PHYSICAL REVIEW E VOLUME 53, NUMBER 5 MAY 1996

Stable solitons in two-component active systems

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(Received 10 October 1995)

As is known, a solitary pulse in the complex cubic Ginzburg-Landau (GL) equation is unstable. We demonstrate that a system of two linearly coupled GL equations with gain and dissipation in one subsystem and pure dissipation in another produces absolutely stable solitons and their bound states. The problem is solved in a fully analytical form by means of the perturbation theory. The soliton coexists with a stable trivial state; there is also an unstable soliton with a smaller amplitude, which is a separatrix between the two stable states. This model has a direct application in nonlinear fiber optics, describing an erbium-doped laser based on a dual-core fiber.

PACS number(s): 42.81.Dp, 03.40.Kf, 42.81.Qb, 52.35.Sb

Localized pulses (solitons) play a central role in a number of nonlinear physical systems, which are now a subject of very broad interest [1,2]. Since real systems are always lossy, it is necessary to have an active element in the system in order to provide gain compensating the losses. In plasma physics and hydrodynamics, the gain is provided by intrinsic instabilities of the system [3]. For solitons in nonlinear optical fibers (NOF's), an effective way to compensate for the losses is by use of erbium-doped amplifiers [2]. Here, however, one encounters a fundamental problem: if the active element is uniformly distributed along the length of the system, it automatically renders the zero solution unstable, thus lending an instability to the whole soliton. A commonly known model which demonstrates this property is the complex cubic Ginzburg-Landau (GL) equation, which in many cases, and especially in application to NOF's, may be regarded as a perturbed nonlinear Schrödinger (NLS) equation [3]:

$$iu_t + \frac{1}{2}u_{xx} + |u|^2 u = i\gamma_0 u + i\gamma_1 u_{xx} - i\gamma_2 |u|^2 u\,, \qquad (1)$$

where $u(x,t)$ is an envelope function (e.g., of electromagnetic waves in the NOF's), t and x are the spatial and temporal variables (in the NOF's their physical meaning is reversed), and γ_0 is the gain, while the coefficients γ_1 and γ_2 account for the dispersive and nonlinear losses, and all the γ's are assumed non-negative. It is well-known that Eq. (1) admits an exact solitary-pulse solution, which in the limit of a vanishing right-hand side of Eq. (1) goes over into the soliton of the NLS equation [3]. Regarding the pulses in the model (1) as perturbed NLS solitons, it has also been demonstrated that they are able to form two-soliton and multisoliton bound states, which are stable against disturbances of the separation and phase differences between the solitons [4]. However, it is obvious that, since $\gamma_0 > 0$, the trivial solution $u = 0$ is unstable in this model; hence an isolated pulse as a whole is unstable too [it was demonstrated numerically that locally stable pulses are possible in Eq. (1) with *negative* γ_0 and γ_2 [5]; we do not consider this case here as the model is then globally unstable]. This circumstance does not render the pulses meaningless objects — they may be *effectively* stable when the system is short enough, or the evolution is considered at finite times. In the general case, however, development of the instability leads to dynamical chaos [6]. An example is the so-called "dispersive chaos" experimentally observed in binary-fluid convection [7].

A problem of fundamental interest is to find sufficiently simple physical models which can produce totally stable pulses. One example is the driven damped NLS equation [8], which has various applications, including those in nonlinear fiber optics [9]. However, in that model the pulses are not truly localized, being supported by an oscillating background. Experimentally, absolutely stable localized pulses of traveling-wave convection were discovered beneath the instability onset in narrow channels filled with a binary fluid heated from below [10]. A distinctive feature of a model supporting stable pulses is bistability, as, being stable, the localized pulses must coexist with the stable trivial solution. The simplest way to provide for the bistability is to introduce a *quintic* GL equation [11]. In terms of nonlinear fiber optics, this equation models a nonlinear amplifier [12], which can be built, e.g., as a combination of linear amplifier and a saturable absorber with an instantaneous response. In this case, the quintic equation can be obtained by means of a truncated Taylor expansion of the gain and loss characteristic of the medium.

Within the framework of the latter model, the existence of stable pulses can be demonstrated analytically

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1063-651X/96/53(5)/5365(4)/$10.00

in two opposite cases: when the quintic GL equation is close to the NLS equation [13], and when it is close to the real GL equation [14]. For intermediate values of the parameters, existence of the stable pulses in the model was demonstrated numerically [15]. Stable localized pulses for which the quintic equation seems to be an appropriate phenomenological model were observed in the form of subcritical pulses in the traveling-wave convection in narrow channels [10]. Similar pulses were produced by numerical simulations of the full system of the corresponding two-dimensional hydrodynamic equations [16]. However, the quintic GL equation proves to be a crude model, even at the phenomenological level, for the pulses observed in the experiment. The most essential feature which finds no explanation in the framework of this model is the very low velocity of the pulse in the laboratory reference frame [17]. As demonstrated by Riecke [18], an essential improvement of the description of the experimentally observed pulses is achieved within the framework of a model coupling the *cubic* GL equation (actually, two such equations for counterpropagating waves) to an additional equation for a real variable, which is a "mean field," representing in this problem a concentration field. It is well known that coupling of the GL equation to a real mean-field mode can drastically change the dynamics of the GL model [19].

