scalar field ψ where $\boldsymbol{\nabla}_2^2 = \partial_z^2 + \partial_y^2$. Since there is only one basic field, the quantities $\boldsymbol{f}, \underline{\boldsymbol{L}}$, etc., have only one component. Assuming a width w in y with periodic BCs and the "transverse" mode $k_y = 0$, we have $C = w$, $\underline{\boldsymbol{O}} = (\tau_0^{\rm SH})^2$, $\underline{\boldsymbol{S}} = \tau_0^{\rm SH}$, $\boldsymbol{f} = \boldsymbol{f}^\dagger = 1$, and the scalar product $[\boldsymbol{\phi}, \boldsymbol{\psi}] = \phi^*\psi$, which yields $Q = Q^{\rm SH}/w$ for the amplitude equation (5). The amplitude is defined by (4), $\psi(z,y,t) = A(z,t)e^{ik_c z} + \text{c.c.}$ Furthermore, the growth rate $\lambda(\epsilon,k) = [\epsilon - \tilde{\xi}_0^4(k_c^2-k^2)^2]/\tau_0^{\rm SH}$ yields the well known relations for the deterministic coefficients $v_g = 0$, $\tau_0 = \tau_0^{\rm SH}$, and $\xi_0^2 = 4k_c^2\tilde{\xi}_0^4$.

B. Application to the Taylor-Couette system

Now we illustrate the above scheme in a nontrivial case and provide the starting point for the analytic calculations to be carried out in Sec. III by calculating the noise term in the amplitude equation of TCF. The expressions obtained are equivalent to the results recently published by SBH, but more suitable for the subsequent calculations.

The system consists of two concentric cylinders of inner and outer radii R_1 and R_2 and infinite length, rotating at angular frequencies Ω_1 and Ω_2, respectively. The basic equations are the NS equations for the fluid velocities and the pressure together with the incompressibility condition $\boldsymbol{\nabla}\cdot\boldsymbol{v} = 0$. In view of the mapping to RBC it is favorable to scale lengths by the gap width $d = R_2 - R_1$ and time by the radial viscosity-diffusion time d^2/ν. Linearization of the scaled stochastic NS equations around the basic Couette flow $U_\phi(r)\hat{\boldsymbol{\phi}}$ with the Couette profile $U_\phi(r)$ from [12] gives (without through flow)

$$\partial_t\boldsymbol{v} = 2\omega(r)\hat{\boldsymbol{r}}v_\phi + 2\frac{\eta^2\omega_1 - \omega_2}{1-\eta^2}\hat{\boldsymbol{\phi}}v_r + \boldsymbol{\nabla}^2\boldsymbol{v} - \boldsymbol{\nabla}p + \boldsymbol{\nabla}\underline{\tilde{\boldsymbol{\sigma}}}, \quad (10)$$

where $\boldsymbol{v}(\boldsymbol{r},t)$ is the deviation of the velocities from the basic flow, $\omega(r) = U_\phi(r)/r$ is the angular velocity of the basic flow, $\omega_1 = \Omega_1 d^2/\nu$ and $\omega_2 = \Omega_2 d^2/\nu$ are the dimensionless cylinder rotations, $\eta = R_1/R_2$ is the radius ratio, and $\hat{\boldsymbol{r}}$ and $\hat{\boldsymbol{\phi}}$ are, respectively, the unit vectors in the radial and azimuthal directions.

We consider a range of the control parameters (corotation), where the first instability with respect to the main control parameter ω_1 is axisymmetric and well separated from the instabilities of the other nonaxisymmetric modes [12,13]. In this case, the r component of the curl of the curl of (10) and the ϕ component of (10) in cylindrical coordinates (r, ϕ, z) represent a 2×2 system for v_r and v_ϕ,

$$-\partial_t\left(\boldsymbol{\nabla}^2 - \frac{1}{r^2}\right)v_r + \left(\boldsymbol{\nabla}^2 - \frac{1}{r^2}\right)^2 v_r + 2\omega(r)\partial_z^2 v_\phi$$

$$= [\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\boldsymbol{\nabla}\underline{\tilde{\boldsymbol{\sigma}}})]_r \equiv D_{1,jk}\tilde{\sigma}_{jk},$$

$$\partial_t v_\phi - 2\frac{\eta^2\omega_1 - \omega_2}{1-\eta^2}v_r - \left(\boldsymbol{\nabla}^2 - \frac{1}{r^2}\right)v_\phi$$

$$= [\boldsymbol{\nabla}\underline{\tilde{\boldsymbol{\sigma}}})]_\phi \equiv D_{2,jk}\tilde{\sigma}_{jk}, \quad (11)$$

where $\boldsymbol{\nabla}^2 = \partial_r^*\partial_r + \partial_z^2$ with $\partial_r^* = r^{-1}\partial_r r$. The matrix-differential operator $\underline{\boldsymbol{D}}$ with the components $D_{i,jk}$ is defined consistently with Eq. (3) by above identities.

