Transient properties of a bistable kinetic model with correlations between additive and multiplicative noises: Mean first-passage time

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The transient properties of a bistable system with correlations between additive and multiplicative noise terms are investigated. The explicit expressions of the mean first-passage time (MFPT) are obtained. The numerical computations show that the MFPT of the system is affected by λ, the strength of correlations between additive (intensity α) and multiplicative (intensity D) noise terms. For the case of perfectly correlated noises ($\lambda=1$), the MFPT corresponding to $\alpha>D$ and $\alpha<D$ exhibits very different behaviors, and the MFPT for $\alpha=D$ diverges to infinity. [S1063-651X(96)07105-3]

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I. INTRODUCTION

Though various noises are presented simultaneously in some stochastic processes, noises are assumed to have different origins and are treated as independent random variables in most of the previous investigations [1–11]. However, in certain situations noises may have a common origin and thus may be correlated with each other as well [12–20].

Fox discussed N-component stochastic processes with correlations between the noises, and obtained an explicit equation for the probability distribution to first order in the correlation time by using the method of ordered operator cumulants [12]. In the limit of the correlation times going to zero, Fox's equation can reduce to a general Fokker-Planck equation for stochastic processes driven by correlated white noises.

Recently the steady-state statistical properties of a bistable kinetic model with correlations between additive and multiplicative noise terms have been discussed in Ref. [18]. They showed that in the α-D parameter plane, the critical curve separating the unimodal and bimodal regions of the stationary probability distribution (SPD) of the model is shown to be affected by λ, the strength of correlations between additive and multiplicative noise terms, the area of the bimodal region in the α-D plane is contracted as λ is increased; when α and D are fixed, the form of SPD changes from a bimodal to a unimodal structure as λ is increased. The positions of the extrema of SPD depend sensitively on the intensity of the multiplicative noise D and weakly depend on the additive noise intensity α; For the case of perfectly correlated noises ($\lambda=1$), the SPD's corresponding to $\alpha/D>1$ and $\alpha/D<1$ exhibit a very different shape of divergence, and $\alpha/D=1$ plays the role of a critical ratio. A natural question is whether these changes of SPD are reflected in the transient properties. However, the transient properties of the bistable kinetic model driven by correlated noises have not been discussed. It must be pointed out that the escape time of the bistable system with independent noise terms ($\lambda=0$) was analyzed in Ref. [11]. It showed that the numerical results do not show any significant change in the MFPT associated with a change of the stationary distribution.

In this paper, the transient properties of the bistable system with correlations between additive and multiplicative noise terms are investigated. Our interest here is to discuss the effects of the strength of correlated noises on mean first-passage time (MFPT). In Sec. II the SPD of the system is presented. The explicit expressions of MFPT are obtained in Sec. III. We end with conclusions from the numerical computations.

II. STEADY-STATE DISTRIBUTION FUNCTION

Consider a one-dimensional bistable kinetic system, which contains correlations between additive and multiplicative white noise, and assume the dimensionless form

$$\dot{x}=x-x^3+x\xi(t)+\eta(t). \tag{1}$$

The statistical properties of the noise terms are characterized by their first and second moments,

$$\begin{gathered}\langle\xi(t)\rangle=\langle\eta(t)\rangle=0,\\ \langle\xi(t)\xi(t')\rangle=2D\delta(t-t'),\\ \langle\eta(t)\eta(t')\rangle=2\alpha\delta(t-t'),\\ \langle\xi(t)\eta(t')\rangle=\langle\eta(t)\xi(t')\rangle=2\lambda\sqrt{\alpha D}\,\delta(t-t')\quad(0\leqslant\lambda\leqslant1),\end{gathered}\tag{2}$$

where α and D are the additive and multiplicative noise intensities, respectively. The parameter λ measures the strength of the correlations between additive and multiplicative noise terms. The potential