In this work, we aim to put forward a model of another type supporting absolutely stable solitons. Actually, the model will generate two solitons, one stable and one unstable, the unstable one being a separatrix between attraction domains of the stable soliton and of the stable trivial solution. It will also be demonstrated that the model may support stable bound states of solitons.

This model describes a dual-core NOF (also known as a directional coupler [20]), in which one core is active (i.e., doped with erbium), while the other is passive (undoped). The couplers have attracted a lot of attention due to the possibility of applications in photonics [21]. In Refs. [22] and [23], a coupler with one active core was proposed. It was demonstrated that, by adding the second passive core with loose ends to the usual doped fiber used in soliton-generating lasers, one can essentially improve the quality of the generated solitons: while the soliton, being a strongly nonlinear object, remains essentially confined to the active core, linear noise can readily couple to the passive core, where it is radiated away through the loose ends. This provides effective filtering of the noise. Independently, in Ref. [24] it was proposed to use a coupler for separating noise from the soliton in an existing pulse. The basic physical idea proposed in Ref. [24] is the same as in Ref. [22]: the soliton stays in the core in which it is propagating, while the noise easily couples to the second core. It was assumed in Ref. [24] that the second core was lossy, so that the noise tunneling to that core would be killed there by the losses. An important result obtained both analytically and numerically in Ref. [24] is that the best efficiency of the filtering is attained, not at very strong losses in the second core, but in the case when the loss coefficient is close to the coupling constant between the two cores.

The numerical simulations reported in Refs. [22]–[24] demonstrate that the soliton seems very stable in these models (moreover, in Ref. [23] it was demonstrated that the model produced a robust soliton even in the case when dispersion in the active core was *normal*, i.e., the solitons could not exist in the usual NLS equation). These observations suggest analyzing the existence and stability of solitary pulses in two-component models. A remarkable fact is that, as will be shown below, this problem can be consistently solved in a fully analytical form.

We adopt the following model, which is a combination of the more special ones from Refs. [22] and [24] [unlike in Eq. (1), here we will use the standard "optical" notation for the spatial and temporal variables z and τ]:

$$iu_z + \frac{1}{2}u_{\tau\tau} + |u|^2u - i\gamma_0 u - i\gamma_1 u_{\tau\tau} + \kappa v = 0, \tag{2}$$

$$iv_z + \frac{1}{2}v_{\tau\tau} + |v|^2v + i\Gamma_0 v + \kappa u = 0, \tag{3}$$

where the variables u and v are envelopes of the electromagnetic waves in the active and passive cores, the coefficients γ_0 and γ_1 are the same as in Eq. (1), the nonlinear dissipation is neglected (it can be easily restored), κ is the coupling constant, and Γ_0 is the loss coefficient in the passive core. An additional loss coefficient Γ_1 in the passive core, similar to γ_1 in the active one, may be added to the model. However, the additional lossy term would render the following analysis more cumbersome without making any essential change. Contrary to this, it will be demonstrated that keeping the term $\sim \gamma_1$ in Eq. (2) is necessary.

As concerns physical applications of this model, it was already mentioned that it was closely related to the ones describing a soliton-generating laser [22] (but in that case, $\Gamma_0 = 0$) and a time-domain fiber filter [24] (however, $\gamma_0 = \gamma_1 = 0$ in the latter model). Very recently, it was demonstrated that the model based on Eqs. (2) and (3) in their full form finds another practically important application: it is the basis for a new type of nonlinear optical amplifier [25].

First of all, we will consider the stability of the trivial solution $u = v = 0$. Inserting into linearized Eqs. (2) and (3) $(u, v) \sim \exp(\sigma z - i\omega t)$, one can immediately obtain the relation between the instability growth rate σ and the perturbation frequency ω:

$$\tilde{\sigma}^2 + \left(\Gamma_0 - \gamma_0 + \gamma_1\omega^2\right)\tilde{\sigma} + \kappa^2 - \Gamma_0\gamma_0 + \gamma_1\Gamma_0\omega^2 = 0,$$
$$\tilde{\sigma} \equiv \sigma + \frac{1}{2}i\omega^2. \tag{4}$$

The stability condition for the trivial solution implies that $\mathrm{Re}\,\tilde{\sigma} \leq 0$ at all real ω, which is equivalent to demanding that the free term and the coefficient in front of the linear term in Eq. (4) are always ≥ 0. Eventually, this amounts to

$$\gamma_0 < \Gamma_0 < \kappa^2/\gamma_0. \tag{5}$$

Obviously, a necessary condition following from Eq. (5) is $\gamma_0 < |\kappa|$, i.e., the coupling must be stronger than the gain. Notice that the coefficient γ_1 does not appear in Eq. (5).