This system is very similar to the basic Boussinesq equations of RBC for the vertical velocity and the temperature deviation and will be the starting point for the mapping to RBC in Sec. III. It is of the form (3) with the deterministic operators

$$\underline{\boldsymbol{L}}(\partial_r, ik, r) = \begin{pmatrix} (k^2 - \partial_r\partial_r^*)^2 & -2k^2\omega(r) \\ -2\frac{\eta^2\omega_1-\omega_2}{1-\eta^2} & k^2 - \partial_r\partial_r^* \end{pmatrix}, \quad (12)$$

$$\underline{\boldsymbol{S}}(\partial_r, ik, r) = \begin{pmatrix} k^2 - \partial_r\partial_r^* & 0 \\ 0 & 1 \end{pmatrix}. \quad (13)$$

The details of calculating the correlation matrix $\underline{\boldsymbol{Q}}$ with Eq. (7) can be found in the Appendix. The result is

$$\underline{\boldsymbol{Q}}(\partial_r, ik, r) = 2Q_0\begin{pmatrix} k^2(k^2 - \partial_r\partial_r^*)^2 & 0 \\ 0 & k^2 - \partial_r\partial_r^* \end{pmatrix}, \quad (14)$$

with the small dimensionless parameter

$$Q_0 = \frac{k_B T}{\rho d \nu^2}. \quad (15)$$

A straightforward insertion of (13) and (14) in (6) gives the noise strength of the amplitude equation for axisymmetric Taylor vortices

$$Q = \frac{Q_0}{\pi\bar{r}|N|^2}\int_{r_1}^{r_2}\frac{r\,dr}{\bar{r}}\{f_r^{\dagger *}k_c^2(k_c^2 - \partial_r\partial_r^*)^2 f_r^\dagger + f_\phi^{\dagger *}(k_c^2 - \partial_r\partial_r^*)f_\phi^\dagger\}, \quad (16)$$

$$N = \int_{r_1}^{x_2}\frac{r\,dr}{\bar{r}}\{f_r^{\dagger *}(k_c^2 - \partial_r\partial_r^*)f_r + f_\phi^{\dagger *}f_\phi\}, \quad (17)$$

where r_1 and r_2 are the scaled radii and $C = 2\pi\bar{r}$ with $\bar{r} = (R_1 + R_2)/(2d)$ is the scaled cross section. Note that in the narrow-gap limit the factor $r/\bar{r}$ approaches unity. Equations (16) and (17) are equivalent to the result Eqs. (2.12)–(2.14) of SBH, which can be seen by identifying F_A in their work with $Q\tau_0/(2\xi_0)$, using $\partial_r^* f_r + ik_c f_z = 0$, $\partial_r^* f_r^\dagger + ik_c f_z^\dagger = 0$, and taking care of the different scalings [25]. It is also equivalent to the result of Deissler, Eq. (8) in [20], if one assumes in Deissler's formula a ratio of the second (or bulk) viscosity to the usual viscosity of 2/3 [26].

Expression (16) is valid also for nonzero through flow since the through flow changes neither $\underline{\boldsymbol{S}}$ nor the stochastic forces in (11). The actual value of Q, however, changes because the through flow changes $\underline{\boldsymbol{L}}$ and thus the transverse eigenfunctions.

III. ANALYTIC EXPRESSIONS FOR THE TAYLOR-COUETTE SYSTEM

In this section we derive for the above TCF system approximate analytic expressions for the rotation of the outer cylinder at threshold, for the coefficients τ_0, ξ_0, and Q of the stochastic amplitude equation, and for the mean kinetic energy contained in the fluctuating vortices, all as functions of η, ω_2, and Q_0. This is performed by mapping the system to second order in the relative gap width onto the much simpler RBC system with a Prandtl number $P = 1$ (Sec. III A), which allows us to express everything in terms of the fixed RBC values for k_c, R_c, τ_0, ξ_0, and Q (Sec. III B). In Sec. III C we insert the RBC values for rigid (no-flux) BCs and compare the approximations with numerically obtained results.

A. Approximate mapping onto the RBC system

It is well known [27] that in the narrow-gap limit $\eta \to 1$ and for nearly equal rotation rates $\mu := \omega_2/\omega_1 \to 1$ with $\mu/\eta^2 < 1$, the dynamics of axisymmetric Taylor vortices becomes equivalent to that of RBC rolls with the wave number restricted to one direction. Starting from this limiting case SBH gave expressions for the critical inner rotation ω_{1c} with ω_2 fixed and for the coefficients of the amplitude equation. Their expressions are of first order in $(1-\eta)$ and the relative error with respect to their numerical results is of the order of 10% for $\eta = 0.738$; see Fig. 3 of SBH.