$$U(x)=-\tfrac{1}{2}x^2+\tfrac{1}{4}x^4 \tag{3}$$

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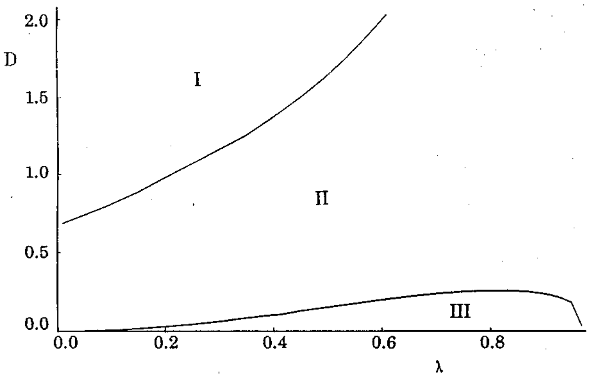

FIG. 1. The valid region for $0 \leqslant \lambda < 1$ in the D-λ parameter plane. $\alpha = 0.1$. $R < 0$ in regions I and III, $R > 0$ in region II.

corresponding to (1) has two stable states $x_1 = -1$, $x_2 = 1$ and an unstable state $x_0 = 0$.

The Fokker-Planck equation corresponding to (1) with (2) can be written as Fox's equation for the probability function [12] in the limit of the correlation times going to zero:

$$\frac{\partial}{\partial t} P(x,t) = -\frac{\partial}{\partial x}\left[x - x^3 + Dx + \lambda\sqrt{\alpha D}\right]P(x,t) + \frac{\partial^2}{\partial x^2}\left[Dx^2 + 2\lambda\sqrt{\alpha D}x + \alpha\right]P(x,t). \quad (4)$$

The steady-state distribution function is

$$P_{\mathrm{st}}(x) = \begin{cases} N[\mathscr{D}(x)]^{-1/2}\exp\left[-\dfrac{\hat{U}(x)}{D}\right] & \text{for } 0 \leqslant \lambda < 1, \\ \tilde{N}[\tilde{\mathscr{D}}(x)]^{-1/2}\exp\left[-\dfrac{\hat{U}(x)}{D}\right] & \text{for } \lambda = 1, \end{cases} \quad (5)$$

where N and $\tilde{N}$ are normalization constants, $\mathscr{D}(x) = Dx^2 + 2\lambda\sqrt{\alpha D}x + \alpha$, $\tilde{\mathscr{D}}(x) = Dx^2 + 2\sqrt{\alpha D}x + \alpha$, and the modified potential $\hat{U}(x)$ is

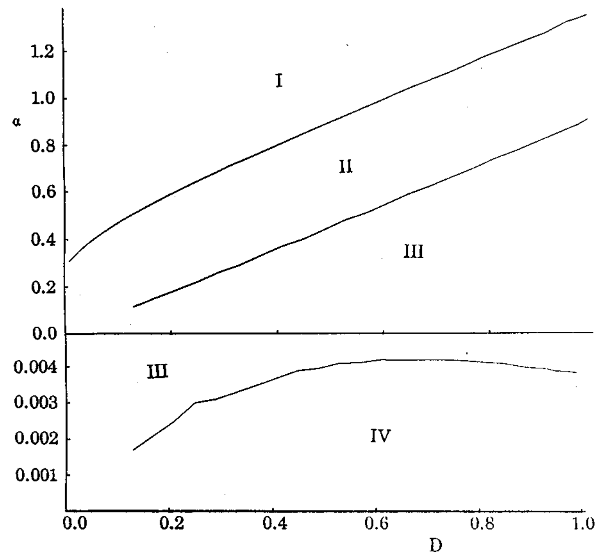

FIG. 3. The valid region for $\lambda = 1$ in α-D parameter plane. $\tilde{R} < 0$ in regions I and III, $\tilde{R} > 0$ in regions II and IV.