Now, let us consider evolution of the soliton in the system of Eqs. (2) and (3). We will treat this problem perturbatively, assuming that the coupling, gain, and losses are all small perturbations to the NLS equation, although different perturbations may have different orders of smallness. Actually, we will assume that the gain and losses in the active core are essentially weaker than the coupling between the cores, while the losses in the passive one may be comparable to the coupling. Anyway, the perturbative treatment of all these terms is quite reasonable in application to the NOF's.

The soliton has two nontrivial parameters, viz., the amplitude and velocity. The simplest way to derive perturbation-induced evolution equations for these parameters is to use the so-called balance equations for the energy and momentum of the soliton [1]. In our model, the evolution equation for the velocity will be exactly the same as in the model (1), which describes the process of "braking" of the soliton by the linear frequency-dependent losses [1]; therefore we will not consider this equation, and will simply set the velocity equal to zero. Then, in the zeroth-order approximation, the soliton resides only in the first core and has the form

$$u = \eta \,\mathrm{sech}(\eta\tau)\, e^{i\phi(z)}, \tag{6}$$

where $d\phi/dz = \frac{1}{2}\eta^2$, and η is the soliton's amplitude. In the next approximation, one seeks for a component ("shadow") of the soliton in the second core [1]. Obviously, it has the form $u(z,\tau) = V(\tau)\exp[i\phi(z)]$, where the real function $V(\tau)$ is determined by the equation following from Eqs. (3) and (6):

$$\frac{d^2V}{d\tau^2} - \eta^2 V = -2\kappa\eta\,\mathrm{sech}(\eta\tau) \tag{7}$$

[at this step, we neglect the lossy term in Eq. (3) which enters only at the next order]. Finally, the slow evolution of the soliton's amplitude under the action of the perturbations is determined by the above-mentioned balance equation for the quantity which plays the role of energy in nonlinear optics:

$$N \equiv \int_{-\infty}^{+\infty} \left[|u(\tau)|^2 + |v(\tau)|^2\right] d\tau. \tag{8}$$

An exact balance equation following from Eqs. (2) and (3) is

$$\frac{dN}{dz} = 2\gamma_0 \int_{-\infty}^{+\infty} |u(\tau)|^2 d\tau - 2\gamma_1 \int_{-\infty}^{+\infty} |u_\tau(\tau)|^2 d\tau - 2\Gamma_0 \int_{-\infty}^{+\infty} |v(\tau)|^2 d\tau. \tag{9}$$

Now, one should find $V(\tau)$ from Eq. (7) and insert it into the last term on the right-hand side of Eq. (9), while the contribution from v to the left-hand side [see Eq. (8)] may be neglected in the lowest nontrivial approximation. Equation (7) can be solved by means of the Fourier transformation, which leads to the following integral representation for $V(\tau)$:

$$|V(\tau)| = |\kappa| \int_{-\infty}^{+\infty} \mathrm{sech}\,(\eta\tau')\, e^{-\eta|\tau-\tau'|} d\tau'. \tag{10}$$

Inserting this expression into the last term of Eq. (9), one can explicitly calculate all the integrals, which eventually yields the evolution equation sought:

$$\frac{d\eta}{dz} = 2\gamma_0\eta - \frac{2}{3}\gamma_1\eta^3 - C\kappa^2\Gamma_0\eta^{-3}, \tag{11}$$

where $C \equiv \frac{1}{6}\pi^2 + \zeta(3) \approx 2.845$. The formal singularity of the last term in Eq. (11) at $\eta \to 0$ is fictitious, as this expression is irrelevant at very small η; see below.

It is straightforward to see that Eq. (11) gives rise to two physical ($\eta^2 > 0$) fixed points, provided that

$$\gamma_0^3 > (3C/8)\kappa^2\Gamma_0\gamma_1^2, \tag{12}$$

and to no fixed points in the opposite case. Thus (12) is the necessary and sufficient condition for existence of solitons in the considered model. It is, of course, important to check if this condition is compatible with the other fundamental condition, (5), which is necessary for the stability of solitons in the model. Because Eq. (5) does not involve the parameter γ_1, one can secure the compatibility simply by choosing γ_1 to be small enough.

Next, it is easy to check that, once the condition (12) is met, the solution with larger η^2 is stable, and the one with smaller η^2 is unstable. It is very plausible that the soliton corresponding to the larger root is a completely stable solution in the full model, while the smaller root corresponds to an unstable soliton which plays the role of a separatrix between the stable soliton and the stable trivial solution.