We give now second-order expressions by mapping the Taylor system to this order onto the Boussinesq equations for RBC. To make the notation consistent, we take in the RBC system $\hat{\boldsymbol{x}}$ as the vertical direction and restrict the modes to be parallel to $\hat{\boldsymbol{z}}$. When we compare the noise strength of the amplitude equations, we consider explicitly a quasi-one-dimensional RBC system with a width w and periodic BCs in the y direction.

The x component of the curl of the NS equation in the Boussinesq approximation and the conservation of heat energy represent two coupled equations for the vertical velocity v_x and the temperature deviation θ from the conductive state [16]. The linear part of the stochastic equations is of the form (3) with

$$\underline{\boldsymbol{L}}^{(\mathrm{RBC})}(\partial_x, ik) = \begin{pmatrix} (k^2-\partial_x^2)^2 & -k^2 \\ -R & k^2-\partial_x^2 \end{pmatrix}, \qquad (18)$$

$$\underline{\boldsymbol{S}}^{(\mathrm{RBC})}(\partial_x, ik) = \begin{pmatrix} k^2-\partial_x^2 & 0 \\ 0 & P \end{pmatrix}, \qquad (19)$$

and Eq. (7) for the noise correlation leads to

$$\underline{\boldsymbol{Q}}^{(\mathrm{RBC})}(\partial_x, ik) = 2Q_0 \begin{pmatrix} k^2(k^2-\partial_x^2)^2 & 0 \\ 0 & R\alpha^{(\mathrm{RBC})}(k^2-\partial_x^2) \end{pmatrix}. \qquad (20)$$

Lengths are scaled by the cell thickness d, time by d^2/ν, and temperature by $P\Delta T/R$, where $\Delta T = T_{\mathrm{bottom}} - T_{\mathrm{top}}$ is the applied temperature difference. The Rayleigh number R with the critical value $R_c = 1708$ (for rigid BCs) and the Prandtl number P are defined as usual [28]. The stochastic force of the velocity equation is the same as in the first equation of (11), so $O_{11}^{(\mathrm{RBC})} = O_{11}$, expressed in Cartesian coordinates. The stochastic force of the temperature equation is the divergence of the heat current (the Onsager current in this case) and leads to the component O_{22} [16], where $\alpha^{(\mathrm{RBC})} = \rho_m \alpha T g d/(c_v \Delta T)$ is usually negligible (for ideal gases it is the ratio of the potential energy to elevate the gas to a height d and the thermal energy to heat the gas by ΔT). Since the corresponding term in TCF is not negligible, we will keep this contribution.

In the above-mentioned limit the rotations ω_1, ω_2, and $\omega(r)$ all become equal and $\partial_r^* \to \partial_r$; the TCF system can be mapped exactly onto RBC. This means that the Taylor number $\tilde{T}$ can be defined in a way [see Eq. (23) in the above limit] that the corresponding matrix operators can be made identical by multiplying lines and columns. The analytic expressions of SBH for small $1-\eta$ and $\omega_2 \geq 0$ are based on a mapping with the approximation $\omega(r) \approx (\omega_1+\omega_2)/2$, $\partial_r^* = \partial_r$. This approximation is of first order in $(1-\eta)$ and becomes exact for all ω_2 in the limit $\eta \to 1$.

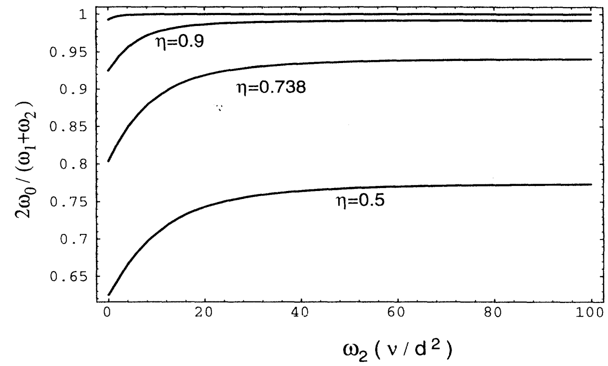

FIG. 1. Ratio of the second-order approximation $\omega(r) \approx \omega_0$ used in this work and $\omega(r) \approx (\omega_1+\omega_2)/2$ used in Ref. [19] as a function of the dimensionless outer rotation $\omega_2 = \Omega_2 d^2/\nu$. The parameter is the radius ratio with values $\eta = 0.99$ (top curve), 0.9, 0.738, and 0.5.