$$\hat{U}(x) = \begin{cases} \frac{1}{2}x^2 - 2\lambda\sqrt{\alpha/D}x + \beta_1 \ln|Dx^2 + 2\lambda\sqrt{\alpha D}x + \alpha| \\ \quad + \beta_2 \tan^{-1}\left[\dfrac{\sqrt{D/\alpha}x + \lambda}{\sqrt{1+\lambda^2}}\right] & \text{for } 0 \leqslant \lambda < 1 \\ \frac{1}{2}x^2 - 2\sqrt{\alpha/D}x + \tilde{\beta}_1 \ln|\sqrt{D}x + \sqrt{\alpha}| \\ \quad + \tilde{\beta}_2 \dfrac{1}{\sqrt{D}x + \sqrt{\alpha}} & \text{for } \lambda = 1, \end{cases} \quad (6)$$

where

$$\beta_1 = \frac{\alpha}{2D}(4\lambda^2 - 1) - \frac{1}{2}, \quad \beta_2 = \frac{\lambda}{\sqrt{1-\lambda^2}}\left[\frac{\alpha}{D}(3 - 4\lambda^2) + 1\right] \quad (7)$$

and

$$\tilde{\beta}_1 = \frac{3\alpha}{D} - 1, \quad \tilde{\beta}_2 = \sqrt{\alpha}\left[\frac{\alpha}{D} - 1\right]. \quad (8)$$

We can show that the extrema of $\hat{U}(x)$ coincide with those of the deterministic potential $U(x)$. However, the extrema of

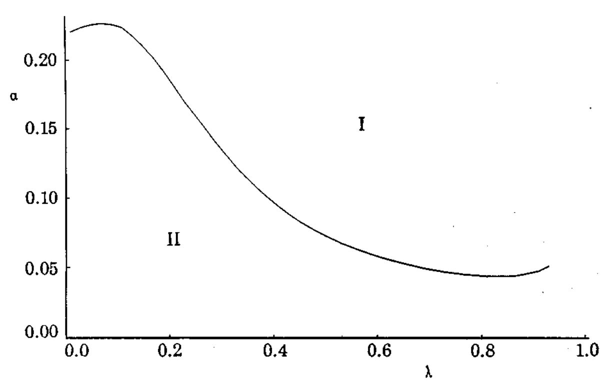

FIG. 2. The valid region for $0 \leqslant \lambda < 1$ in the α-λ parameter plane. $D = 0.1$. $R < 0$ in region I, $R > 0$ in region II.

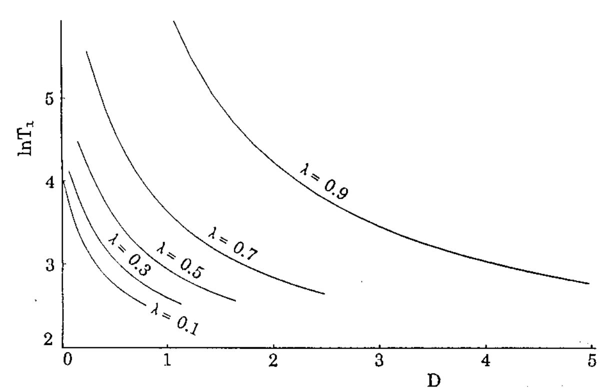

FIG. 4. The MFPT of the bistable system for $0 \leqslant \lambda < 1$ (15) as a function of D for $\alpha = 0.1$. $\lambda = 0.1$, 0.3, 0.5, 0.7, and 0.9, respectively.

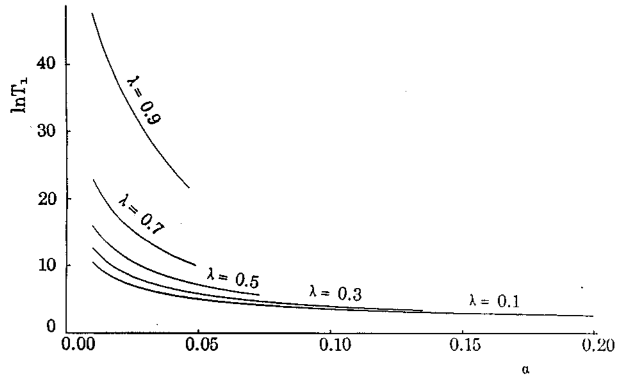

FIG. 5. The MFPT of the bistable system for $0 \leqslant \lambda < 1$ (15) as a function of α for $D = 0.1$. $\lambda = 0$, 1, 0.3, 0.5, 0.7, and 0.9, respectively.