Usually, existence of solitons is related to modulational instability of continuous wave (cw) solutions [2]. The cw modulational (in)stability in the present model will be considered in detail elsewhere. However, in the regime which is akin to the case considered in this work, i.e., when the coefficients of the gain and loss are small, the coupling constant is small too, and the field in one core (v) is therefore much weaker than in the other (u); the modulational instability in the present model is, evidently, close to that in the single NLS equation. Thus, as well as in the usual NLS equation, one may qualitatively regard the solitons in the present model as pulses produced by the modulational instability of the cw.

It is now relevant to discuss conditions guaranteeing application of the perturbation theory to this problem. The "primary" conditions are $\gamma_1 \ll 1$, $|\kappa| \ll \eta^2$, and $\Gamma_0 \ll \eta^2$. Since, in a typical case, $\eta^2 \sim \gamma_0/\gamma_1$, the final set of applicability conditions takes the form

$$\gamma_1 \ll 1, \quad \gamma_1\Gamma_0 \ll \gamma_0, \quad \gamma_1|\kappa| \ll \gamma_0. \tag{13}$$

Obviously, these conditions are compatible with the underlying inequalities (5) and (12).

When formulating the model, we have omitted the dispersive loss term $i\Gamma_1 u_{\tau\tau}$ in Eq. (3). If kept, it will generate an additional term $\sim -\kappa^2\Gamma_1\eta^{-1}$ in Eq. (11). It is easy to check that the latter term will not produce any qualitative difference in the properties of the fixed points. On the other hand, if the term $\sim \gamma_1$ is omitted in Eq. (2), the result will be disastrous: there will remain a single unstable fixed point. Actually, the model is globally unstable in that case. Finally, if the nonlinear lossy term $\sim \gamma_2$ (the two-photon absorption, in terms of the nonlinear optics) is added to Eq. (2) [cf. Eq. (1)], it will merely renormalize the coefficient γ_1 in the final results displayed above.

Now, we will briefly consider bound states (BS's) of the solitons in this model, following the lines of Ref. [4]. The BS's may exist due to the fact that the small linear terms accounting for the gain and dissipation render solitons' tails oscillatory, which, in turn, gives rise to local minima in the effective potential of the soliton-soliton interaction, produced by overlapping of the "head" of each soliton with the tail of the other one. However, in the framework of perturbation theory, the BS's are fragile, although stable: the distance between the solitons in the BS is large, and, accordingly, the corresponding binding energy is exponentially small. Nevertheless, the existence and stability of the BS's predicted by the perturbation theory was confirmed by direct numerical simulations [26] of the driven damped NLS equation [8] (see also [9]); recently, this prediction was also confirmed, with fairly good accuracy, for the cubic GL equation (1) [27]. In the present model, the BS's can be rendered more robust by increasing the dissipative constant Γ_0 in the passive core. Therefore, we will consider the case $\Gamma_0 \gg \gamma_0, \gamma_1\eta^2$, which is compatible with all the above conditions necessary for the existence of stable solitons. In this case, we again consider linearized Eqs. (2) and (3); however, instead of the plane-wave solution leading to the dispersion equation (4), we are now interested in an exponentially decaying solution describing the soliton's tail:

$$(u,v) \sim \exp(-\eta|\tau| + i\chi|\tau| + iqz), \tag{14}$$

where we consider η as a given soliton's inverse size [cf. Eq. (6)], while χ and q must be found. The linearized equations immediately yield (with regard to the assumed dominance of Γ_0)

$$\chi = \Gamma_0/2\eta, \quad \left(q - \frac{1}{2}\eta^2 + \frac{1}{2}\chi^2\right)^2 = \kappa^2 - \frac{1}{4}\Gamma_0^2. \tag{15}$$

According to Ref. [4], the minimum separation T between solitons in the BS is determined by the coefficent χ in Eq. (14): $T = \pi/2\chi = \pi\eta/\Gamma_0$, where we have made use of Eq. (14). On the other hand, the second relation in Eq. (15) imposes a fundamental limitation $\Gamma_0 < 2|\kappa|$, otherwise the soliton simply does not exist. This, in turn, leads to a limitation on the minimum separation between the bound solitons: $T > \pi\eta/2\kappa$.

In conclusion, we have demonstrated that a simple analytically tractable model, based on the linearly coupled cubic GL equations, admits a fully stable soliton coexisting with the stable trivial state. The model finds a direct physical realization in terms of the fiber laser and suggests a way to stabilize soliton generation in the laser.

Note added in proof. The existence of the solitons predicted in the present work analytically has been very recently corroborated by direct numerical simulations [J. Atai and B. A. Malomed (unpublished)].

This research was partially supported by the National Science Foundation through the Center for Ultrafast Optical Science at the University of Michigan, Ann Arbor.

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