We propose the approximation $\partial_r^* \to \partial_r$ and $\omega(r) \to \omega_0$, where

$$\omega_0 = (\omega_1+\omega_2)/2 - (\omega_1-\omega_2)\,\delta/2, \qquad (21)$$

$$\delta = \frac{3}{4}(1-\eta). \qquad (22)$$

This means essentially that the Couette profile $\omega(r)$ is approximated by its value in the middle of the gap, $\omega(\bar{r}) = \omega_0 + O(1-\eta)^2$, instead of the average of the cylinder rotations. Figure 1 shows this difference at threshold. The rationale why $\omega(\bar{r})$ is a better approximation than $(\omega_1+\omega_2)/2$ is that, due to the no-flux BCs, the dynamics of the velocity fields is mainly determined by the volume elements near the middle of the gap. Examining the Couette profile together with approximate symmetry properties of the eigenfunctions shows that, apart from a term with negligible prefactor [29], this approximation yields linear growth rates $\lambda(\epsilon, k)$ that differ only in second order of $(1-\eta)$ from the exact solutions. Since the deterministic coefficients of the amplitude equation are given in terms of Taylor-expansion coefficients of $\lambda(\epsilon, k)$, their approximations are second order as well. With these approximations, the deterministic RBC and TCF operators become equivalent by relating the geometries, fields, and parameters of the two systems according to Table I and

TABLE I. Mapping of the Taylor system to thermal convection.

System properties	RBC with $P=1$	TCF
coordinates	(x,y,z)	(r,ϕ,z)
size and BC in the transverse directions	thickness d, rigid BC in x width w, periodic BC in y	gap d, rigid BC in r circumference $2\pi\bar{r}$ in ϕ
fields	(v_x,θ)	$(v_r, 2\omega_0 v_\phi)$
considered modes	$k_y=0$	axisymmetric
control parameter	R	$\tilde{T}$
distance from threshold	$\frac{R}{R_c}-1 =: \epsilon^{(\mathrm{RBC})}$	$\frac{\tilde{T}}{\tilde{T}_c}-1 = \left(\frac{\omega_1^2}{\omega_{1c}^2}-1\right)_{\eta,\mu}$
relative noise contribution of the second field	$\alpha^{(\mathrm{RBC})} \approx 0$	$\alpha^{(\mathrm{TCF})} = \frac{4\omega_0^2}{R_c}$

defining the Taylor number $\tilde{T}$ as

$$\tilde{T} = \frac{4(\eta^2\omega_1 - \omega_2)}{1-\eta^2}\omega_0. \tag{23}$$

Since also the BCs are the same (rigid BCs for the temperature correspond to ideally conducting plates), the linear analysis gives the same growth rates $\lambda(\epsilon, k)$, eigenvectors, and critical parameters at threshold. The correlation operator of the stochastic forces of TCF in the mapped variables v_r and $2\omega_0 v_\phi$ has the same functional dependence on r and differs from that of RBC only in the scalar prefactor of the component O_{22}, which is $\alpha^{(\mathrm{TCF})}$ instead of $\alpha^{(\mathrm{RBC})}$. In contrast to the negligible contribution of the temperature fluctuations, $\alpha^{(\mathrm{TCF})}$ is not negligible and becomes even dominant for large corotations; see the discussion after Eq. (31). The RBC operators have the same form for one and two dimensions (provided that $k_y = 0$ and there are periodic BCs in y), so the above considerations hold for both cases. The noise strength Q in the amplitude equations, however, depends on the dimensionality and the transverse size. In summary, the mapping procedure in Table I ensures that the threshold values for k_c and $R_c(\tilde{T}_c)$ and the deterministic coefficients of the respective amplitude equations differ only to second order in the gap width, while the noise strength has a contribution proportional to $\alpha^{(\mathrm{RBC})}(\alpha^{(\mathrm{TCF})})$, which depends on the system parameters.

B. Analytic expressions

The critical inner rotation $\omega_{1c}(\eta, \omega_2)$ can be calculated from Eq. (23) by inserting for $\tilde{T}$ the critical value $\tilde{T}_c = R_c$ and solving for ω_1. For resting outer cylinder we obtain

$$\omega_{1c}^2(\omega_2 = 0) = \frac{R_c(1-\eta^2)}{2\eta^2(1-\delta)}. \tag{24}$$

The analytic expressions for $\omega_2 \neq 0$ are given in Eq. (B1) in Appendix B.

The definition in Table I for the threshold distance in the Taylor system implies that one increases the control parameter by simultaneously increasing both cylinder rotations with the ratio kept constant. Experimentally, however, it is more favorable to increase only the rotation of the inner cylinder and to define, in accordance with SBH,

$$\epsilon = \left(\frac{\omega_1}{\omega_{1c}} - 1\right)_{\eta, \omega_2 = \mathrm{const}} \tag{25}$$

as the relevant distance from threshold. With $\frac{1}{\tau_0} = \frac{\partial \lambda}{\partial \tilde{T}}\frac{\partial \tilde{T}}{\partial \epsilon}$, $\frac{1}{\tau_0^{(\mathrm{RBC})}} = \frac{\partial \lambda}{\partial R}\frac{\partial R}{\partial \epsilon^{(\mathrm{RBC})}}$, and $\frac{\partial \lambda}{\partial \tilde{T}} = \frac{\partial \lambda}{\partial R}$, one obtains

$$\frac{\tau_0}{\tau_0^{(\mathrm{RBC})}} = \left(\frac{\xi_0}{\xi_0^{(\mathrm{RBC})}}\right)^2 = \frac{\tilde{T}_c}{\omega_{1c}}\left(\frac{\partial \tilde{T}}{\partial \omega_1}\right)^{-1}_{\eta, \omega_2}. \tag{26}$$

The first equality follows from the fact that the diffusion constant $\xi_0^2/\tau_0 = -\frac{1}{2}\partial_k^2 \lambda$ does not depend on ϵ (at lowest order) and thus is, for any definition of ϵ, the same in both systems.