$P_{\mathrm{st}}(x)$ and $U(x)$ do not coincide. The extrema of the stationary distribution (5) are determined by the algebraic equation

$$x^3 - (1-D)x + \lambda\sqrt{\alpha D} = 0 \tag{9}$$

and the critical curve separating the unimodal and bimodal regions in the α-D plane is described by

$$\tfrac{1}{4}\lambda^2 D\alpha + \tfrac{1}{27}(D-1)^3 = 0. \tag{10}$$

The α-D plane has been plotted in Ref. [18]. For the bimodal region, $P_{\mathrm{st}}(x)$ has three extrema: $\tilde{x}_1 = A\cos[(\theta+2\pi)/3]$, $\tilde{x}_2 = A\cos(\theta/3)$, and $\tilde{x}_0 = A\cos[(\theta+4\pi)/3]$ with

$$A = 2\left[\frac{1+D}{3}\right]^{1/2}, \quad \theta = \cos^{-1}\left[\frac{\lambda\sqrt{\alpha D}}{2}\left(\frac{3}{1+D}\right)^{3/2}\right]. \tag{11}$$

In the case of independent noises ($\lambda = 0$), the positions of the extrema of $P_{\mathrm{st}}(x)$ are only dominated by the intensity of the multiplicative noise. However, the presence of correlations between noises ($\lambda \neq 0$) changes this picture. The positions of the extreme values of $P_{\mathrm{st}}(x)$ depend on the intensity of the multiplicative noise, the intensity of the additive noise, and the parameter λ, which denotes the degree of correlation between noise terms. For small α and D, $\tilde{x}_1$, $\tilde{x}_2$, and $\tilde{x}_0$ are close to x_1, x_2, and x_0, respectively.

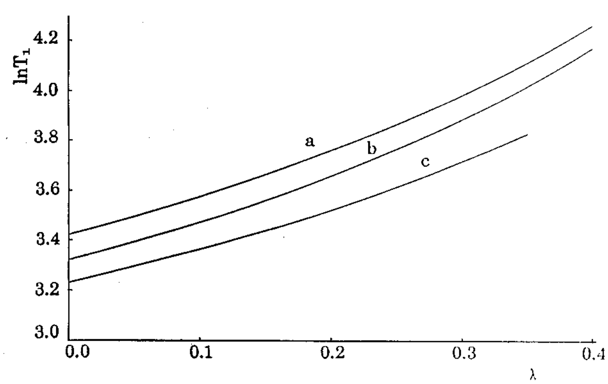

FIG. 6. The MFPT of the bistable system for $0 \leqslant \lambda < 1$ (15) as a function of λ. Curve a: the case of $\alpha = D$ ($\alpha = 0.1$, $D = 0.1$); curve b: the case of $\alpha < D$ ($\alpha = 0.1$, $D = 0.128$); curve c: the case of $\alpha > D$ ($\alpha = 0.114$, $D = 0.1$).

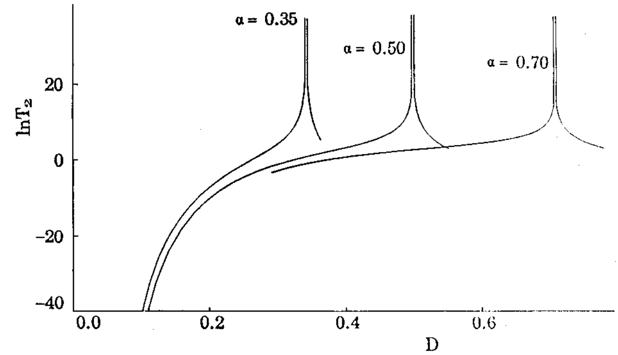

FIG. 7. The MFPT of the bistable system for $\lambda = 1$ (16) as a function of D. $\alpha = 0.35$, 0.5, and 0.7, respectively.