In 1D systems, fluctuations of the amplitude are usually proportional to $Q\tau_0/\xi_0$. To compare the results, we follow SBH and give expressions for the dimensionless noise strength

$$F_A = \frac{Q\tau_0}{2\xi_0} \tag{27}$$

rather than for Q itself. Since the operators and eigenfunctions in Eq. (6) are the same for RBC and the mapped Taylor system, the noise strength Q is that of RBC with $\alpha^{(\mathrm{RBC})}$ replaced by $\alpha^{(\mathrm{TCF})}$, provided that the amplitudes are related to equivalent physical fields and the cross sections are the same. While the definition $\tilde{T}/\tilde{T}_c - 1$ for the threshold distance would lead to $F_A = F_A^{(\mathrm{RBC})}$ with $\alpha^{(\mathrm{RBC})}$ replaced by $\alpha^{(\mathrm{TCF})}$, the actual use of Eq. (25) leads with (26) to

$$\frac{[f_r, f_r]F_A}{[f_x, f_x]F_A^{(\mathrm{RBC})}(P=1, w=2\pi\bar{r})} = \sqrt{\frac{\tau_0}{\tau_0^{(\mathrm{RBC})}}}\left(1 + \alpha^{(\mathrm{TCF})}\right). \tag{28}$$

As discussed in Sec. II, the scalar products in front of F_A and $F_A^{(\mathrm{RBC})}$ ensure independence from the normalizations of the eigenvectors and can be set to one by a corresponding choice of the normalizations.

Finally, we calculate from Eq. (28) the average line density of the kinetic energy (energy per unit length) of the velocity fluctuations. The result in physical units is

$$\frac{\partial\langle E\rangle}{\partial z} = \frac{\rho_m}{2}\int_c d^2 r_\perp \langle v_r^2 + v_\phi^2 + v_z^2\rangle = \frac{k_B T}{4\xi|\epsilon^{(\mathrm{RBC})}|}\left(1 + \frac{\alpha^{(\mathrm{TCF})} - 1}{\tau_0^{(\mathrm{RBC})}|\lambda_0|}\right) \times \left(\frac{1+\alpha^{(\mathrm{TCF})}}{\alpha^{(\mathrm{TCF})}}\right), \tag{29}$$

where $\xi = d\xi_0|\epsilon|^{-1/2}$ is the actual correlation length of the fluctuating pattern and λ_0 is the equilibrium relaxation rate ($\tilde{T} = 0$) of the critical mode with $|\lambda_0|\tau_0^{(\mathrm{RBC})} = 2.95$ for rigid BCs and $|\lambda_0|\tau_0^{(\mathrm{RBC})} = 2$ for free BCs. The result is obtained by expressing the projection integrals of the linear operators in Eq. (6) in terms of λ_0 and $\tau_0^{(\mathrm{RBC})}$ and using $\epsilon/\epsilon^{(\mathrm{RBC})} = \tau_0/\tau_0^{(\mathrm{RBC})}$. Note that the natural distance from threshold seems to be $\epsilon^{(\mathrm{RBC})}$ rather than ϵ. The first factor leads to the equipartition result $\langle E\rangle = k_B T/(2|\epsilon^{(\mathrm{RBC})}|)$ per degree of freedom if one defines as a dynamically active degree of freedom a fluid element with the volume C times the double correlation length. Note that with this definition of the volume element, its energy is the same as that of the critical Fourier mode in k space.

C. Results for rigid boundary conditions

As already mentioned, the threshold values are (within the approximation) the same as in RBC. For rigid BCs we have $\tilde{T}_c = R_c = 1708$ and $k_c = k_c^{(\mathrm{RBC})} = 3.117$, remarkably independent of η and ω_2. The numerical results for $\omega_2 = 0$, $\eta = 0.738$, and without through flow are [8,30] $k_c = 3.136$ and $\mathcal{T}_c = 2131.8$, where $\mathcal{T}_c = \tilde{T}_c/(1-\delta)$; the definition of the Taylor number in Ref. [8] is, in our approximation, equal to 2126. Figure 2(a) shows that the deviation of the threshold formula (B1) with respect to the numerical results of SBH is less than 0.3% for $\eta = 0.738$ and all $\omega_2 \geq 0$. For $\omega_2 \to \infty$ the asymptotes are given by $\omega_{1c} = \omega_2/\eta^2$, i.e., instability sets in as soon as Rayleigh's stability criterion $\partial_r(r^2\omega)^2 > 0$ [12] for an inviscid fluid is violated (Euler limit). The approximations (and also the approximation of SBH) become exact in this limiting case.