III. MEAN FIRST-PASSAGE TIME

Our aim is to discuss the effects of the strength of correlations between additive and multiplicative noise terms on the MFPT. A rigorous definition of escape time out of x_1 is provided by the MFPT $T_{x_2}(x_1)$ of the process $x(t)$ to reach the point x_2 with initial condition $x(t=0) = x_1$. This is given by [21–23]

$$T_{x_2}(x_1) = \int_{x_1}^{x_2} \frac{dx}{\mathscr{D}(x)P_{\mathrm{st}}(x)} \int_{-\infty}^{x} dy\; P_{\mathrm{st}}(y). \tag{12}$$

In the case in which the intensity of the two types of fluctuations, measured by α and D, is small in comparison with the energy barrier

$$D < \hat{U}(x_0) - \hat{U}(x_1), \tag{13}$$

$T_{x_2}(x_1)$ becomes independent of the initial condition $x(t=0) < x_0$ and of x_2 [11]. The steepest-descent approximation [24–26] to (12) becomes

$$T_{x_2}(x_1) \approx T = 2\pi[|U''(x_1)U''(x_0)|]^{-1/2}\exp\{[\hat{U}(x_0) - \hat{U}(x_1)]/D\}. \tag{14}$$

It is interesting to note that in (14) the dependence on the noise intensities and the strength of correlations between the noises is contained in the exponential factor, while the pref-

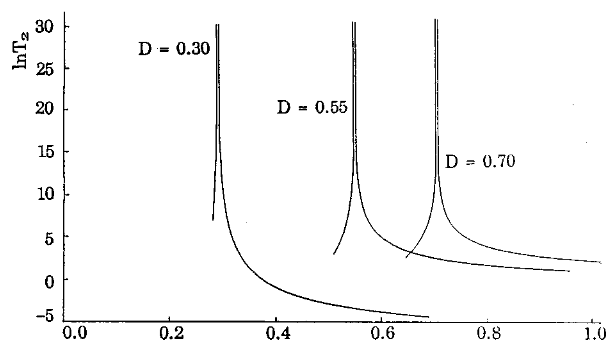

FIG. 8. The MFPT of the bistable system for $\lambda = 1$ (16) as a function of α. $D = 0.3$, 0.55, and 0.7, respectively.

actor $2\pi[|U''(x_1)U''(x_0)|]^{-1/2}$ is independent of α, D, and λ. From (3), (6), and (14) we obtain the explicit expressions of MFPT:

$$T_1=\sqrt{2}\pi\left|\frac{D}{\alpha}-2\lambda\left(\frac{D}{\alpha}\right)^{1/2}+1\right|^{-\beta_1/D}\exp\left[-\frac{1}{2D}-\frac{2\lambda\sqrt{\alpha D}}{D^2}+\frac{\beta_2}{D}\left(\tan^{-1}\frac{\lambda}{\sqrt{1-\lambda^2}}-\tan^{-1}\frac{\lambda-\sqrt{D/\alpha}}{\sqrt{1-\lambda^2}}\right)\right],\quad \text{for } 0\leqslant\lambda<1, \tag{15}$$

$$T_2=\sqrt{2}\pi\left|1-\left(\frac{D}{\alpha}\right)^{1/2}\right|^{-\tilde{\beta}_1/D}\exp\left[-\frac{1}{2D}-\frac{2\sqrt{\alpha D}}{D^2}+\frac{\tilde{\beta}_2}{D\sqrt{\alpha}-\alpha\sqrt{D}}\right],\quad \text{for } \lambda=1. \tag{16}$$

It must be pointed out that our results are confined by (13). Therefore, the MFPT (15) is valid for $R>0$ and

$$R=-D-\frac{1}{2}-2\lambda\left(\frac{\alpha}{D}\right)^{1/2}+\beta_1\ln\frac{\alpha}{|D-2\lambda\sqrt{\alpha D}+\alpha|}+\beta_2\left[\tan^{-1}\frac{\lambda}{\sqrt{1+\lambda^2}}-\tan^{-1}\frac{\lambda-\sqrt{D/\alpha}}{\sqrt{1+\lambda^2}}\right]. \tag{17}$$

The MFPT (16) is valid for $\tilde{R}>0$ and

$$\tilde{R}=-D-\frac{1}{2}-2\left(\frac{\alpha}{D}\right)^{1/2}+\tilde{\beta}_1\ln\frac{\sqrt{\alpha}}{|\sqrt{\alpha}-\sqrt{D}|}+\tilde{\beta}_2\left[\frac{1}{\sqrt{\alpha}}-\frac{1}{\sqrt{\alpha}-\sqrt{D}}\right]. \tag{18}$$

These provide the restriction on the noise intensities α and D. In Figs. 1 and 2, we display the valid region for $0\leqslant\lambda<1$ in the D-λ parameter plane and in the α-λ parameter plane, respectively. The valid regions in the α-D parameter plane for $\lambda=1$ are displayed in Fig. 3.

IV. CONCLUSIONS

By virtue of the expressions of the MFPT (15) for $0\leqslant\lambda<1$ and (16) for $\lambda=1$, we have plotted the curves of the MFPT in Figs. 4–9. These curves of MFPT are plotted in the valid regions as shown in Figs. 1–3. The conclusions that can be drawn from these figures are as follows.

When we fix the value of the intensity of additive noise α (e.g., α=0.1), Fig. 4 shows that the MFPT (15) increases as the strength of correlations between additive and multiplicative noises λ increases but decreases as the intensity of the multiplicative noise D increases. When we fix the value D (e.g., $D=0.1$), Fig. 5 shows that the MFPT (15) increases with increasing λ but decreases with increasing α. The effect of λ on the MFPT can be understood by the change of the SPD of the system with varying λ, which has been discussed in Ref. [18]. The form of the SPD changes from a bimodal to a unimodal structure as λ increases, and the position of the extremum of SPD shifts to the stable state $x_1=-1$, so that the effect of λ makes the MFPT increase. However, the effects of α and D make the MFPT decrease as in the case of uncorrelated noises [11].

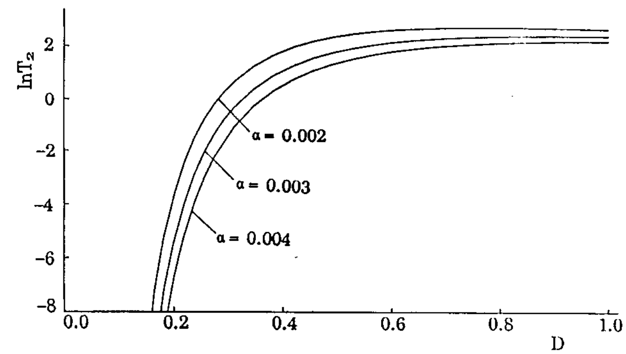

FIG. 9. The MFPT of the bistable system for $\lambda=1$ (16) as a function of D. α=0.002, 0.003, and 0.004, respectively.

When we take the values α and D in the neighborhood of $\alpha=D$ in the valid region, it is interesting to point out that the MFPT (15) for $\alpha=D$ is the biggest in the three cases $\alpha>D$, $\alpha=D$, and $\alpha<D$, under the same λ as shown in Fig. 6.