The deterministic coefficients of the amplitude equation are, with (26) and (23),

$$\frac{0.0769}{\tau_0} = \frac{0.148}{\xi_0^2} = \left(1 - \frac{\omega_2}{\omega_1 \eta^2}\right)^{-1} + \left(1 + \frac{(1+\delta)\omega_2}{(1-\delta)\omega_{1c}}\right)^{-1}. \qquad (30)$$

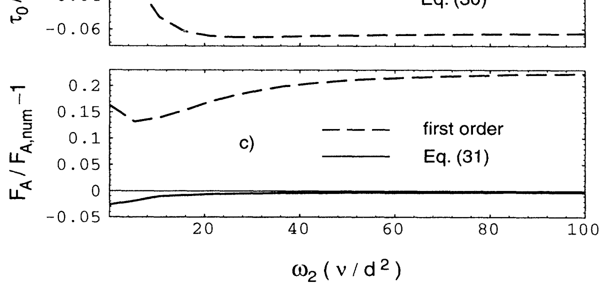

FIG. 2. Comparison of analytic approximations for the coefficients of the stochastic amplitude equation with numerical results for a radius ratio $\eta = 0.738$: (a) critical outer rotation ω_{1c}, (b) time constant τ_0, and (c) noise strength F_A of the amplitude equation. Plotted are the relative deviations $\delta x = (x_{\mathrm{approx}} - x_{\mathrm{num}})/x_{\mathrm{num}}$, where $x = \omega_{1c}, \tau_0$, or F_A, for the approximations of this work, Eqs. (B1), (30), and (31), respectively (straight lines), and for the approximations of Ref. [19] (broken lines) as a function of the dimensionless outer rotation ω_2.

For zero outer rotation, $\tau_0 = \tau_0^{(\mathrm{RBC})}/2 = 0.0385$ and $\xi_0^2 = \xi_0^{(\mathrm{RBC})2}/2 = 0.074$ are, in our approximations, independent of η. The numerical results of [19] for $\eta = 0.738$ are $\tau_0 = 0.0379$ and $\xi_0^2 = 0.0725$. Figure 2(b) shows the deviation of Eq. (30) from the numerical values for the same gap width as a function of the corotation ω_2. The magnitude of the relative error is less than 1.3% for all $\omega_2 \geq 0$ and becomes asymtotically 0.3% for large ω_2.

To calculate the noise strength, we relate the amplitude fluctuations $\langle |A|^2 \rangle = Q\tau_0/(8\xi_0\sqrt{|\epsilon|})$ to the cross-section-averaged fluctuations $\overline{\langle v_r^2 \rangle}$ of the radial velocity, which with (4) is equal to $[f_r, f_r]F_A/(2\sqrt{|\epsilon|})$. This corresponds to relating the RBC amplitude to the cross-section-averaged fluctuations of the vertical velocity and $[f_r, f_r]F_A$ to $[f_x, f_x]F_A^{(\mathrm{RBC})} = 0.39Q_0$, the usual result for 2D RBC. Inserting this into Eq. (28) leads, with (26), to

$$[f_r, f_r]F_A = \frac{0.39Q_0}{2\pi\bar{r}}\sqrt{\frac{\tau_0}{0.0769}}\left(1 + \alpha^{(\mathrm{TCF})}\right). \qquad (31)$$

Expression (31) is, together with (29), the main analytic result of this paper. It is the same for the different scalings of SBH and in this paper [31]. If $\boldsymbol{f}$ is normalized to $[f_r, f_r] = \bar{r}^{-1}\int r dr |f_r|^2 = 1$, then the right-hand side gives directly the noise strength F_A. In Fig. 2(c) the comparison of the analytic expression (31) with numerical results [19] shows that the relative error is about 2.5% for $\omega_2 = 0$. Moreover, the error decreases for increasing ω_2 and approaches asymptotically about 0.3%, while a residual error of about 20% remains in the first-order approximation [Fig. 3(c) of SBH].