For the case of perfectly correlated noises ($\lambda=1$), there are two valid regions (regions II and IV) in Fig. 3. In the valid region II the MFPT (16) corresponding to $\alpha>D$ and $\alpha<D$ exhibits very different behaviors. When α is fixed, the MFPT for $\alpha>D$ increases with increasing D but that for $\alpha<D$ decreases with increasing D in Fig. 7. When D is fixed, the MFPT for $D>\alpha$ increases as α increases, but that for $D<\alpha$ decreases as α increases in Fig. 8. For the particular case of $\lambda=1$, and $\alpha=D$, the MFPT diverges to infinity due to the factor

$$|1-\sqrt{D/\alpha}|^{-\tilde{\beta}_1/D}\rightarrow\infty$$

in (16) as shown in Figs. 7 and 8. In fact, with two identical noises, the Langevin equation (1) can be written as

$$\dot{x}=x-x^3+x\xi(t)+\xi(t),$$

changing the variable to $u=(x+1)$,

$$\dot{u}=-u(u-1)(u-2)+u\xi(t),$$

which corresponds to pure multiplicative noise acting on a bistable system whose minima are now at $u=0$ and $u=2$. Since we are considering escape from the potential well at $u=0$ (i.e., $x=-1$), where the multiplicative noise has no effect at all, so that the MFPT is infinite. In the valid region IV, because $D\gg\alpha$, there is no the phenomenon as mentioned above. The MFPT increases monotonically with increasing D as can be seen in Fig. 9.

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[1] R. Roy, A. W. Yu, and S. Zhu, Phys. Rev. Lett. **55**, 2794 (1985).
[2] S. Zhu, A. W. Yu, and R. Roy, Phys. Rev. A **34**, 4333 (1986).
[3] R. F. Fox and R. Roy, Phys. Rev. A **35**, 1838 (1987).
[4] E. Peacock-Lopez, F. J. de la Rubia, B. J. West, and K. Lindenberg, Phys. Rev. A **39**, 4026 (1989).
[5] S. Kai, S. Wakabayashi, and M. Imasaki, Phys. Rev. A **33**, 2612 (1986).
[6] A. Diaz-Guilera and J. M. Rubi, Phys. Rev. A **34**, 462 (1986).
[7] M. F. Schumaker and W. Horsthemke, Phys. Rev. A **36**, 354 (1987).
[8] P. Reineker, B. Kaiser, and A. M. Jayannavar, Phys. Rev. A **39**, 1469 (1989).
[9] J. J. Brey, J. M. Casado, and M. Morillo, Z. Phys. B **66**, 263 (1987).
[10] N. G. Stocks, R. Mannella, and P. V. E. McClintock, Phys. Rev. A **42**, 3356 (1990).
[11] E. Guardia and M. San Miguel, Phys. Lett. **109A**, 9 (1985).
[12] R. F. Fox, Phys. Lett. **94A**, 281 (1983).
[13] I. I. Fedchenia, J. Stat. Phys. **50**, 1043 (1988).
[14] I. I. Fedchenia, J. Stat. Phys. **52**, 1005 (1988).
[15] A. Fulinski and T. Telejoko, Phys. Lett. A **152**, 11 (1991).
[16] S. Zhu, Phys. Rev. A **47**, 2405 (1993).
[17] L. Cao and D. J. Wu, Phys. Lett. A **185**, 59 (1994).
[18] D. J. Wu, L. Cao, and S. Z. Ke, Phys. Rev. E **50**, 2496 (1994).
[19] Y. Jia, Commun. Theor. Phys. **23**, 45 (1995).
[20] Y. Jia, L. Cao, and D. J. Wu, Phys. Rev. A **51**, 3196 (1995).
[21] R. L. Stratonovich, *Topics in the Theory of Random Noise*, Vol. 1 (Gordon and Breach, New York, 1963).
[22] K. Lindenberg and B. J. West, J. Stat. Phys. **42**, 201 (1986).
[23] J. Masoliver, B. J. West, and K. Lindenberg, Phys. Rev. A **35**, 3086 (1987).
[24] C. W. Gardiner, *Handbook of Stochastic Methods*, Springer Series in Synergetics Vol. 13 (Springer-Verlag, Berlin, 1983).
[25] P. Hanggi, F. Marchesoni, and P. Grigolini, Z. Phys. B **56**, 333 (1984).
[26] R. F. Fox, Phys. Rev. A **33**, 467 (1986).