The result is very intuitive. The first factor $0.39Q_0/(2\pi\bar{r})$ is the noise strength for 1D RBC with a width equal to the circumference of the Taylor system in the middle of the gap. The second factor is due to the nonequivalent definitions of the threshold distances in the two systems. The third factor gives the increase of the fluctuations due to the v_ϕ fluctuations. The relative contribution of the v_ϕ fluctuations $\alpha^{(\mathrm{TCF})} = 4\omega_0^2/1708$ has its origin in the Coriolis force $2\omega v_\phi \hat{\boldsymbol{r}}$, which leads in the equation for the radial velocity to a stochastic force proportional to $4\omega_0^2\langle v_\phi^2 \rangle$. It vanishes in the narrow-gap limit but is nonzero otherwise, even for $\omega_2 = 0$, where $\alpha^{(\mathrm{TCF})} = (1-\eta^2)(1-\delta)/2\eta^2$. For large ω_2, where $\omega_{1c}(\omega_2, \eta)$ approaches the Euler limit $\omega_{1c} \to \omega_2/\eta^2$, $\alpha^{(\mathrm{TCF})}$ is proportional to ω_2^2. Figure 3 gives F_A as a function of ω_2 for different values of η. In the limit of large ω_2, the factor $\tau_0/\tau_0^{(\mathrm{RBC})}$ decreases proportionally to $1/\omega_2$, making F_A proportional to ω_2.

The influence of the v_ϕ field is also reflected in the kinetic energy (29). The terms in parentheses increase the equilibriumlike fluctuations of the first factor, especially in the Euler limit of large ω_2 ($\alpha^{(\mathrm{TCF})} \gg 1$) or for small gaps ($\alpha^{(\mathrm{TCF})} \ll 1$). Both cases involve high rotation rates. For large ω_2, the main effect is the Coriolis force as described earlier. For small gaps the magnitude of the v_ϕ component in the critical mode is much larger than that of v_r and v_z; the measurable fluctuations of the radial velocities remain small, as seen from Eq. (31). For the parameters in the experiment of Ref. [8], Eq. (29)

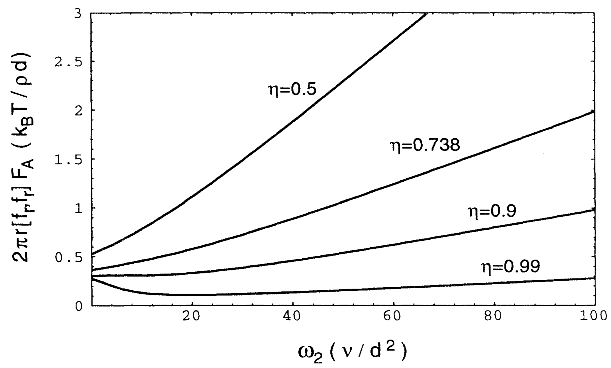

FIG. 3. Cross-section-integrated noise intensity $2\pi\bar{r}[f_r, f_r]F_A = \int_C d^2r_\perp |f_r|^2 F_A$ as a function of ω_2 with the radius ratio η as parameter. The cross-section-integrated fluctuations of the radial velocity in scaled units are $(2\sqrt{\epsilon})^{-1}$ times this quantity.

gives, in physical units ($\epsilon^{(\mathrm{RBC})} = 2\epsilon$ for $\omega_2 = 0$),

$$\frac{\partial \langle E \rangle}{\partial z}(\omega_2 = 0, \eta = 0.738) = 0.385 \frac{k_B T}{\xi|\epsilon|} = 1.00 \frac{k_B T}{\sqrt{|\epsilon|}}. \quad (32)$$

IV. CONCLUSION

The mean kinetic energy of the fluctuating Taylor vortices per (suitably defined) dynamically active degree of freedom was found to be $k_B T/(2|\epsilon^{(\mathrm{RBC})}|)$ times a factor [the two terms in parentheses in (29)] that is usually of the order of unity in a wide range of the parameters R_1/R_2 and Ω_2. This looks remarkably similar to equipartition fluctuations in equilibrium systems. Indeed, the energy of the fluctuations in equilibrium systems (such as director fluctuations of a nematic liquid crystal in a long and narrow cell near the Fréedericksz transition) and interestingly also in RBC with $P = 0$ and in EHC in the thick-cell limit is exactly $k_B T/(2|\epsilon|)$ per an equivalently defined degree of freedom [23]. The factor comes from the dynamics of the azimuthal velocity field v_ϕ and makes the fluctuations different from equilibrium fluctuations. Specifically, it increases the energy for large corotations, which could be explained by the Coriolis force exerted by the v_ϕ in the radial direction.

This can be put into a broader context. Since the dynamics of v_ϕ contains not only fluctuations but also dissipations, the coupling of the radial velocity v_r to v_ϕ can be seen as a macroscopic analog of the coupling to microscopic degrees of freedom. While the microscopic coupling described by the fluctuating forces of the fluctuation-dissipation theorem (1) leads in equilibrium systems always to macroscopic fluctuations in accordance with the equipartition theorem, the fluctuations and dissipations of the macroscopic coupling of fields in nonequilibrium systems are generally no longer balanced. In some cases, the coupling increases the fluctuations as shown for the Taylor system, but also in EHC, where the coupling of the director field to the electrical degrees of freedom (charge-density fluctuations) increases the director fluctuations [18]. In other cases, as in RBC with $P \neq 0$, the coupling of the velocities to the temperature field decreases the fluctuation energy [23]. When the relaxation of the main field (director field in EHC, vertical velocity in RBC) becomes much slower than that of any coupled field (small Prandtl number in RBC, thick cells in EHC), fluctuations and dissipations remain balanced.

The crucial point in our work is the assumption of local equilibrium, which allows us to start with the stochastic hydrodynamics of Landau and Lifshitz [21]. This ansatz seems to describe correctly the stationary fluctuations observed in the direct experiments [1,3,4]. A preliminary estimate starting from Eq. (6) for realistic rigid lateral BCs [32] leads in the geometry of the Ref. [4] to a decrease of the theoretical noise strength by about a factor of 2 with respect to the case of periodic lateral BCs; this agrees at least semiquantitatively with the observations in [4].

In contrast, in the indirect experiments using convective instabilities or time ramps that measure nonstationary fluctuations (for convective instabilities nonstationary with respect to the frame moving with the group velocity), the measured noise intensities were two to four orders of magnitude larger than predicted. This seems to indicate that in nonequilibrium systems the assumption of stochastic hydrodynamics in the usual form is valid only in stationary situations.

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APPENDIX A: CALCULATION OF THE NOISE STRENGTH IN THE STOCHASTIC AMPLITUDE EQUATION FOR AXISYMMETRIC TCF VORTICES

The only source of dissipation and thus the only noise source comes from the viscous stress tensor σ_{ij}. The Onsager matrix in the scaled units of Sec. II is

$$k_B M_{ij,kl} = Q_0(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}). \quad (\mathrm{A1})$$

The stochastic forces $\nabla\tilde{\underline{\sigma}}$ of the Navier-Stokes equations are in cylindrical coordinates [21]

$$(\nabla\tilde{\underline{\sigma}})_i = \left[\delta_{ik}\frac{1}{r}\partial_j r + \frac{1}{r}\delta_{k2}(\delta_{i2}\delta_{j1} - \delta_{i1}\delta_{j2})\right]\tilde{\sigma}_{jk}, \quad (\mathrm{A2})$$

where all indices take the values r, ϕ, and z. With this, the components of $\underline{\boldsymbol{D}}$, defined in (11), read explicitly

$$D_{1,jk} = -\epsilon_{2lk}\partial_z\partial_l\frac{1}{r}\partial_j r + \delta_{j2}\delta_{k2}\frac{1}{r}\partial_z^2, \tag{A3}$$

$$D_{2,jk} = \delta_{2k}\left(\frac{1}{r}\partial_j r + \frac{1}{r}\delta_{j1}\right), \tag{A4}$$

where ϵ_{ijk} is the totally antisymmetric unit tensor. The Hermitian conjugates with respect to the scalar product (8) are

$$D^\dagger_{jk,1} = \epsilon_{2lk}\partial_z\partial_j\frac{1}{r}\partial_l r + \delta_{j2}\delta_{k2}\frac{1}{r}\partial_z^2, \tag{A5}$$

$$D^\dagger_{2,jk} = -\delta_{2k}\left(\partial_j - \frac{1}{r}\delta_{j1}\right). \tag{A6}$$

Equation (7) for the correlation matrix of the stochastic forces has the explicit form

$$O_{ij} = k_B D_{i,jk}(M_{jk,lm} + M_{lm,jk})\left(D^\dagger\right)_{lm,j}, \tag{A7}$$

leading to the correlation matrix (14) in the main text.

APPENDIX B: OUTER CYLINDER ROTATION AT THRESHOLD

Inserting $\tilde{T}_c = R_c = 1708$ (rigid BC) in (23) gives a quadratic equation in ω_1 with the solution

$$\begin{aligned} \omega_{1c}(\omega_2,\eta) &= P + \sqrt{P^2+U}, \\ P &= \frac{1-\eta^2-(1+\eta^2)\delta}{2\eta^2(1-\delta)}\omega_2, \\ U &= \frac{(1+\delta)\omega_2^2 + 854(1-\eta^2)}{\eta^2(1-\delta)}. \end{aligned} \tag{B1}$$

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[26] This is plausible because for this ratio the Landau expressions [21] for fluctuations of the stress tensor in compressible fluids, used in [20], give vanishing fluctuations of the diagonal components. In incompressible fluids, these pressure fluctuations cannot lead to fluctuations of the observable quantities.

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[30] Nonzero through flow has very little influence on these parameters. Within the range where axisymmetric modes become first unstable, the influence on $\mathcal{T}_c$ and k_c is smaller than 0.7% and 0.06%, respectively [8]. Similar orders of magnitude apply for τ_0 and ξ_0^2.

[31] Note, however, that in [19] the amplitude is related to v_z instead of v_r and the scalar product is defined differently as well.

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